Isostatic Structures: Using Richard Rado's Matroid Matchings

Henry Crapo, *Les Moutons matheux*, La Vacquerie
Joint work with Tiong Seng Tay, Nat. Univ. Singapore,
and Emanuela Ughi, Univ. Perugia

Workshop on Rigidity, Fields Institute, 11-14 October, 2011

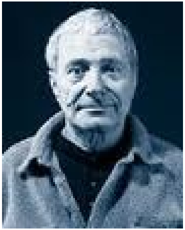
Outline

Dedication

I would like to dedicate this talk to two persons, both of whom are architects and engineers.

Dedication

To *Janos Baracs*,



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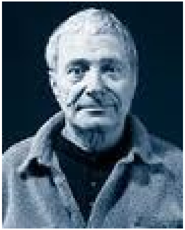

instigator and cofounder of
the research group *Topologie Structurale*,
who learned projective geometry
from his high school math teacher in Budapest,
and who introduced Ivo Rosenberg and myself to
three dimensional space and rigidity
during a workshop for members of the
Centre de recherches mathématiques
in January 1973, over 38 years ago,

Dedication

. . . posing, among other problems:

- to characterize generically 3-isostatic graphs
- to predict special positions of non-rigidity for generically 3-isostatic graphs,
- to specify the correct placements of cross-braces in grid frameworks.
- to analyze the rigidity of tensegrity frameworks.
- to analyze the relation between stresses and lifting of plane polyedral frameworks.
- to develop a theory of periodic filling of space by copies of one or more associated zonohedra.

Dedication

To *Richard Gage*,

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founder and leading member of
the association *Architects and Engineers for 911 Truth*,
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Dedication

Everything you ever wanted to know
about the 9/11 conspiracy theory
in under 5 minutes.

http://www.informationclearinghouse.info/article29110.htm

(surely the central rigidity problem of our era)

Dedication

With special thanks to Walter Whiteley and Bob Connelly,
Ileana Streinu and Tibor Jordán,
who have so energetically
kept this beautiful subject alive and well,
expanding its horizons,
training the researchers of this new generation,
and making it possible for us to be together today.

Main Points

(1) If a graph G has a shellable semi-simplicial map to the d-simplex K_{d+1}, then it is generically d-isostatic.

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Conjecture: A graph is generically d-isostatic if and only if
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(4) We investigate further restrictions of the class of maps to maps that are fewer in number and easier to construct: maps whose vertex packets are *broken paths*.

Generically Isostatic Graphs

A graph $G(V, E)$ is generically d-isostatic if and only if it is edge-minimal among graphs that are rigid in some (and therefore in almost every) position in real Euclidean or projective space of dimension d.

Generically Isostatic Graphs

We shall deal only with generic behavior of graphs as structures, so we will speak simply of "d-isostatic" graphs, dropping the adjective "generic".

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d-Isostatic Graphs

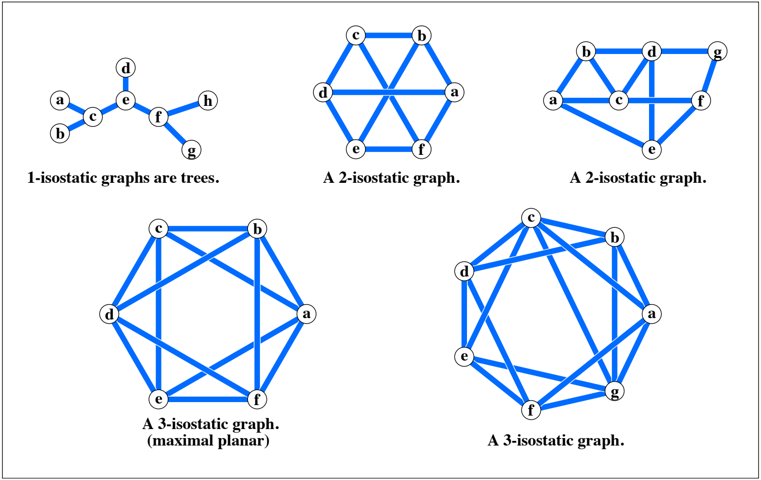

Figure: d-Isostatic graphs, for $d = 1, 2, 3$.

Definition

A *semi-simplicial map* $f : G(V, E) \to K_{d+1}$,

where

$$K_{d+1} = K(I, J)$$

and

$$I = \{1, 2, \dots\}, \quad J = \{12, 13, \dots\},$$

consists of a pair of maps

$$f_0 : V \to I, f_1 : E \to J,$$

that preserve incidence.

Definition

That is, an edge $e = ab$ whose vertices a and b have distinct values $f_0(a) = i,\ f_0(b) = j$ in I must be sent by f_1 to $ij \in J$.

We call such an edge $e = ab$ an ij-bridge.

Definition

An edge $e = ab$ whose end vertices go to the same vertex,
say $f_0(a) = i = f_0(b)$,
must be sent to an edge ij of K incident to i.

We call such an edge $e = ab$ a *loop at i toward j*.

Definition

The subset $f_0^{-1}(i)$, for any vertex $i \in I$,
we call the *i^{th} vertex packet* of f, denoted V_i.

Definition

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($\mathcal{P}_0$) *Edge independence:* The inverse image $f_1^{-1}(ij)$, denoted T_{ij}, of any edge ij of K is a tree spanning the union $V_i \cup V_j$ of its two related vertex packets.

Definition

(the combined statement:)

A *semi-simplicial map* $f : G(V, E) \longrightarrow K_{d+1}(I, J)$,
consists of a pair of maps $f_0 : V \longrightarrow I, f_1 : E \longrightarrow J$,
that preserve incidence, and ...

($\mathcal{P}_0$) *Edge independence:* The inverse image $f_1^{-1}(ij)$,
denoted T_{ij}, of any edge ij of K
is a tree spanning the union $V_i \cup V_j$
of its two related vertex packets.

Visual Representation of Maps

Semi-simplicial maps have very satisfactory visual representations, using *colors* taken from a standard edge-coloring of K_{d+1} to specify the images of each edge.

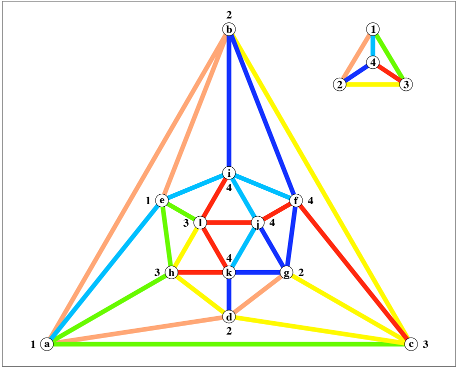

Figure: A d-isostatic graph, with semi-simplicial map.

