Lecture 2: Regular Maps

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Maps

A map M is an embedding of a connected graph or multi-graph X into a surface S, having the property that all of the components of $S \setminus X$ (obtained by removing X from S) are homeomorphic to unit disks — called the faces of the map.

Orientability and genus

A map is called orientable or non-orientable depending on whether it lies on an orientable surface (such as the sphere, the torus or the double torus) or a non-orientable surface (such as the real projective plane or the Klein bottle).

Any such map on the sphere (or equivalently, the Euclidean plane) is called a planar map, or a map of genus 0.

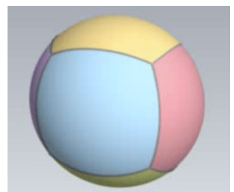
The Four-Colour Theorem is all about planar maps.

Generally, the genus of a map is the genus of the supporting surface. For an orientable surface S, this is the number of 'handles' that need to be attached to a sphere in order to obtain S. For example, maps on the torus have genus 1.

Regular maps

The five Platonic solids 

can be viewed as embeddings of symmetric graphs on the sphere

... e.g. the cube 

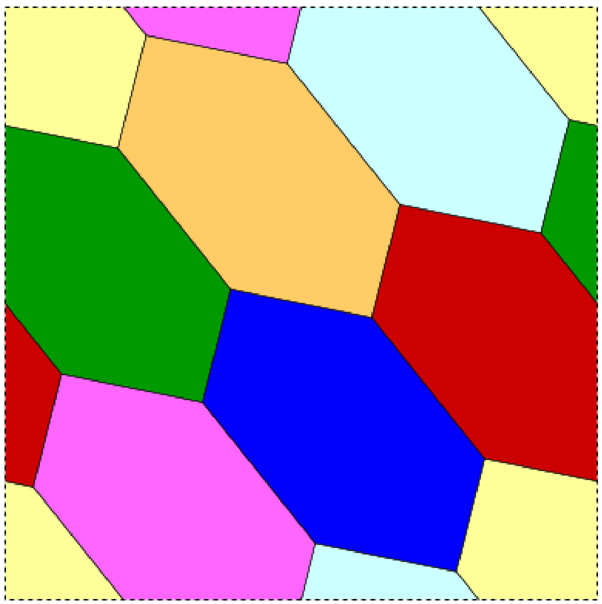
In each case, the automorphism group of the map has a single orbit on incident vertex-edge-face triples (or 'flags').

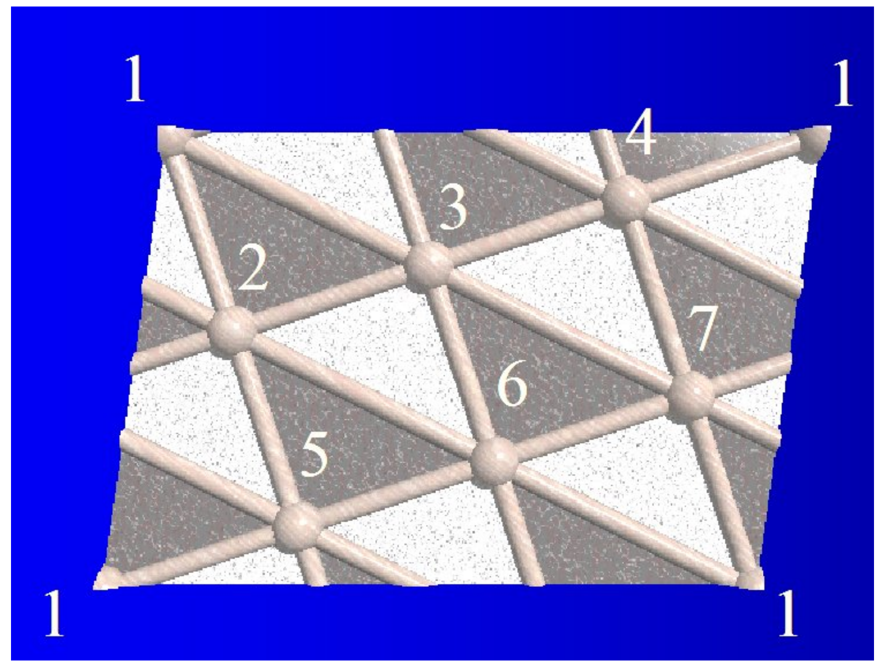
Any such graph embedding is called a regular map.

