Existentially Polytime Theorems (EP Theorems)

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and many other people.

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An **EP theorem** is an existence theorem
where each possible instance of what exists (a possible output)
is algorithmically easy to recognize relative to the size of the input.

We like proofs of EP theorems which are algorithms for finding an instance of what exists. We like the algorithms to be polynomial time (i.e.,“polytime”). **Open:** If you know it is there, can it be hard to find?

EP theorems are everywhere. You know many. You often know them with non-algorithmic proofs. We give here a tiny number of examples.

The Marriage Theorem

For any set of boy-nodes and any set of girl-nodes
and an edge between boy and girl if she loves him,
either there exists a way for all the boys to marry girls who love them
or there exists a subset of boys who aren’t loved by enough girls. (Not both)

The most beautiful and practical proof of this is a simple algorithm for finding a matching of boys to girls or finding an insufficiently loved subset of boys.

We don't know a good characterization in general
of graphs with Hamiltonian cycles,
but we know some EP theorems giving sufficient conditions
for Hamiltonian cycles.

For example, **Dirac's Theorem:**
If every node in graph G has degree at least half the number n of nodes, there exists a Hamiltonian cycle.

This is proved by a polytime algorithm which finds a Hamiltonian cycle or a node of degree less than n/2.

In fact, we have the **Dirac-Ore Lemma**:
For a graph G containing a Hamiltonian cycle H containing edge d = (u,v) if the (degree of u) + (degree of v) in G-v is at least the number of nodes, then there exists another Hamitonian cycle H' in G which does not contain edge d.

Another Theorem which looks similar:

For any graph G containing a Hamiltonian cycle H containing edge d, if G-E(H) is connected, there exists in G another Hamiltonian cycle H' which contains edge d.

The proof is a lovely algorithm.

It is an **open** question whether or not this algorithm is polytime.

A non-standard way of saying
the **Strong Perfect Graph Theorem**
(which until recently was a conjecture):

For any graph G, either there exists a Berge obstruction or there exists a colouring of nodes and a clique which are the same size.

Berge obstruction: an odd hole or an odd antihole.

Favourite Open Question: Does there exist a reasonably simple direct polytime algorithm for finding a Berge obstruction or a clique and colouring of the same size?

Partial results in this direction:

For a 0-1 matrix, a **balance obstruction** is an odd order submatrix with two 1's in each row and each column.
A "balanced matrix" means a 0-1 matrix without a balance obstruction.

Old Theorem of Berge and Las Vergnas:

A matrix of 0's and 1's contains a balance obstruction
or there exists a colouring of the columns
such that the ones of any row are differently coloured and such that
the number of colours is the largest number of 1's in a row.

Proved by a very simple algorithm for finding such a colouring
or finding an instance of a balance obstruction.
There is also known a very deep theory and algorithm for recognizing whether or not a given graph is balanced. However if what you want is hopefully
a desired colouring of columns regardless of balance,
then you don't need to first recognize whether or not the matrix is balanced.

A **Meyniel obstruction** is an induced subgraph which is a simple odd cycle with at most one chord. A **Meyniel graph** is a graph without a Meyniel obstruction.

A graph, G, is called **strongly perfect** (do not confuse with the SPGT)
when for every node v in every induced subgraph G' of G,
there is an independent set of nodes in G' (called a "strong independent set")
which hits every maximal clique of G'.
Using an algorithm for this, it is obvious how algorithmically to find in G
a colouring of the nodes and a clique which are the same size.

Open: Virtually nothing is known about strongly perfect graphs
other than that ***Meyniel graphs are strongly perfect***.
That is, we only know the following theorem:
For any graph G, either there is a Meyniel obstruction
or there is a strong independent set.

<u>This is not an EP theorem</u> since in general there is no easy way in general to recognize a strong independent set of nodes. (The predicate ``G contains a strong stable set'' may not be an NP-predicate because the definition of strong independent set is not a polytime certificate.)

To get an EP theorem we have discovered **“nice independent sets”**
which are strong and which are easy to recognize.
We prove that ***Meyniel graphs are strongly perfect*** by a polytime algorithm which in any graph, and for any node v,
finds either a nice independent set S containing v or else a Meyniel obstruction.