Visual Representation of Maps

The tree-decomposition is then easily comprehended.

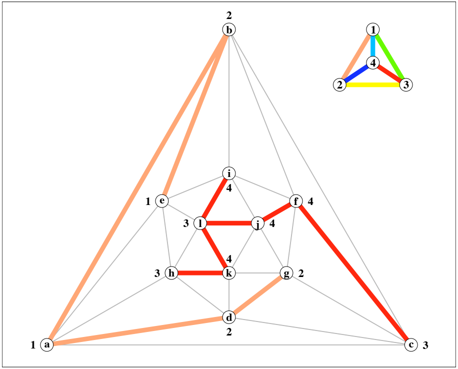

Figure: The trees T_{12} and T_{34}.

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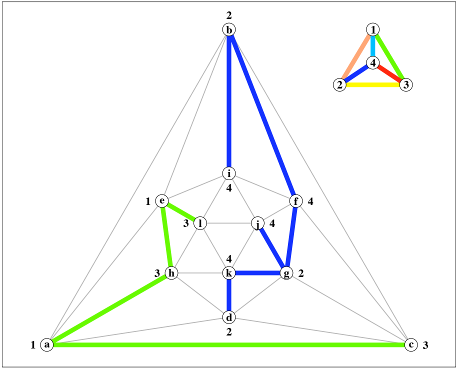

Figure: The trees T_{13} and T_{24}.

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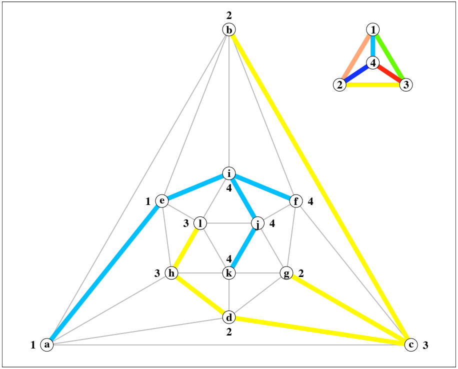

Figure: The trees T_{14} and T_{24}.

Visual Representation of Maps

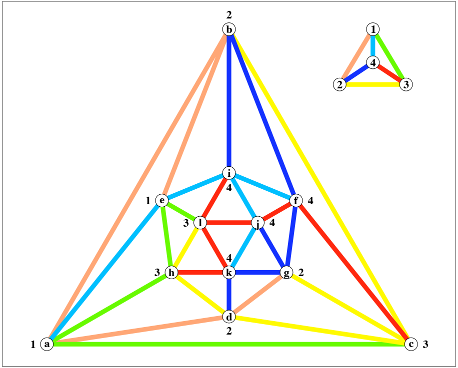

Figure: All together now!.

Path Connectivity

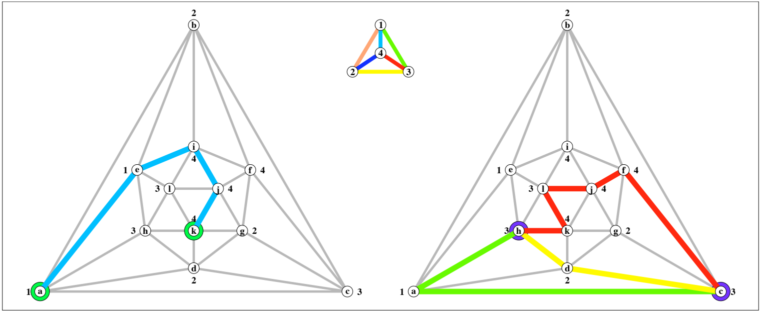

Figure: Paths between vertices having distinct/identical images.

If a and b have distinct images i, j under f_0,
then a and b are connected along a unique path in the tree T_{ij}.

Path Connectivity

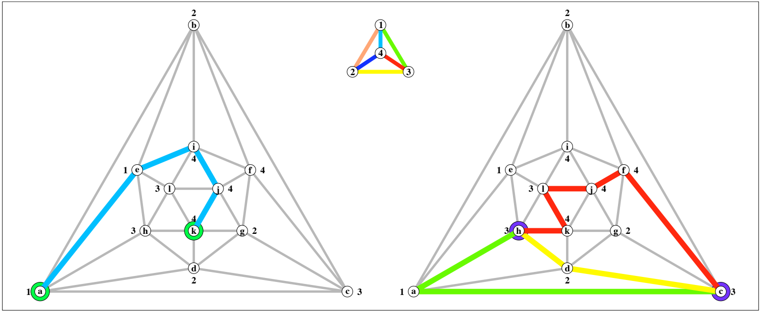

Figure: Paths between vertices having distinct/identical images.

If a and b have the same image i under f_0, then they are connected along unique paths in each of the d trees T_{ij}, for $j \neq i$.

Shelling

A vertex packet can be shelled
if there is a sequence of monochromatic cuts
that reduces it to a subgraph with no edges.

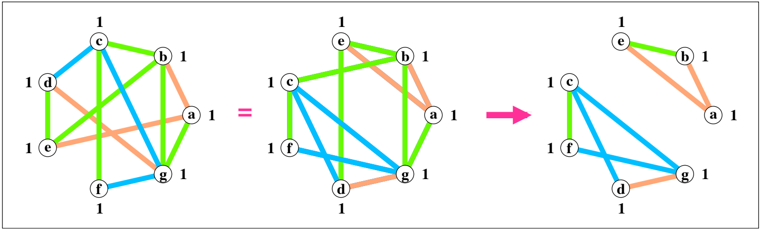

Figure: A sequence of monochromatic cuts.

Special Placement

In the special position given by a semi-simplicial map,
any external equilibrium test load
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any external equilibrium test load
applied at two vertices a, b is uniquely resolvable.

If $f(a) \neq f(b)$, the external load is resolved (and uniquely so)
along the path between a and b in the tree T_{ij},
all those edges being *collinear* along the line $i \vee j$.

Special Placement

In the special position given by a semi-simplicial map,
any external equilibrium test load
applied at two vertices a, b is uniquely resolvable.

If $f(a) = f(b) = i$, the external load
can be uniquely represented as a sum
of $d + 1$ equilibrium loads applied to a, b,
one in each of the (independent) directions $i \vee j$ at i.

These individual loads are then uniquely resolvable
along the paths from a to b in the trees T_{ij}

Theorem

A graph G is generically d-isostatic graph if it has a shellable semi-simplicial map to the d-simplex.