The face-size m and the vertex-degree k give its type $\{m, k\}$.

The types of the five Platonic maps are $\{3, 3\}$, $\{4, 3\}$, $\{3, 4\}$, $\{5, 3\}$ and $\{3, 5\}$. Note that they are all reflexible.

Symmetric maps on the torus



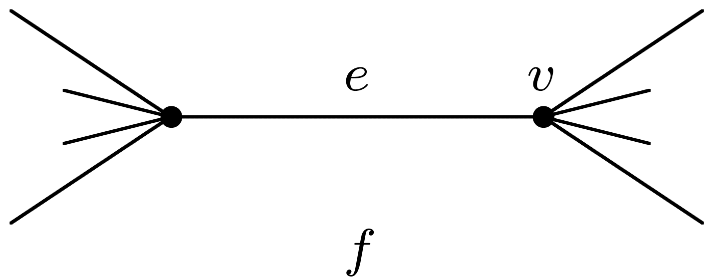


The one on the right is a symmetric embedding of the complete graph K_7. It has type $\{3, 6\}$, and is dual to the one on the left, which has type $\{6, 3\}$. Both maps are chiral.

Automorphisms of maps

An automorphism of a map M is an incidence-preserving permutation of each of the (three) sets of vertices, edges and faces of M.

Any automorphism θ of a map is completely determined by its effect on a given flag (incident vertex-edge-face triple):



Neighbours of the vertex v are taken by θ to neighbours of v^θ, in order, and vertices of the face f are taken by θ to vertices of f^θ, and so on. By connectedness, θ is uniquely determined by the image $(v^\theta, e^\theta, f^\theta)$ of the flag (v, e, f).

Regular maps and rotary maps

Let M be a map, with edge-set $E = E(M)$. Because every automorphism of M is determined by its effect on a given flag, and each edge lies in at most 4 flags, it follows that

$$|\text{Aut}\, M| \le 4|E|.$$

If this bound is attained, Aut M is regular (sharply-transitive) on the flags of M, and M is called a fully regular map.

Similarly, if the carrier surface is orientable, and $\text{Aut}^{\circ}M$ is the group of all orientation-preserving automorphisms of M, then $|\text{Aut}^{\circ}M| \le 2|E|$. When this bound is attained, $\text{Aut}^{\circ}M$ acts regularly on the arcs of M, and the map M is called orientably-regular (or rotary, or sometimes regular!).

‘Regular maps’: some history

The theory of regular maps has been developed by Brahana (1920s), Coxeter, Wilson, Jones & Singerman, et al.

Deep connections to algebraic geometry & Galois theory — Belyi (1979), Grothendieck (1997), Jones et al (2007).

Possible applications to structural chemistry (fullerenes and nanotubes) are being investigated.

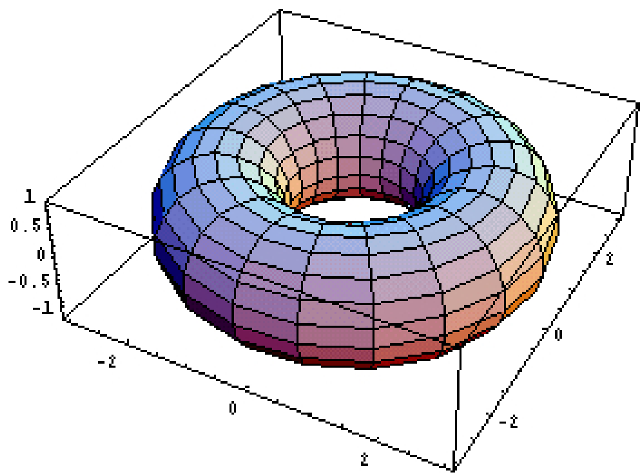
Regular maps are viewed from 3 main perspectives:

- Classification by *underlying graph*
- Classification by *surface*
- Classification by *automorphism group.*

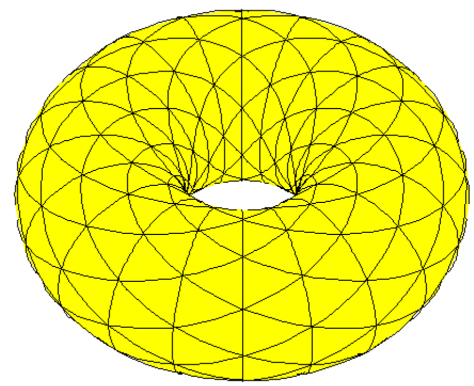
The type of a regular/rotary map

If M is regular or rotary, then M is arc-transitive and face-transitive, so every vertex of M has the same valency/degree (say k) and every face of M has the same size (say m).

We call $\{m, k\}$ the type of M.



Type $\{4, 4\}$



Type $\{3, 6\}$

Genus calculation

If M is an orientably-regular map, with $|V|$ vertices, $|E|$ edges and $|F|$ faces, then by arc-transitivity, we have

$$k|V| = 2|E| = m|F| = |\mathrm{Aut}^{\mathrm{o}} M|$$

so by the Euler-Poincaré formula, the characteristic χ and genus g of the carrier surface (and the map) are given by

$$2 - 2g = \chi = |V| - |E| + |F| = |\mathrm{Aut}^{\mathrm{o}} M|\,(\frac{1}{k} - \frac{1}{2} + \frac{1}{m}).$$

This relates the group order to the genus, via the type.

Classification of regular maps on the sphere

For the sphere, the genus formula gives

$$2 = |V| - |E| + |F| = |\mathrm{Aut}^{\mathrm{o}} M| \, (\frac{1}{k} - \frac{1}{2} + \frac{1}{m}),$$

and since the LHS is positive, we find that $\frac{1}{k} + \frac{1}{m} > \frac{1}{2}$ and so the only possibilities with $k, m \geq 2$ are as follows:

Type $\{m, 2\}$ – the cycle graph C_m drawn on the equator

Type $\{2, k\}$ – with k edges between 2 antipodal vertices

Type $\{3, 3\}$ – the tetrahedral map

Types $\{3, 4\}$ and $\{4, 3\}$ – the octahedral and cube maps

Types $\{3, 5\}$ and $\{5, 3\}$ – icosahedral and dodecahedral maps

Automorphism groups of the Platonic maps

Object	Rotation group	Automorphism group
Tetrahedron	A_4	S_4
Cube	S_4	$S_4 \times C_2$
Octahedron	S_4	$S_4 \times C_2$
Icosahedron	A_5	$A_5 \times C_2$
Dodecahedron	A_5	$A_5 \times C_2$

Classification of regular maps on the torus

For the torus (which is orientable, with genus 1), we have

$$0 = |V| - |E| + |F| = |\mathrm{Aut}^{\mathrm{o}} M|\,(\frac{1}{k} - \frac{1}{2} + \frac{1}{m}),$$

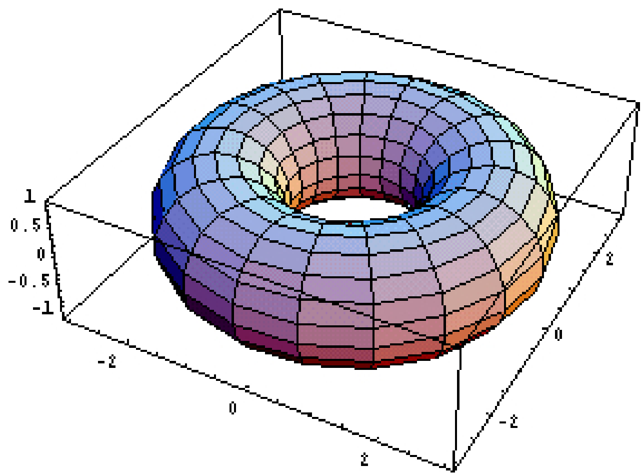
and hence $\frac{1}{k} + \frac{1}{m} = \frac{1}{2}$, which has only three solutions, namely $(m, k) = (3, 6)$, $(4, 4)$ and $(6, 3)$, giving these possibilities:

Type $\{3, 6\}$ – regular triangulations

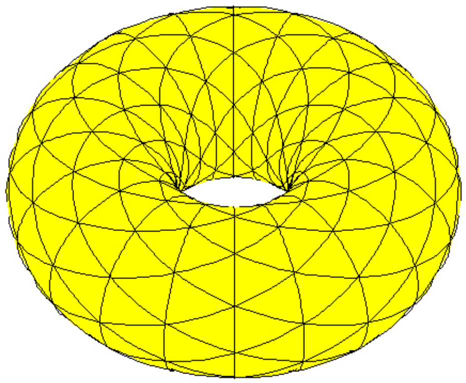
Type $\{6, 3\}$ – honeycomb maps

Type $\{4, 4\}$ – regular quadrangulations.

Classification of regular maps on the torus (cont.)



Type $\{4,4\}$



Type $\{3,6\}$

Now ... What about regular maps of larger genera?

Regular maps of higher genera

For an orientably-regular map M of genus g, we have

$$2g-2=|E|-|V|-|F|=|\mathrm{Aut}^{\mathrm{o}}M|\,(\frac{1}{2}-\frac{1}{k}-\frac{1}{m}).$$

Now since $\frac{1}{2}-\frac{1}{k}-\frac{1}{m}$ is bounded above by $\frac{1}{2}$ and below by $\frac{1}{42}$ (when $(m,k)=(3,7)$ or $(7,3)$), we have

$$4(g-1)<|\mathrm{Aut}^{\mathrm{o}}M|\leq 84(g-1)$$

and since there are only finitely many given groups of a given order, we can expect only finitely many orientably-regular maps of given genus g, when $g>1$.

Similar formulae and inequalities hold for flag-transitive maps.

But: How do we find them?

Exercise

We have just considered the special cases of the sphere and the torus (orientable surfaces of genus 0 and 1).

What about non-orientable surfaces of small genus?

- For the projective plane (non-orientable, genus 1) we have

$$1 = |\mathrm{Aut}\, M|\,(\frac{1}{2k} - \frac{1}{4} + \frac{1}{2m}).$$

  What possibilities does this give for k and m and $|\mathrm{Aut}\, M|$?

- And for the Klein bottle (non-orientable, genus 2)?

And what are the possibilities for k and m and $|\mathrm{Aut}\, M|$ for an orientable surface of genus 2?

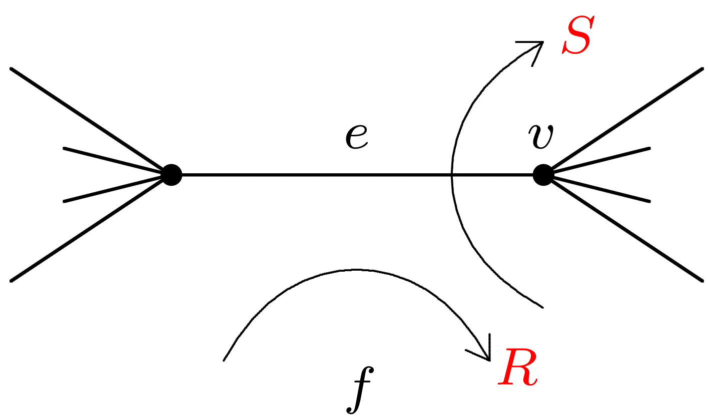
Some group-theoretic analysis

If M is a rotary map of type $\{m, k\}$, then for any flag (v, e, f) there exist automorphisms R and S such that

— R cyclically permutes consecutive edges of the face f

— S cyclically permutes consecutive edges incident to v

— RS reverses the edge e (and moves both v and f)



These satisfy $R^m = S^k = (RS)^2 = 1$, and also they generate an arc-transitive group, which must be $\mathrm{Aut}^{\mathrm{o}} M$ or $\mathrm{Aut}\, M$.

Some group-theoretic analysis (cont.)

