Volumes of Polyhedra in Hyperbolic and Spherical Spaces.

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The calculation of volume of polyhedron is very old and difficult problem. Probably, the first result in this direction belongs to Tartaglia (1499–1557) who found the volume of an Euclidean tetrahedron. Nowadays this formula is more known as Caley-Menger determinant. Recently it was shown by I. Kh. Sabitov (1996) that the volume of any Euclidean polyhedron is a root of algebraic equation whose coefficients are functions depending of combinatorial type and lengths of polyhedra. In hyperbolic and spherical spaces the situation is much more complicated. Gauss used the word "die Dschungel" in relation with volume calculation in non-Euclidean geometry. In spite of this, Janos Boyai, Nicolay Lobachevsky and Ludwig Schlafli obtained very beautiful formulae for non-Euclidean volume of a biorthogonal tetrahedron (orthoscheme). The volume of the Lambert cube and some other polyhedra were calculated by R. Kellerhals (1989), D. A. Derevnin, A. D. Mednykh (2002), A. D. Mednykh, J. Parker, A. Yu. Vesnin (2004), E. Molnar, J. Szirmai (2005) and others. The volume of hyperbolic polyhedra with at least one vertex at infinity was found by E. B. Vinberg (1992). The general formula for volume of tetrahedron remained to be unknown for a long time. A few years ago Y. Choi, H. Kim (1999), J. Murakami, U. Yano (2005) and A. Ushijima (2006) were succeeded in finding of a such formula. D. A. Derevnin, A. D. Mednykh (2005) suggested an elementary integral formula for the volume of hyperbolic tetrahedron. We note that the volume formula for symmetric tetrahedra whose opposite dihedral angles are mutually equal is rather simple. For the first time this phenomena was discovered by Lobachevsky for ideal hyperbolic tetrahedra, which is automatically symmetric. The respective result in quite elegant form was presented by J. Milnor (1982). For general case of symmetric tetrahedron the volume was given by D. A. Derevnin, A. D. Mednykh and M. G. Pashkevich (2004). Surprisedly, but a hundred years ago, in 1906 an essential advance in volume calculation for non-Euclidean tetrahedra was achieved by Italian mathematician Gaetano Sforza. It came to light during discussion of the author with Jose Maria Montesinos-Amilibia at the conference in El Burgo de Osma (Spain), August 2006.

The aim of this lecture is to give a survey of the above mentioned results.