Maps on Dependent Graphs

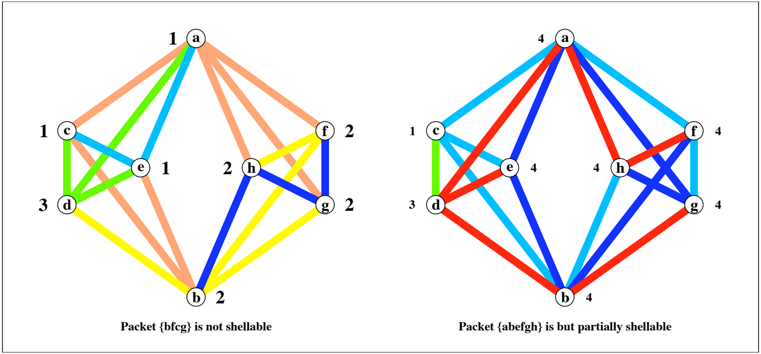

Figure: Non-shellable maps on a 3-dependent graph.

Converse, $d = 2$

For $d = 2$: Any non-shellable map
has an obstacle to shelling
in the form of a set of 3 or more vertices
co-spanned by sub-trees of two trees.

This is a dependent subgraph.

Converse, $d = 2$

Theorem:

A graph G is generically 2-isostatic graph

if and only if

it has a shellable semi-simplicial map to the triangle,

if and only if

all semi-simplicial maps to the triangle are shellable.

Converse, $d = 3$?

This is far from being the case in dimension 3.

A 3-isostatic graph may have
many non-shellable maps to the tetrahedron.

Converse, $d = 3$?

Existence of a non-shellable map establishes only that
there is a subset Q of some vertex packet i
that is spanned by sub-trees of any *pair*
of the three trees T_{ij} for $j \neq i$.

Converse, $d = 3$?

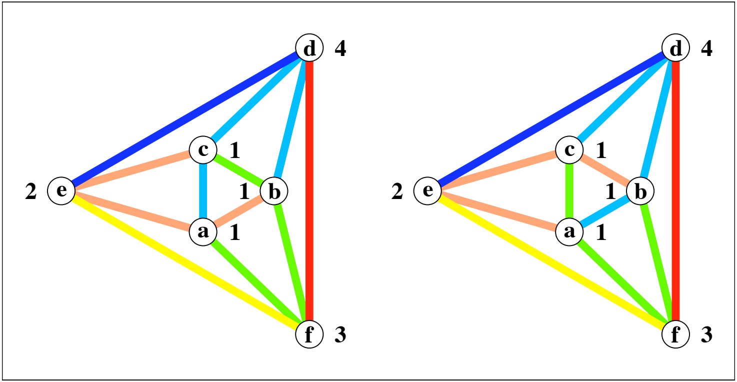

Figure: The packet V_1 contains an obstacle to shelling.

These are the only two edge maps with this vertex map.

Converse, $d = 3$?

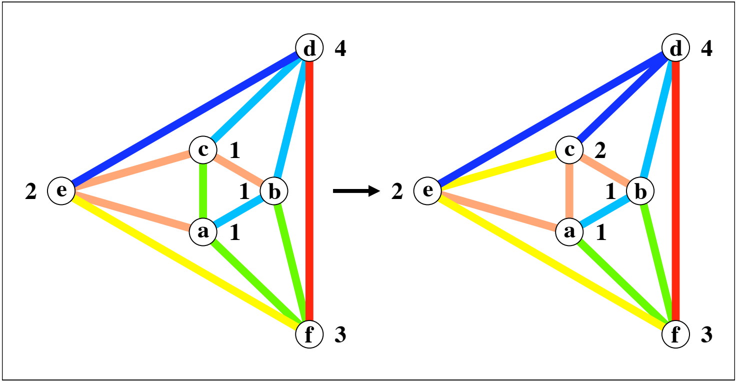

Figure: A change of one vertex image produces a shellable map.

This vertex map has a unique compatible edge map.

Eliminate Obstacles - Eliminate Shelling

Perhaps the best way to deal with obstacles to shelling will be to look for maps in which obstacles cannot occur,

Eliminate Obstacles - Eliminate Shelling

that is, those for which the vertex packets
induce *independent* subgraphs,
that is, cycle-free subgraphs, or forests.

Eliminate Obstacles - Eliminate Shelling

These maps are *freely shellable*:

Eliminate Obstacles - Eliminate Shelling

Simply proceed edge by edge,
each single edge being a monochromatic cut!

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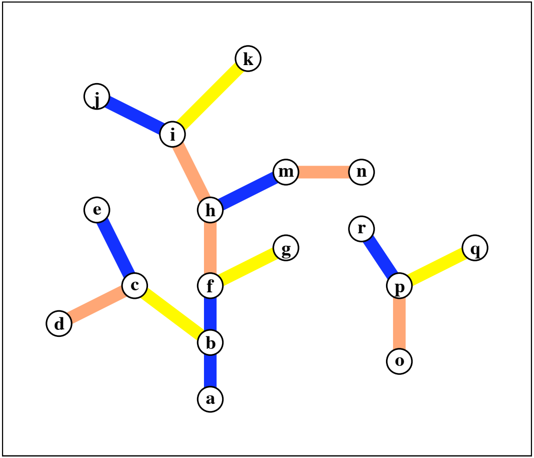


Figure: A forest as induced subgraph of packet V_2.

Eliminate Obstacles - Eliminate Shelling

Conjecture:

A graph G is generically 3-isostatic if and only if it has a semi-simplicial map to the tetrahedron in which all vertex packets induce subgraphs that are independent (ie: forests) as subgraphs of G.

Eliminate Obstacles - Eliminate Shelling

There are four interesting classes of such maps:
those in which the vertex packets induce:

- $\mathcal{F}$ forests
- $\mathcal{T}$ trees
- $\mathcal{B}$ broken paths
- $\mathcal{P}$ paths

Eliminate Obstacles - Eliminate Shelling

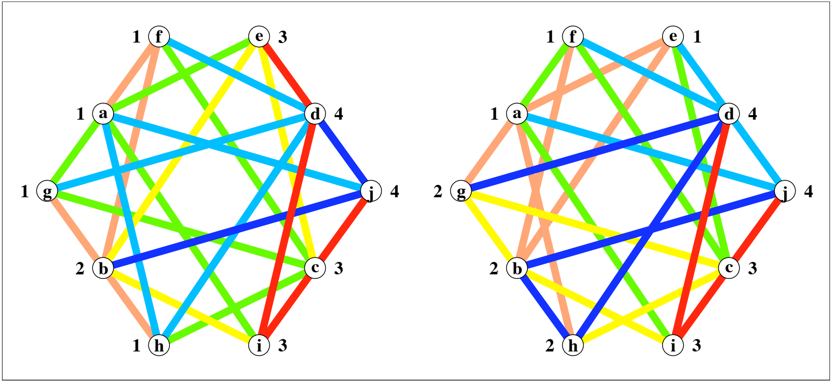
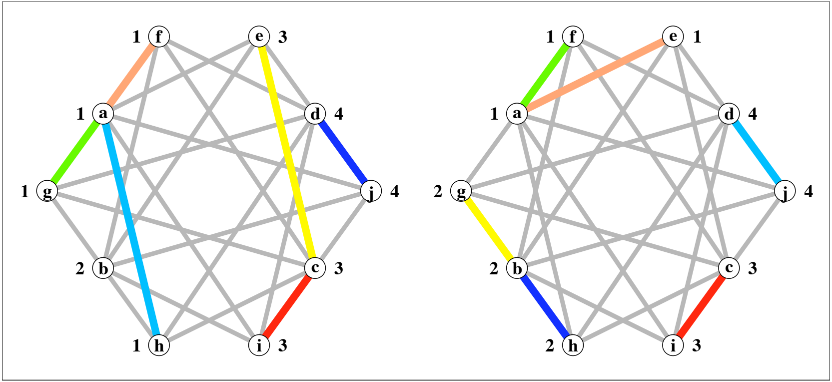


Figure: Vertex packets are trees (l), paths (r).