Hence either $\mathrm{Aut}^{\mathrm{o}} M$ or $\mathrm{Aut}\, M$ is generated by two 'rotary' automorphisms R and S, about the face f and the incident vertex v from some flag (v, e, f), respectively. Note that

- R preserves f, but moves v and e,
- S preserves v, but moves e and f,
- RS preserves e, but moves v and f.

Moreover, if M is orientable, then $\langle R, S\rangle = \mathrm{Aut}^{\mathrm{o}} M$ and has order $2|E|$ (since $\langle R, S\rangle$ acts transitively on arcs and preserves orientation). On the other hand, if M is non-orientable then $\langle R, S\rangle = \mathrm{Aut}\, M$ and is transitive on flags, so has order $4|E|$.

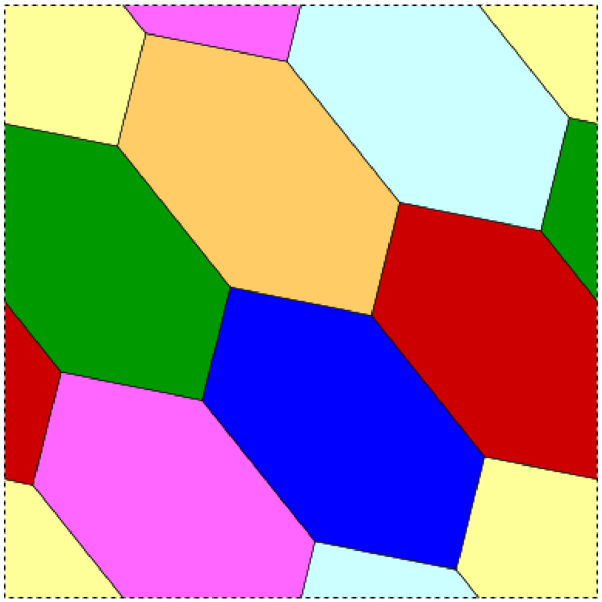
What about reflections?

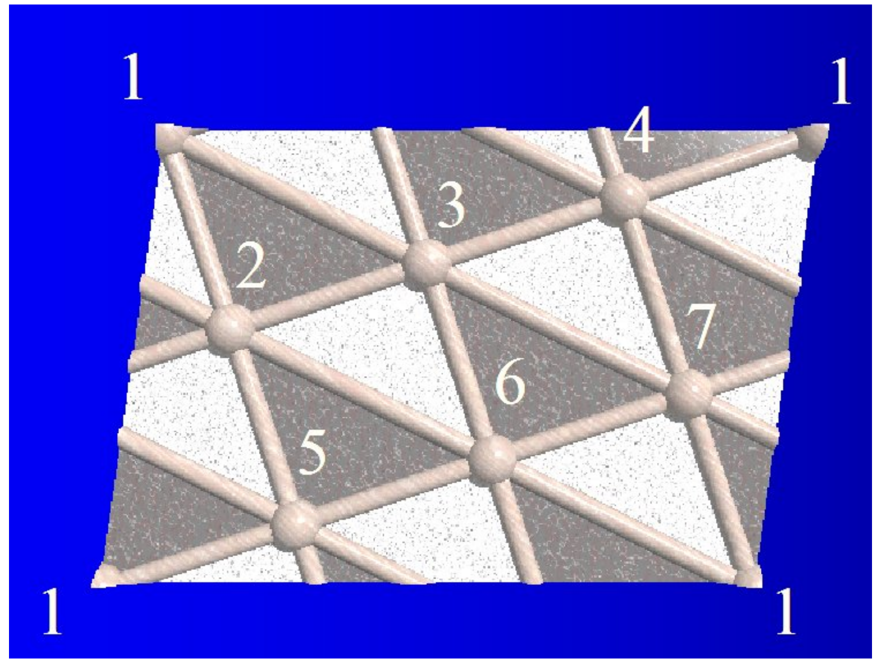
If M is orientable, then it might also admit reflections, which are orientation-reversing automorphisms of order 2. If so, then M is called reflexible; if not, then M is chiral.

