Ana - "Simple Shimura Varieties" 3/14/12

Motivation

if f is a cusp form $/\mathbb{Q}$, then we can view this as π_f, a cuspidal automorphic rep'n of $Gl_2(\mathbb{A}^\infty)$.

Then we can associate $\rho_{f,\ell}$, an ℓ-adic rep'n of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ (which is done through cohomology of modular curves.)

We would like to general this to other groups: Gl_n.

We would also like to generalize local Langlands to Gl_n (Gl_2 was done in Herzig's talk)

The higher-dimensional analogues of modular curves that we'll need are Shimura varieties.

General Setup:

(G, X) where G - real alg group $/\mathbb{Q}$, X is a $G(\mathbb{R})$-conj class of homs $h: \mathbb{C}^\times \longrightarrow G(\mathbb{R})$

(G,X) satisfy certain axioms giving X the structure of complex manifold

$\mathcal{U} \subset G(\mathbb{A}^\infty)$ open compact subgroup (level)

canonical model over $\mathbb{C}$ $\quad {}_{G(\mathbb{Q})}\backslash G(\mathbb{A}^\infty) \times X /_{\mathcal{U}}$

example: modular curve, $G = Gl_2$, $X = \mathcal{H}^\pm = \mathbb{C} \setminus \mathbb{R}$, trivial level structure

$$ {}_{G(\mathbb{Q})}\backslash Gl_2(\mathbb{A}^\infty) \times X /_{Gl_2(\hat{\mathbb{Z}})} \cong {}_{Sl_2(\mathbb{Z})}\backslash \mathcal{H}^+ \quad \text{via strong approximation}$$

V - 2-dimensional vector space $/\mathbb{Q}$

$X = \mathbb{C} \setminus \mathbb{R}$ - parametrize complex structures on $V \otimes_{\mathbb{Q}} \mathbb{R}$

$Gl_2/\mathbb{Q}$ - automorphism group of V

model is $H_1(E, \mathbb{Q})$ where E is an elliptic curve

Nice properties: (about modular curves)

- model over $\mathbb{Q}$
- integral model over $\mathbb{Z}_{(p)}$
- moduli interpretations: elliptic curves + level structure.

<u>But</u> (not so nice properties)
- modular curves are not compact
- both cusp forms + Eisenstein series (induced from $Gl_1 \times Gl_2$)

higher-dim'l generalization is Siegel modular varieties
$X = \mathcal{H}_g = \{\tau \in M_g(\mathbb{C}) \mid \tau^T = \tau, \ Im(\tau) \text{ pos definite}\}$
$G = Sp_{2g}$
- not compact

- do parametrize Abelian varieties, but probably too big.

We are focusing on Harris-Taylor type Shimura varieties:
- G - unitary similitude group $/\mathbb{Q}$
- <u>simple</u> - don't have to worry about aut rep'n of other G' showing up parametrizing simple Abelian varieties (A <u>simple</u> if not isogenous to product of abelian varieties $A_1 \times A_2$)

"no endoscopy"

Take E - imaginary quadratic field, c complex conjugation
F^+ - totally real field, deg $d/\mathbb{Q}$
$F = F^+E$ - CM field (aut rep'ns of $Gl_n(\mathbb{A}_F)$)
$B/_F$ - division algebras of dim n^2/F ($n=2$, quaternion algebras as in Mok's talk)
- want F to be the center of B
- $B^{op} \simeq B \otimes_{F,c} F$
- want involution $*: B \longrightarrow B$ extending c on F where $tr(xx^*) > 0 \ \forall x \in B^\times$

$V = B$ thought of as $B \otimes_{\mathbb{Q}} B^{op}$-module + extra structures
↑ vectorspace over $\mathbb{Q}$ (want this to model $H_1(A, \mathbb{Q})$)