Meyniel graphs, in fact, illustrate three levels of algorithm:

First level: we assume the input graph has no Meyniel obstruction.
Thus the input to the algorithm is a Meyniel graph G and a vertex v of G,
and the output is a nice stable set S containing v.

Second level (a ``robust'' algorithm in the sense of Spinrad):
The input is any graph G and any vertex v of G.
The output is a nice stable set S containing v
or ***a declaration*** that G is not a Meyniel graph.

Third level (an EP search algorithm where each possible output
must be ``easily recognizable'').
The input is any graph G and any vertex v of G.
The output is a nice stable set containing v or a Meyniel obstruction.

Euler complexes (OIKs):

A **d-oik** C = (V, F), d >1, is a finite set V of elements called the vertices of C and a family of (d+1)-element subsets of V , called the **rooms** of C, such that every d-element subset of V is in an even number of the rooms.

A **wall** of a room means a set obtained by deleting one vertex of the room. Any wall of a room in an oik is the wall of a positive even number of rooms.

A **room-partition** R for oik M means a room-family whose rooms partition V, i.e., each vertex is in exactly one room of R.

Theorem: For any oik and any given room-partition, there exists another different room-partition. In fact, any oik has an even number of room partitions.

An OIK is called a pseudo-manifold when each wall is in exactly two rooms. There is a very simple algorithm for proving this theorem which we state for a pseudo-manifold.

Pick a vertex, w.
Unchoose the room r of our chosen starting room-partition which contains w. But keep its wall, r-w.
There is a unique other room s containing wall r-w.
Choose room s. Room s either contains w,
or else shares a vertex v with another chosen room, say r'.
Unchoose r'. Choose the room s'
which is the unique other room containing wall r'-v.
Room s' either contains w
or else shares a vertex v' with another chosen room, say r''.
Unchoose r''. Choose the room s''
which is the unique other room containing wall r''-v'. And so on.
This algorithm eventually stops when the newly chosen room contains w instead of having a vertex in common with another chosen room.

Surprisingly, we show that this algorithm is exponential.
Open: we do not know whether there exists a polytime algorithm.

Polytopal oiks:
Let Ax = b be non-degenerate linear system with positive b.
A feasible basis is maximal linearly independent set, B, of columns of A such that the solution, x, of Bx = b, is positive.

Theorem. The column-subsets of A, which are compliments of feasible bases of A, are the rooms of an oik manifold, in fact “the facets of a simplicial polytope”.

John Nash won a Noble prize, and had a movie made about him, by showing non-algorithmically that a two-person (bimatrix) game has a Nash equilibrium.
Lemke gave an algorithm applied to a pair of simplicial polytopes, which is essentially the same as our algorithm for finding a second room-partition.

The exponentiality of that algorithm depends on the polytopes having an exponential number of rooms relative to the size matrix A.
Our algorithm for finding a second room partition is exponential relative to the number of rooms.

A problem q is said to be **g-complete** for a class g of problems which includes q when a polytime algorithm for q would provide a polytime algorithm for any problem in g.

There is a class of problems called PPAD which includes
finding a second room-partition and finding a 2-person Nash equilibrium.

Finding a 2-person Nash equilibrium has been shown to be PPAD-complete
by a very difficult and technical proof involving matrices.

Open: Show that some simpler problem,
like finding a second room-partition for **explicitly listed** rooms,
is PPAD-complete.

Curiously, another **open question** is to find a theory that includes finding
a second room-partitioning and finding a second Hamiltonian path.
The algorithms are similar.

Necklace Theorem: *Two thieves have an open necklace with k colours of beads, an even number of each colour. No matter how the beads are linearly arranged, it is possible, by making only k cuts, for the thieves to share so that each gets the same number of each colour.*

This is easily proved from
the k-dimensional Ham-Sandwich Theorem
(discussed by Luis Montejano in the 1st talk)
(see www.math.cornell.edu/~eranevo/homepage/TopMethNotes-2Margarita.pdf)

Open: Find a reasonable algorithmic proof!