Eliminate Obstacles - Eliminate Shelling



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Understanding these drawings:

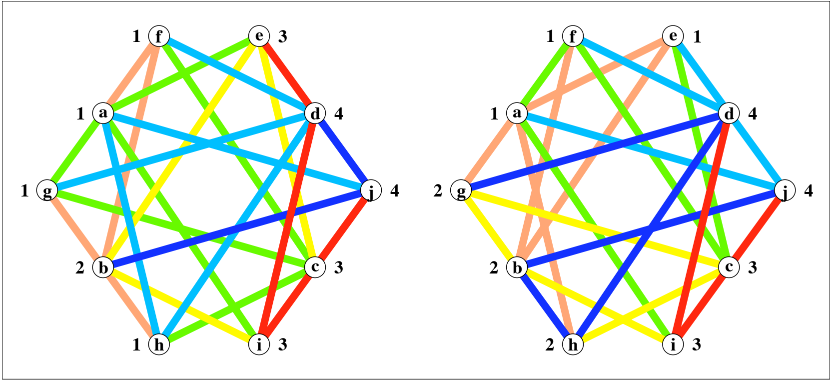

Figure: These drawings may seem complicated, but are easily analyzed.

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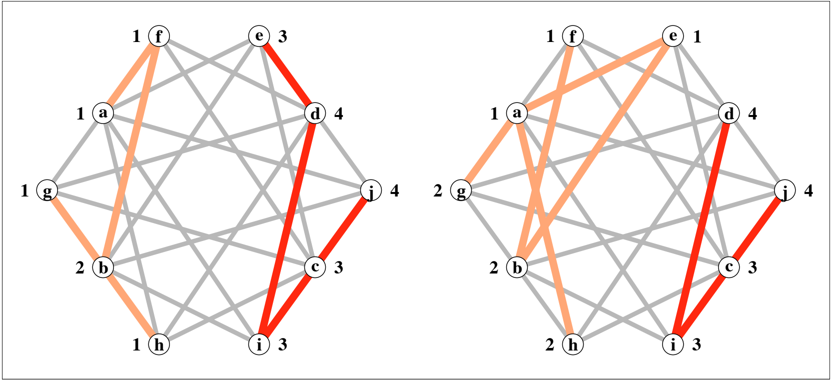

Figure: Trees T_{12}, T_{34}.

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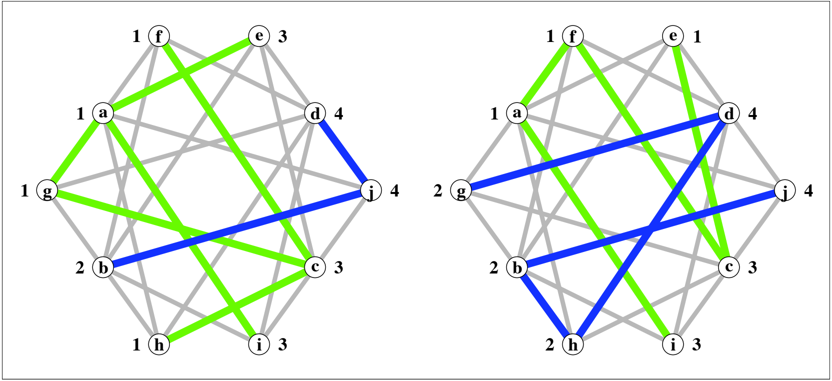

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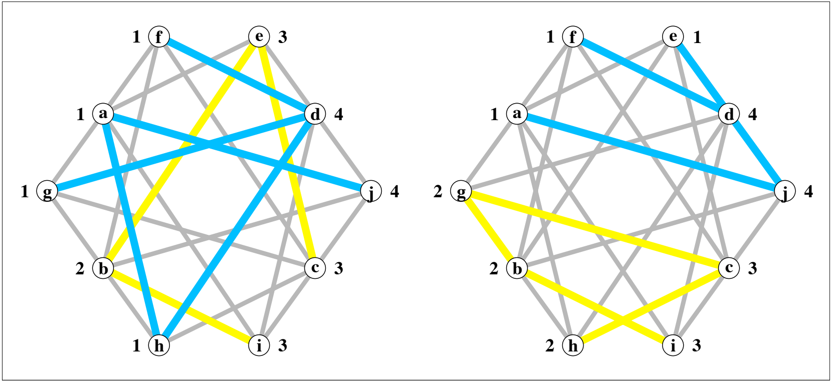


Figure: Trees T_{14}, T_{23}.

The vertex set can not always be partitioned into paths.

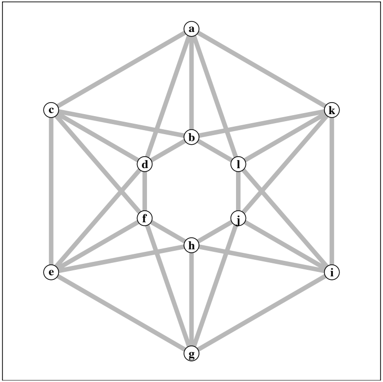
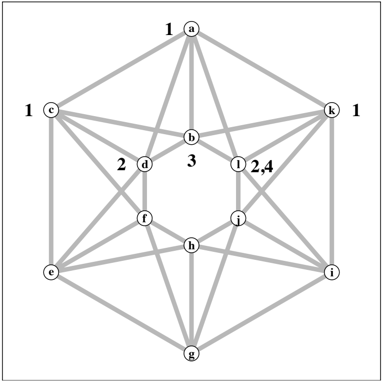


Figure: A hinged ring of tetrahedra.