Let $(\ ,\ ): V \times V \longrightarrow \mathbb{Q}$ be alternating, $*$-Hermitian form on V
$C = End_B(V) \simeq B^{op}$
$\varphi \longmapsto \varphi(1)$
$(\ ,\ )$ defines involutions on C
$(gx, y) = (x, g^\# y)$
$g \longmapsto g^\#$ involution

G - algebraic group $/\mathbb{Q}$ defined as follows:
R - $\mathbb{Q}$-algebra, $G(R) = \{(\lambda, g) \in R^\times \times (R \otimes_{\mathbb{Q}} C)^\times : gg^\# = \lambda\}$

$G(\mathbb{R})$ is group of automorphism of V preserving B-module structure, Herm form up to scalar multiple

<u>Claim:</u> we can choose $(\ ,\ )$ on V such that

$$G(\mathbb{R}) \simeq G(\underset{\substack{\uparrow \\ \text{1-embedding} \\ \tau: F^+ \to \mathbb{R}}}{\mathcal{U}(1,n-1)} \times \underset{\substack{\uparrow \\ \text{other embeddings} \\ \tau: F^+ \hookrightarrow \mathbb{R}}}{\mathcal{U}(0,n)^{d-1}})$$

$$C \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{\tau: F^+ \hookrightarrow \mathbb{R}} (C \otimes_{F^+} \mathbb{R})$$

Reason why claim true: (Hasse principle) & $\sum$ local invariants $\equiv 0 \pmod{d}$ (may have to change ramification conditions for B at finite places)

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if R is an E-algebra, $G(R) = \{(g,h) \in (R \otimes_E C)^\times \times (R \otimes_{E,c} C)^\times \mid gh^\# \in R^\times\}$ via $(g,h) \longmapsto gh^\# g$, can identify above w/ $R^\times \times (R \otimes_E C)^\times$

over E, G is just $G_m \times$ grp defined by B^{op}

If p splits in E, w a prime of F above p, we can ask $B_w \simeq M_n(F_w)$

Then $G/\mathbb{Q}_p \simeq G_m \times \mathrm{Res}_{F_w/\mathbb{Q}_p} Gl_n \times$ other groups

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extend $\tau: F^+ \hookrightarrow \mathbb{R}$ (embedding w/ $\mathrm{sgn}(1,n-1)$) to $\tau: F \hookrightarrow \mathbb{C}$.

X_τ : parametrize complex structures on Herm $B_{\mathbb{R}}$-module $V_{\mathbb{R}}$

$\Updownarrow$

$h: \mathbb{C} \longrightarrow C_{\mathbb{R}}$ inducing positive definite Hermitian $(B_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})$-module structure on $V_{\mathbb{R}}$.

in other words: $h(i) \in C_{\mathbb{R}}$ s.t. (i): $h(i)^2 = -1$

(ii): $h(i)^\# = -h(i)$

(iii): $(x, yh(i))$ is a positive definite symmetric Hermitian form

(iv): $\dim_{\mathbb{C}} (V \otimes_{F,\tau'} \mathbb{C})^{h(i)=i} = \begin{cases} n & \text{for } \tau \\ 0 & \text{for } \tau' \neq \tau \end{cases}$

"Kottwitz determinant condition" $\qquad \tau'|_E = \tau|_E$

Shimura variety $/\mathbb{C}$ level $\mathcal{U}$
canonical model is $G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)\times X_\tau/\mathcal{U}$

<u>Advantages</u>: • if $F^+ \neq \mathbb{Q}$, $X_\mathcal{U}$ compact
• division algebra $\Rightarrow$ "no endoscopy"
(neg) • π has to be discrete series at finite place

• model over F
• moduli space for (A, λ, i, η) (PEL-type)
- A abelian variety $/F$ of dim dn^2
- $\lambda: A \longrightarrow A^\vee$ (V model $H_1(A, \mathbb{Q})$)
- $i: B \hookrightarrow \mathrm{End}(A)\otimes_\mathbb{Z} \mathbb{Q}$ s.t. (A, i) compatible condition on $\mathrm{Lie}\, A$ (analogue of (iv))
- $\bar{\eta}$ - $\mathcal{U}$-invariant orbit of isom $\eta: V\otimes_\mathbb{Q} \mathbb{A}^\infty \xrightarrow{\sim} H_1(A, \mathbb{Q})\otimes_\mathbb{Q} \mathbb{A}^\infty$
• complex points of $X_\mathcal{U}$ coincide w/ $G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)\times X_\tau/\mathcal{U}$

