The number of complex realisations of a rigid graphs

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- Jackson, Jordán, Szabadka (2006) showed that $r(G, p)$ is the same for all generic rigid (G, p) when the rigidity matroid of G is connected and gave a formula for $r(G, p)$ in this case. This implies that $r(G, p) \leq 2^{n/2} \approx 1.14^n$ when G has a connected rigidity matroid.

Complex Realisations

- $r(G,p)$ is the number of real solutions to a system of quadratic equations. In this context it is natural to consider the number of complex solutions. This number should be better behaved than $r(G,p)$, and it will give an upper bound on $r(G,p)$.

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- Let $d : \mathbb{C}^2 \to \mathbb{C}$ by $d(x, y) = x^2 + y^2$.
- Two realisations (G, p) and (G, q) of a graph G in $\mathbb{C}^2$ are **equivalent** if $d(p(u) - p(v)) = d(q(u) - q(v))$ for all $e = uv \in E$.

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- Given a realisation (G, p) of a graph G in $\mathbb{C}^2$, let $c(G, p)$ denote the number of distinct equivalent realisations.

Genericness

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Problem

Can we determine $c(G)$ for a given rigid graph G?

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Suppose G is obtained from H by a type two Henneberg move peformed on a redundant edge of H. Then $c(G) \leq c(H)$.

Conjecture

If G is obtained from H by a type two Henneberg move peformed on a non-redundant edge of H then $c(G) > c(H)$.

Global Rigidity

Theorem

A graph G has $c(G) = 1$ if and only if either G is 3-connected and redundantly rigid or $G \in \{K_2, K_3\}$.

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Corollary

A graph G has $r(G, p) = 1$ for some generic real p if and only if $c(G) = 1$.

Separable graphs

Theorem

Suppose $G = G_1 \cup G_2$ for two edge-disjoint subgraphs G_1, G_2 with $V(G_1) \cap V(G_2) = \{u, v\}$. Let $H_i = G_i + uv$ for $i = 1, 2$.

- If G_1, G_2 are both rigid, then $c(G) = 2c(H_1)c(H_2)$.
- If G_1 is rigid and G_2 is not rigid, then $c(G) = 2c(G_1)c(H_2)$.

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Suppose $G = G_1 \cup G_2 \cup \{e_1, e_2, e_3\}$ for two disjoint subgraphs G_1, G_2 and three disjoint edges e_1, e_2, e_3. Then $c(G) = 12c(G_1)c(G_2)$.

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It follows that we can reduce the problem of determining $c(G)$ to the case when G is 3-connected and all 3-edge-cuts are 'trivial'.

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Can we determine the smallest α such that $c(G) = O(\alpha^n)$ for all rigid graphs G on n vertices? (We know that $2.28 \leq \alpha \leq 4$ by Borcea and Streinu.)

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Problem (Thurston)

Does every rigid graph G have a generic realisation (G, p) in $\mathbb{R}^2$ such that $r(G, p) = c(G)$?