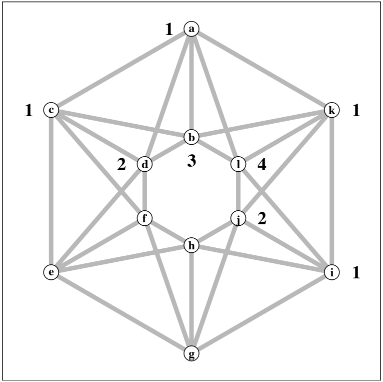
3-isostatic graphs do not necessarily have
maps to K_4 in which
($\mathcal{P}$) induced graphs on vertex packets are paths.

The vertex set can not always be partitioned into paths.



There must be a path of length ≥ 3,
not within a single tetrahedron.
The vertex b is isolated with its image 3.
There must be a path of length ≥ 4.

The vertex set can not always be partitioned into paths.



l must be 4, otherwise there is no 2-path from *d* to *l*.
Then values 3 and 4 are isolated at *b* and *l*,
So only 1 and 2 are available for tetrahedron *efgh*.

Freely shellable semi-simplicial maps

In practice, freely-shellable maps seem to abound,
and seem much easier to find "by hand"
than more general maps
for which you must check shellability.

Freely shellable semi-simplicial maps

What is more, freely-shellable maps
have relatively few loops that need to be assigned.

Partitions that Produce Freely Shellable Maps

To prove a graph $G(V, E)$ is isostatic,
it suffices to exhibit a partition π of the vertex set V
having three properties $\mathcal{P}_i$ (see below).

The main criterion $\mathcal{P}_3$ is
Richard Rado's matroid basis matching condition.

Partitions that Produce Freely Shellable Maps

Theorem: Rado's Basis Matching Theorem

Given any relation R from a set X
to a set S of elements of a matroid $M(S)$,
then there is matching in R from X
to a basis for the matroid $M(S)$
if and only if
the cardinality $|X| = \text{rank } \rho(S)$ of the matroid M,
and, for every subset $A \subset X$,
the cardinality $|A| \leq \rho(A)$,
the rank of its image $R(A)$ in $M(S)$.

Bibliography on Matroid Matching

Richard Rado,
A Theorem on Independence Relations,
Quarterly J. of mathematics, Oxford **13** (1942), 83-89.

Bibliography on Matroid Matching

Joseph P. S. Kung, Gian-Carlo Rota, Catherine H. Yan,
Combinatorics: The Rota Way,
Cambridge University Press, 2009.

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Kazuo Murota,
Matrices and Matroids for Systems Analysis
Springer Verlag,
Algorithms and Combinatorics **20** (2000),(revised 2010).

Bibliography on Matroid Matching

And an article which led us to the possibility of insisting that vertex packets induce paths:

Roger K. S. Poh,
On the Linear Vertex-Arboricity of a Planar Graph
Journal of Graph Theory, **14 No. 1** (1990), 73-75.

A Matroid Union

Given a partition of the vertex set of G,
define bridges and loops,
and for each ij construct the matroid minor:
restrict to the induced subgraph
on the union of the two packets,
and contract by its bridges.

Then take the matroid union over all pairs ij

A Matroid Union

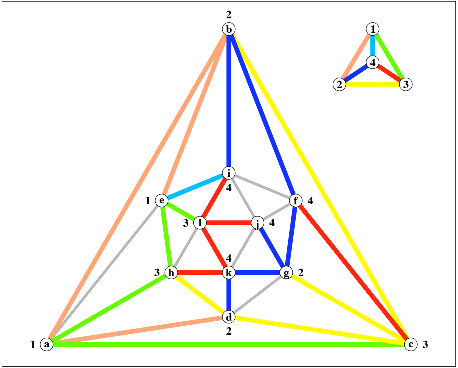

Figure: The bridges of a map on the icosahedron.

A Matroid Union

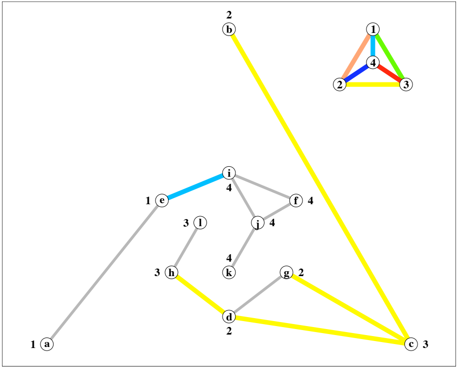

Figure: Restrictions to packet unions $V_1 \cup V_4$ and $V_2 \cup V_3$.

Characterization of Partitions for Freely-Shellable Maps

Theorem: A partition π of the vertex set of a graph $G(V, E)$
is the inverse image partition
of a freely-shellable semi-simplicial map
$f : G \longrightarrow K_{d+1}$ if and only if
the partition π has the following three properties $\mathcal{P}_i$

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($\mathcal{P}_1$) The induced subgraph G_i on any part π_i of π is independent (circuit-free).

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($\mathcal{P}_2$) For any pair ij, the bridge subgraph $G(V_i \cup V_j, B_{ij})$ is independent.

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($\mathcal{P}_2$) For any pair ij, the bridge subgraph $G(V_i \cup V_j, B_{ij})$ is independent.

($\mathcal{P}_3$) The relation $\mathcal{R}$ between the set of loops of G and the set of elements of the matroid union M satisfies the Rado condition for basis matching: $|L| = \rho(M)$ and

$$\forall A \subseteq E, |A| \leq \rho(\mathcal{R}(\mathcal{A})).$$

A Partition Not Satisfying the Rado Condition

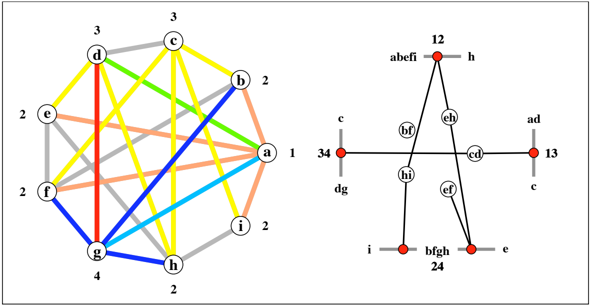

Figure: Partition $(a)(befhi)(cd)(g)$ does not satisfy the Rado condition.

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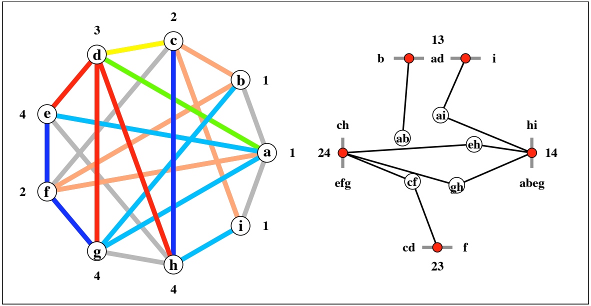


Figure: Partition $(a)(befhi)(cd)(g)$ has 2 compatible loop maps.