• if G satisfies Hasse principle
$$\left.\begin{array}{l} V\otimes \mathbb{A}^\infty \xrightarrow{\sim} H_1(A, \mathbb{Q})\otimes \mathbb{A}^\infty \\ V_\mathbb{R} \simeq H_1(A, \mathbb{R}) \simeq \mathrm{Lie}\, A \end{array}\right\} \Rightarrow \exists\, \eta_0: V \xrightarrow{\sim} H_1(A, \mathbb{Q})$$
well defined up to $G_\mathbb{Q}$

$$V\otimes \mathbb{A}^\infty \xrightarrow[\eta_0\otimes \mathbb{A}^\infty]{} H_1(A, \mathbb{Q})\otimes \mathbb{A}^\infty \xrightarrow[\eta]{} V\otimes \mathbb{A}^\infty$$

$\rightsquigarrow$ get element in $G(\mathbb{A}^\infty)$
elt in X_τ

to define over ring of integers over p-adic field.
p split in $l \ni u\, u^c$
w prime above u of F (induced from $\tau: F \hookrightarrow \mathbb{C} \cong \overline{\mathbb{Q}}_p$)
$\mathcal{O}_K \subset K = F_w$ ring of integers

now have $X_\mathcal{U}$ as moduli space $(A, \lambda, i, \eta^p, \eta_p)$
where $\bar{\eta}^p$ is orbit of isom $\eta: V\otimes_\mathbb{Q} \mathbb{A}^{\infty,p} \xrightarrow{\sim} H_1(A, \mathbb{Q})\otimes_\mathbb{Q} \mathbb{A}^{\infty,p}$

example: $(\mu_{p^n})_n$ $\mathbb{Q}_p/\mathbb{Z}_p$

A abelian variety action by $\mathcal{O}_F$
let $\mathcal{G}$ be the inductive system of $A[p^n]$ $n \to \infty$ } p-divisible group

Serre-Tate: deformations of A $\simeq$ deformations of $\mathcal{G}$
locally, structure of X_u given by deformations on $\mathcal{G}$.

in char p: $(\overline{\mathbb{F}}_p)$ { Lie $A \otimes_{\mathbb{Z}_p \otimes \mathcal{O}_E} \mathcal{O}_{E,u}$ has rank n
compatible
actions of $\mathcal{O}_F$ on Lie A via end and structure maps are the same

$A[p^\infty] = A[u^\infty] \times A[(u^c)^\infty]$

interchanged by polarization

focus on one, $A[u^\infty] = \prod_{w_i \text{ above } u} A[w_i^\infty]$ Lie $A[w_i^\infty]$ is 0-dim'l for $w_i \neq w$

$\mathcal{O}_F \longrightarrow \mathcal{O}_{F_w} \cong \mathcal{O}_K$ and n-dim'l at w

$A[w_i^\infty]$ is 0-dim'l $\Rightarrow$ as group scheme, (given by signature)
étale, $\Rightarrow$ deformations are unique

recall w above p, Assume $B_w \simeq M_n(F_w)$

$A[w^\infty] \longrightarrow \varepsilon A[w^\infty]$ - 1 dim'l p-divisible group
$M_n(\mathcal{O}_{F_w})$ using an idempotent $e = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & \ddots & \\ \vdots & & & \\ 0 & & & 0 \end{pmatrix}$

1-dim'l p-divisible groups w/ $\mathcal{O}_K$-action (studied by Drinfeld)
Deformation ring $\mathcal{O}_K[[X_1, \dots, X_{n-1}]]$ (no p-level structure)

for n=2 similar structure