Hence there are three kinds of rotary/regular maps:

- orientable and reflexible
  (with $|\mathrm{Aut}^{\mathrm{O}} M| = |\langle R, S\rangle| = 2|E|$ and $|\mathrm{Aut}\, M| = 4|E|$)
- orientable but chiral
  (with $|\mathrm{Aut}\, M| = |\mathrm{Aut}^{\mathrm{O}} M| = |\langle R, S\rangle| = 2|E|$)
- non-orientable
  (with $|\mathrm{Aut}\, M| = |\langle R, S\rangle| = 4|E|$).

Recall: examples of chiral maps on the torus





The one on the right is an embedding of the complete graph K_7. It has type $\{3, 6\}$, and is dual to the one on the left, which has type $\{6, 3\}$.

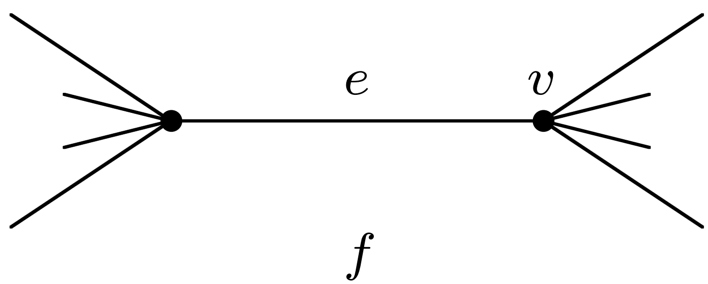
More analysis in the flag-transitive case

Let M be a flag-transitive map of type $\{m, k\}$. Then for any flag (v, e, f), there exist reflections a, b and c such that

— a preserves e and f, but moves v (... vertical axis)

— b preserves v and f, but moves e (... oblique axis)

— c preserves v and e, but moves f (... horizontal axis)



These satisfy $a^2 = b^2 = c^2 = (ab)^m = (bc)^k = (ac)^2 = 1$ and generate the flag-transitive group $\mathrm{Aut}\,M$. Also $ab = R$ and $bc = S$ (where R and S are as defined previously).

Connection with Triangle Groups

Our automorphisms R and S satisfy $R^m = S^k = (RS)^2 = 1$, so $\langle R, S \rangle$ is a quotient of the ordinary $(m, k, 2)$ triangle group

$$\Delta^o(m,k,2) = \langle\, x, y, z \mid x^m = y^k = z^2 = xyz = 1 \,\rangle$$

via a smooth epimorphism taking (x, y, z) to (R, S, RS).

Similarly, in the flag-transitive case, $\mathrm{Aut}\, M$ is a quotient of the full $(m, k, 2)$ triangle group

$$\Delta(m,k,2) = \langle\, a, b, c \mid a^2 = b^2 = c^2 = (ab)^m = (bc)^k = (ac)^2 = 1 \,\rangle.$$

As we will soon see, this also works in reverse!

Labelling by cosets

In each case, Aut M acts transitively on vertices, edges and faces, so for any given flag (v, e, f), we can label vertices, edges and faces of M respectively by (right) cosets of

- the stabilizer in Aut M of v (either $\langle S \rangle$ or $\langle b, c \rangle$)
- the stabilizer in Aut M of e (either $\langle RS \rangle$ or $\langle a, c \rangle$)
- the stabilizer in Aut M of f (either $\langle R \rangle$ or $\langle a, b \rangle$).

The action of Aut M on M is given by right multiplication.

Incidence (between vertices and edges, or between vertices and faces, or between edges and faces) is given by non-empty intersection of cosets. For example, the neighbour of v along the edge e is labelled with the coset $\langle S \rangle RS$ or $\langle b, c \rangle a$, which has non-empty intersection with $\langle RS \rangle$ or $\langle a, c \rangle$.

Construction for orientably-regular maps

Let G be any finite group that is generated by elements R and S (of order at least 2) such that RS has order 2.

Now define a map $M = M(G, R, S)$ by taking

$$\begin{aligned} \text{vertices} &= \text{right cosets in } G \text{ of } \langle S \rangle \\ \text{edges} &= \text{right cosets in } G \text{ of } \langle RS \rangle \\ \text{faces} &= \text{right cosets in } G \text{ of } \langle R \rangle \end{aligned}$$

with incidence given by non-empty intersection of cosets.