A Partition Not Satisfying the Rado Condition

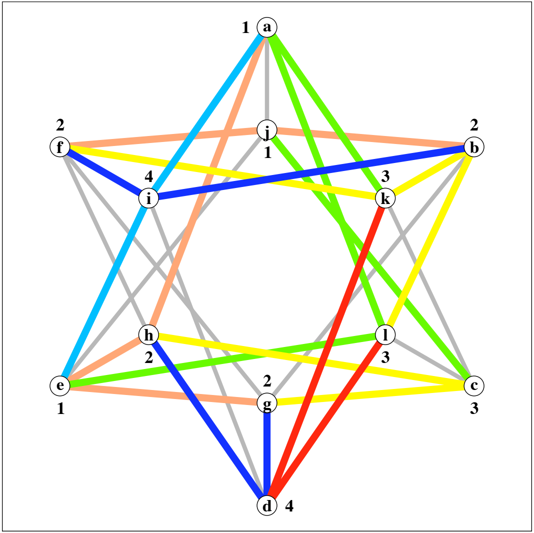


Figure: A non-Rado partition for $K_{6,6}$ less 6 edges. (edge *di*!)

A Partition Not Satisfying the Rado Condition

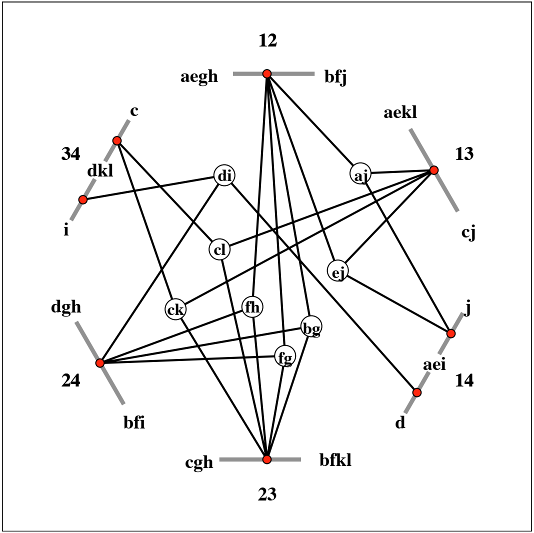


Figure: The Rado relation $\mathcal{R}$ for that partition.

A Partition Satisfying the Rado Condition

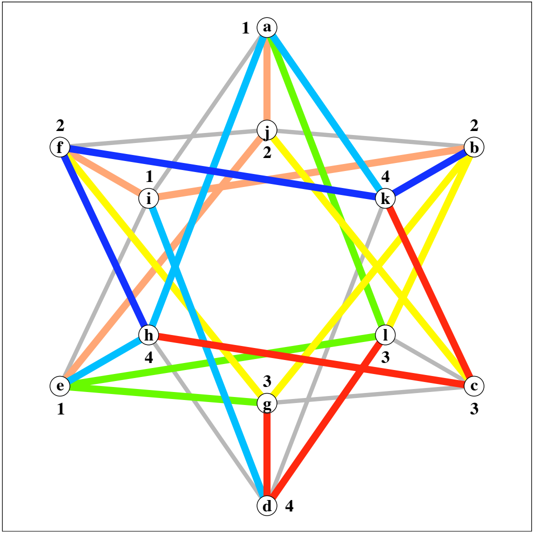
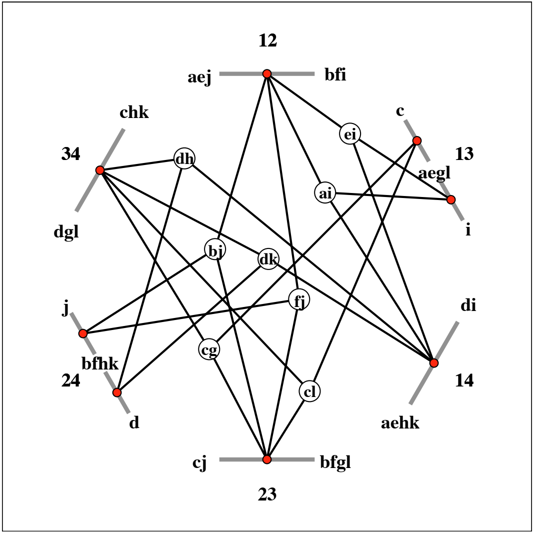


Figure: A partition with 32 compatible loop maps.

A Partition Satisfying the Rado Condition



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A Partition Satisfying the Rado Condition

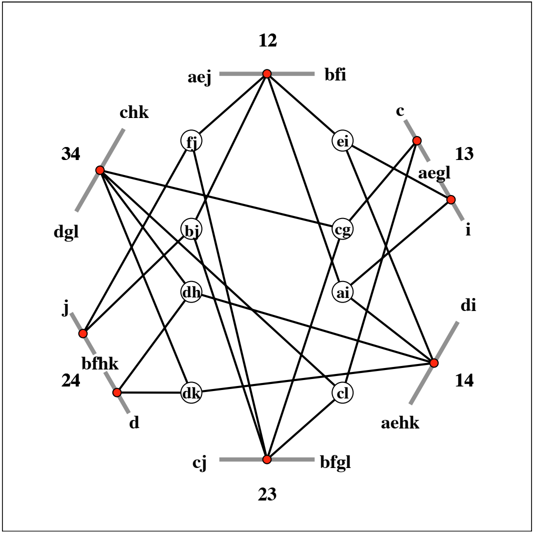

Figure: The symmetry of $\mathcal{R}$ is perhaps more visible here.

A Partition Satisfying the Rado Condition

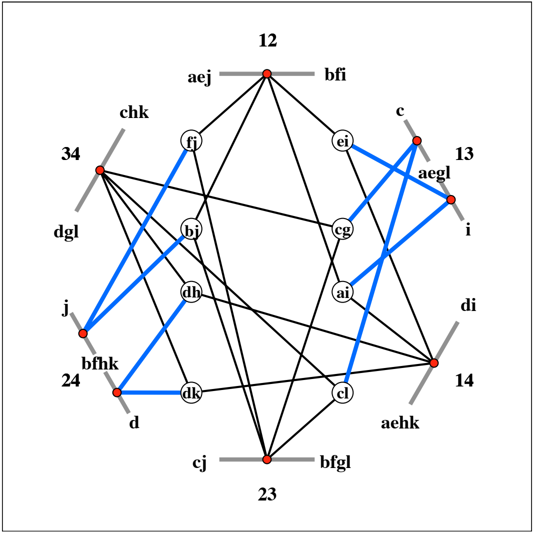

Figure: Four independent binary choices, ...

A Partition Satisfying the Rado Condition

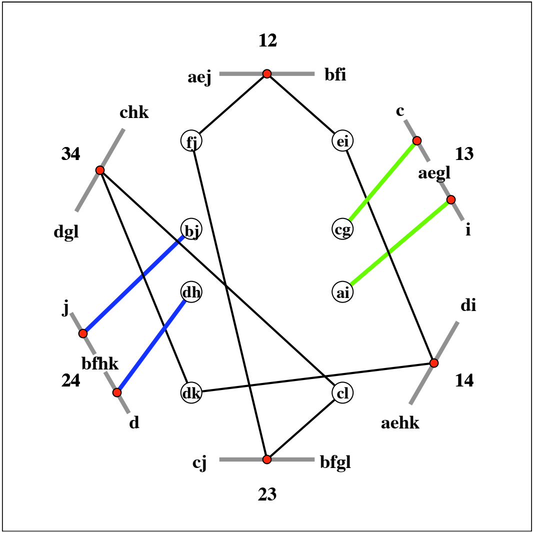

Figure: After four independent binary choices, a cycle remains.

The Road Ahead

It remains to prove that any 3-isostatic graph
has a freely-shellable semi-simplicial map to the simplex K_4.

The Road Ahead

This has always been the hard part of the problem!

The Road Ahead

What is likely to happen?

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Either:

There will be a relatively simple proof,
I would guess during the next few months, . . .

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(an example on the complete graph K_{56})
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like the biplane
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Followed by a rapid retreat
from an untenable position!

The Road Ahead

Which properties of isostatic graphs
might permit us to prove the conjecture?

The Road Ahead

Which properties of isostatic graphs
might permit us to prove the conjecture?

We lean toward an analogue in $d = 3$
of Tay's proof for $d = 2$.

Toward an Analogue of Tay's Proof for $d = 2$

We use the $(3v - 6) \times 6v$
projective rigidity matrix R,
and the $(3v + 6) \times 6v$ matrix S
whose rows span the orthogonal complementary subspace.

Toward an Analogue of Tay's Proof for $d = 2$

By Hodge star complementation,
the determinants of full-size minors of R
are equal to the determinants
of the complementary full-size minors of S
up to a sign ± 1 of the bipartition of the column set,
and up to a fixed polynomial quantity Q,
called the pure condition or resolving bracket,
which is non-zero exactly when the graph is isostatic.

Toward an Analogue of Tay's Proof for $d = 2$

The column matroids of R and of S are dual to one another, and are independent of the graph G in question!

Toward an Analogue of Tay's Proof for $d = 2$

The column matroids of R and of S are dual to one another, and are independent of the graph G in question!

($Q \neq 0$ exactly when the rows of R form a basis for the space of external equilibrium loads on the set V of vertices of G, regarded as a single rigid body.)

The Orthogonal Complementary Matrices S and R

a12	b12	c12	d12	e12	a13	b13	c13	d13	e13	a14	b14	c14	d14	e14	a23	b23	c23	d23	e23	a24	b24	c24	d24	e24	a34	b34	c34	d34	e34	
1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	C12
0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	C13
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	C14
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	C23
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	C24
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	C34
a3	0	0	0	0	-a2	0	0	0	0	0	0	0	0	0	a1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	a123
a4	0	0	0	0	0	0	0	0	0	-a2	0	0	0	0	0	0	0	0	0	a1	0	0	0	0	0	0	0	0	0	a124
0	0	0	0	0	a4	0	0	0	0	-a3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	a1	0	0	0	0	a134
0	b3	0	0	0	0	-b2	0	0	0	0	0	0	0	0	0	b1	0	0	0	0	0	0	0	0	0	0	0	0	0	b123
0	b4	0	0	0	0	0	0	0	0	0	-b2	0	0	0	0	0	0	0	0	0	b1	0	0	0	0	0	0	0	0	b124
0	0	0	0	0	0	b4	0	0	0	0	-b3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b1	0	0	0	b134
0	0	c3	0	0	0	0	-c2	0	0	0	0	0	0	0	0	0	c1	0	0	0	0	0	0	0	0	0	0	0	0	c123
0	0	c4	0	0	0	0	0	0	0	0	0	-c2	0	0	0	0	0	0	0	0	0	c1	0	0	0	0	0	0	0	c124
0	0	0	0	0	0	0	c4	0	0	0	0	-c3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	c1	0	0	c134
0	0	0	d3	0	0	0	0	-d2	0	0	0	0	0	0	0	0	0	d1	0	0	0	0	0	0	0	0	0	0	0	d123
0	0	0	d4	0	0	0	0	0	0	0	0	0	-d2	0	0	0	0	0	0	0	0	0	d1	0	0	0	0	0	0	d124
0	0	0	0	0	0	0	0	d4	0	0	0	0	-d3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	d1	0	d134
0	0	0	0	e3	0	0	0	0	-e2	0	0	0	0	0	0	0	0	0	e1	0	0	0	0	0	0	0	0	0	0	e123
0	0	0	0	e4	0	0	0	0	0	0	0	0	0	-e2	0	0	0	0	0	0	0	0	0	e1	0	0	0	0	0	e124
0	0	0	0	0	0	0	0	0	e4	0	0	0	0	-e3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	e1	e134
ac12	0	-ac12	0	0	ac13	0	-ac13	0	0	ac14	0	-ac14	0	0	ac23	0	-ac23	0	0	ac24	0	-ac24	0	0	ac34	0	-ac34	0	0	ac
ad12	0	0	-ad12	0	ad13	0	0	-ad13	0	ad14	0	0	-ad14	0	ad23	0	0	-ad23	0	ad24	0	0	-ad24	0	ad34	0	0	-ad34	0	ad
ae12	0	0	0	-ae12	ae13	0	0	0	-ae13	ae14	0	0	0	-ae14	ae23	0	0	0	-ae23	ae24	0	0	0	-ae24	ae34	0	0	0	-ae34	ae
0	bc12	-bc12	0	0	0	bc13	-bc13	0	0	0	bc14	-bc14	0	0	0	bc23	-bc23	0	0	0	bc24	-bc24	0	0	0	bc34	-bc34	0	0	bc
0	bd12	0	-bd12	0	0	bd13	0	-bd13	0	0	bd14	0	-bd14	0	0	bd23	0	-bd23	0	0	bd24	0	-bd24	0	0	bd34	0	-bd34	0	bd
0	be12	0	0	-be12	0	be13	0	0	-be13	0	be14	0	0	-be14	0	be23	0	0	-be23	0	be24	0	0	-be24	0	be34	0	0	-be34	be
0	0	cd12	-cd12	0	0	0	cd13	-cd13	0	0	0	cd14	-cd14	0	0	0	cd23	-cd23	0	0	0	cd24	-cd24	0	0	0	cd34	-cd34	0	cd
0	0	ce12	0	-ce12	0	0	ce13	0	-ce13	0	0	ce14	0	-ce14	0	0	ce23	0	-ce23	0	0	ce24	0	-ce24	0	0	ce34	0	-ce34	ce
0	0	0	de12	-de12	0	0	0	de13	-de13	0	0	0	de14	-de14	0	0	0	de23	-de23	0	0	0	de24	-de24	0	0	0	de34	-de34	de

Figure: Columns grouped by trees T_{ij}.