This makes M an orientably-regular map, with G acting by right multiplication as $\mathrm{Aut}^{\mathrm{o}} M$, and multiplication by S giving the ordering of edges around each vertex. The type of M is $\{m, k\}$ where $m = o(R)$ and $k = o(S)$.

Similar construction for (fully) regular maps

Let G be any finite group that is generated by three involutions a, b and c such that ac has order 2, and ab and bc have order at least 2.

Now define a map $M = M(G, a, b, c)$ by taking

$$\begin{aligned} \text{vertices} &= \text{right cosets in } G \text{ of } \langle b, c\rangle \\ \text{edges} &= \text{right cosets in } G \text{ of } \langle a, c\rangle \\ \text{faces} &= \text{right cosets in } G \text{ of } \langle a, b\rangle \end{aligned}$$

with incidence given by non-empty intersection of cosets.

This makes M a regular map, with G acting by right multiplication as $\operatorname{Aut} M$, and multiplication by $S = ca$ giving the ordering of edges around each vertex.

The type of M is $\{m, k\}$ where $m = o(ab)$ and $k = o(bc)$.

Reflexibility of orientably-regular maps

Let M be an orientably-regular map of type $\{m, k\}$, with $\mathrm{Aut}^{\circ}M$ generated by R and S s.t. $R^m = S^k = (RS)^2 = 1$.

Then M is reflexible if and only if Aut M is generated by three involutions a, b, c with $ab = R$ and $bc = S$. Whenever that happens we have $a = Rb$ and $c = bS$, with conjugation by b giving $R^b = (ab)^b = ba = R^{-1}$ and $S^b = (bc)^b = cb = S^{-1}$.

Thus: M is reflexible if and only if $G = \mathrm{Aut}^{\circ}M$ has an involutory automorphism β such that $R^{\beta} = R^{-1}$ and $S^{\beta} = S^{-1}$.

If no such automorphism β exists, then M is chiral, and the orientably-regular map $M' = M(G, R^{-1}, S^{-1})$ is a mirror image of M. In that case (M, M') is a chiral pair.

Orientability of flag-transitive maps

Let M be a flag-transitive map of type $\{m, k\}$, with Aut M generated by elements a, b and c such that $a^2 = b^2 = c^2 = (ab)^m = (bc)^k = (ac)^2 = 1$.

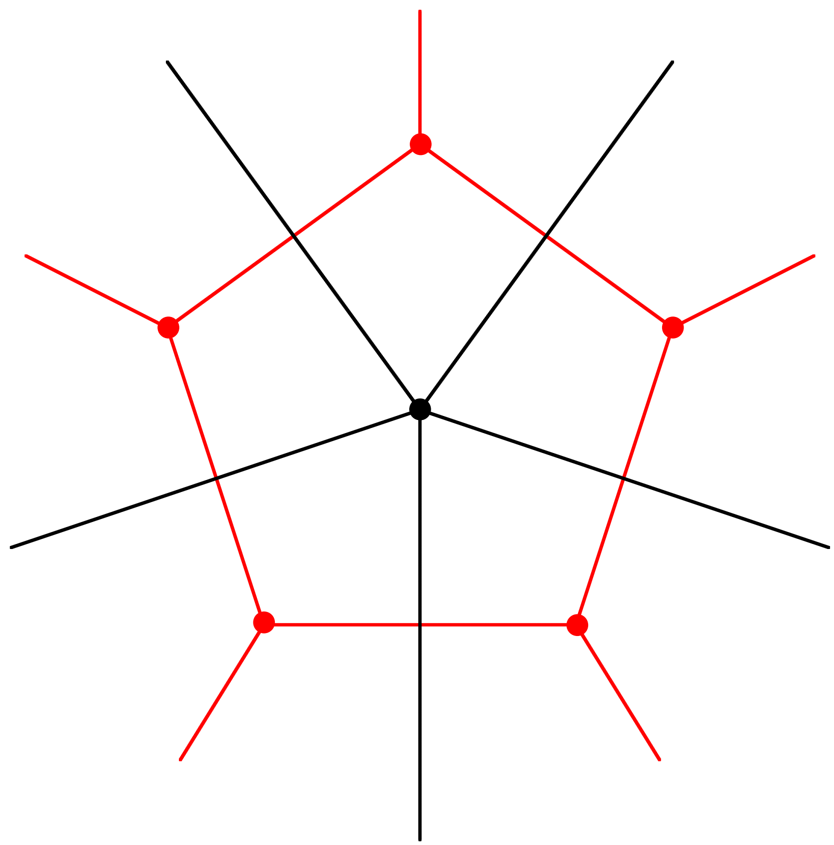
Then M is orientable if and only if the subgroup generated by $R = ab$ and $S = bc$ has index 2 in Aut M.