The Orthogonal Complementary Matrices S and R

a12	a13	a14	a23	a24	a34	b12	b13	b14	b23	b24	b34	c12	c13	c14	c23	c24	c34	d12	d13	d14	d23	d24	d34	e12	e13	e14	e23	e24	e34	
1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	C12
0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	C13
0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	C14
0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	C23
0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	C24
0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	C34
a3	-a2	0	a1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	a123
a4	0	-a2	0	a1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	a124
0	a4	-a3	0	0	a1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	a134
0	0	0	0	0	0	b3	-b2	0	b1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b123
0	0	0	0	0	0	b4	0	-b2	0	b1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b124
0	0	0	0	0	0	0	b4	-b3	0	0	b1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	b134
0	0	0	0	0	0	0	0	0	0	0	0	c3	-c2	0	c1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	c123
0	0	0	0	0	0	0	0	0	0	0	0	c4	0	-c2	0	c1	0	0	0	0	0	0	0	0	0	0	0	0	0	c124
0	0	0	0	0	0	0	0	0	0	0	0	0	c4	-c3	0	0	c1	0	0	0	0	0	0	0	0	0	0	0	0	c134
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	d3	-d2	0	d1	0	0	0	0	0	0	0	0	d123
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	d4	0	-d2	0	d1	0	0	0	0	0	0	0	d124
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0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	e3	-e2	0	e1	0	0	e123
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	e4	0	-e2	0	e1	0	e124
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	e4	-e3	0	0	e1	e134
ac12	ac13	ac14	ac23	ac24	ac34	0	0	0	0	0	0	-ac12	-ac13	-ac14	-ac23	-ac24	-ac34	0	0	0	0	0	0	0	0	0	0	0	0	ac
ad12	ad13	ad14	ad23	ad24	ad34	0	0	0	0	0	0	0	0	0	0	0	0	-ad12	-ad13	-ad14	-ad23	-ad24	-ad34	0	0	0	0	0	0	ad
ae12	ae13	ae14	ae23	ae24	ae34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-ae12	-ae13	-ae14	-ae23	-ae24	-ae34	ae
0	0	0	0	0	0	bc12	bc13	bc14	bc23	bc24	bc34	-bc12	-bc13	-bc14	-bc23	-bc24	-bc34	0	0	0	0	0	0	0	0	0	0	0	0	bc
0	0	0	0	0	0	bd12	bd13	bd14	bd23	bd24	bd34	0	0	0	0	0	0	-bd12	-bd13	-bd14	-bd23	-bd24	-bd34	0	0	0	0	0	0	bd
0	0	0	0	0	0	be12	be13	be14	be23	be24	be34	0	0	0	0	0	0	0	0	0	0	0	0	-be12	-be13	-be14	-be23	-be24	-be34	be
0	0	0	0	0	0	0	0	0	0	0	0	cd12	cd13	cd14	cd23	cd24	cd34	-cd12	-cd13	-cd14	-cd23	-cd24	-cd34	0	0	0	0	0	0	cd
0	0	0	0	0	0	0	0	0	0	0	0	ce12	ce13	ce14	ce23	ce24	ce34	0	0	0	0	0	0	-ce12	-ce13	-ce14	-ce23	-ce24	-ce34	ce
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	de12	de13	de14	de23	de24	de34	-de12	-de13	-de14	-de23	-de24	-de34	de

Figure: Columns grouped by vertices v.

The Orthogonal Complementary Matrices S and R

Any set of columns in R labeled by a single vertex,
say by a
and by a circuit in K_4,
such as 12, 23, 34, 14,
are dependent.

The Orthogonal Complementary Matrices S and R

Any set of columns in R labeled by a single vertex,
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Any set of columns in R labeled by a edge of K_4,
say by 12
and by all vertices a,
are dependent.

The Orthogonal Complementary Matrices S and R

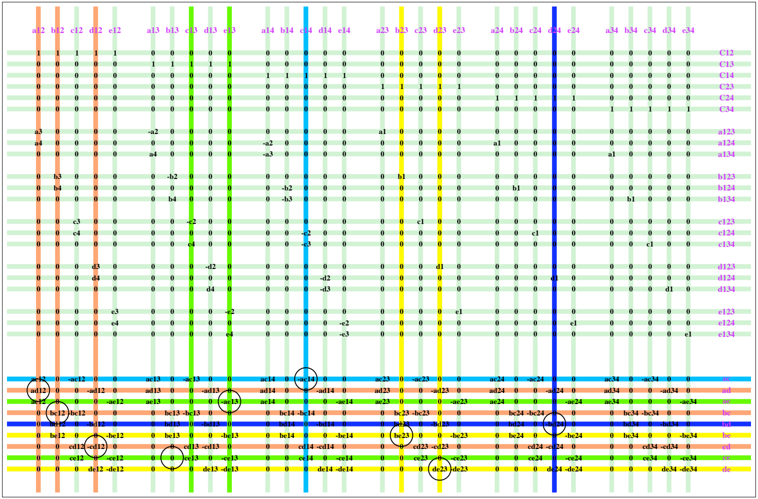

Figure: From a non-zero diagonal to a (rooted) freely shellable map.

The Orthogonal Complementary Matrices S and R

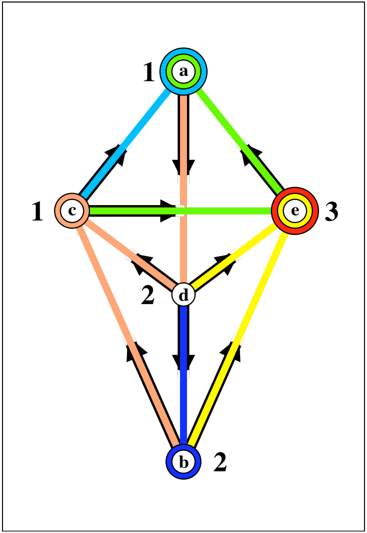

Figure: The corresponding rooting of a freely shellable map.

An analogue of Henneberg reduction?

Is it possible to reduce any isostatic graph to an isostatic graph on one fewer vertex, by a procedure that, when repeated, leads, step-by-step, to a map?

Grazie

Thank you for your attention.

This paper should be up on the arXiv soon:

Isostatic Structures:
Using Richard Rado's Independent Matchings