When this happens, the orientation-preserving group $\mathrm{Aut}^{\mathrm{o}} M$ is $\langle R, S\rangle = \langle ab, bc\rangle$, which consists of all the elements of Aut M that are expressible as words of even length in the generators a, b, c. [Those of odd length reverse orientation.]

On the other hand, the map M is non-orientable if and only if Aut $M = \langle a, b, c\rangle = \langle ab, bc\rangle = \langle R, S\rangle$, which happens if and only if there exists a relator of odd length in a, b and c.

Duality of rotary/regular maps

The geometric/topological dual M^* of a map M is obtained by taking faces of M as vertices of M^*, and vice versa:



Duality of rotary/regular maps (cont.)

Under duality, the stabilizer of a vertex of M is interchanged with the stabilizer of a face of M^*, and vice versa.

Algebraically, this is achieved by the correspondence $R \leftrightarrow S$, or in the flag-transitive case, by $(a, b, c) \leftrightarrow (c^b, b, a^b)$, which interchanges $\langle a, b\rangle$ with $\langle b, c\rangle$.

[Note: the more natural correspondence $(a, b, c) \leftrightarrow (c, b, a)$, which is used in the definition of polytope duals, takes M to the mirror image of M^*; hence the polytope dual of M is not isomorphic to M^* when M is chiral.]

The map M is called self-dual if M^* is isomorphic to M, or equivalently, if the correspondence $R \leftrightarrow S$ or $(a, b, c) \leftrightarrow (c^b, b, a^b)$ induces an automorphism of $\mathrm{Aut}\, M$.

Examples (on the sphere)

- The dual of the equatorial map (of type $\{m, 2\}$) is the antipodal map (of type $\{2, m\}$), and vice versa
- The tetrahedral map (of type $\{3, 3\}$) is self-dual
- The dual of the cube map (of type $\{4, 3\}$) is the octahedral map (of type $\{3, 4\}$), and vice versa
- The dual of the dodecahedral map (of type $\{5, 3\}$) is the icosahedral map (of type $\{3, 5\}$), and vice versa.

Finding regular maps of higher genera

Recall that if M is a rotary/regular map of type $\{m, k\}$ on a surface of Euler characteristic χ, then $G = \text{Aut}\, M$ is a quotient of the corresponding $(2, m, k)$ triangle group, and

$$\chi = |E| - |V| - |F| = |G|\,(\tfrac{1}{2} - \tfrac{1}{k} - \tfrac{1}{m}) \text{ or } |G|\,(\tfrac{1}{4} - \tfrac{1}{2k} - \tfrac{1}{2m}),$$

depending on whether M is flag-transitive (fully regular).

Hence finding all regular maps of given Euler characteristic χ reduces to finding all (smooth) quotients of relevant triangle groups of particular orders. This can be done using algebra and computation, to build up a census of examples ...

Summary of approach

- Take the ordinary or full $(2, m, k)$ triangle group
- Decide on the maximum desired order of Aut M (using the genus formula)
- Use computational methods to find all quotients of the triangle group of up to that order
- For each one, confirm the type, and then check for reflexibility, orientability, duality, and so on.

We now have lists of all regular maps of genus 2 to 300, and even have beautiful pictures of many of these, thanks to Jarke van Wijk (a computer scientist at Eindhoven) ...