Fields Institute Workshop
Galois Representations, Shimura Varieties and Automorphic Forms

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Lecture 2: Automorphic representations

1 Automorphic representations

Let $G =$ a connected reductive group over a number field k. We denote by $\mathbb{A}$ the ring of adeles of k. We will give three different definitions of automorphic representations, underlining the standard one, and then we will explain the connections between them. When no confusion arises, we set $G = G(\mathbb{A})$ (for instance, $\widehat{G}$ denotes the unitary dual – the set of isomorphism classes of irreducible unitary representations (not to be confused with the dual group $\check{G}$) – of $G(\mathbb{A})$; and $\Gamma = G(k)$.

Definition. An automorphic representation is an irreducible unitary representation that appears (i.e. is in the support of Plancherel measure) in a Plancherel decomposition:

$$L^2(\Gamma\backslash G) = \int_{\widehat{G}} \mathcal{H}_\pi \mu(\pi).$$

(Every unitary representation of G has an essentially unique such decomposition into a direct integral of irreducibles with multiplicities. Compare: the decomposition of $L^2(\mathbb{R}/\mathbb{Z})$ or $L^2(\mathbb{R})$ into direct sums/integrals of one-dimensional spaces spanned by unitary exponentials. However, here we will have both a continuous and a discrete (modulo center) spectrum.

Definition. An automorphic representation is an admissible representation π of $G(\mathbb{A})$, together with an injective morphism:

$$\nu : \pi \hookrightarrow C^\infty(\Gamma\backslash G).$$

An *automorphic form* is a (K_∞-finite) vector in the image of such a ν.

Admissible means: "each K-type has finite multiplicity" (where K denotes a "good" maximal compact subgroup of $G(\mathbb{A})$), but it also implicitly means that we are in the category of representations π such that: as a representation of $G(\hat{k})$, π is *smooth*, i.e. every vector has an open stabilizer; and for fixed level (=compact open subgroup of $G(\hat{k})$) K_f, π^{K_f} is a *smooth representation of moderate growth* (defined in the first lecture) of $G(k_\infty)$. Such a π is topologized as the direct limit of the Fréchet spaces π^{K_f} over a basis of neighborhoods of the identity in $G(\hat{k})$. Because of admissibility, every K-finite vector generates a subrepresentation of finite length over $\prod_{v\in S} G(k_v)$, for any finite set S of places.

This definition implies the usual moderate growth condition for automorphic forms, as well as the fact that they are annihilated by an ideal of finite codimension in $\mathfrak{z}(\mathfrak{g})$.

Definition. An *automorphic representation* is an *irreducible subquotient* of a representation π as in the previous definition.

I now explain how to go from the first to the third (more standard) definition. Langlands has proven that a representation is automorphic if and only if it is a subquotient of a parabolically induced cuspidal automorphic representation of a Levi subgroup. Since cuspidal representations are, up to twist by an idele class character of the group,

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in the discrete (modulo center) spectrum of $L^2(\Gamma\backslash G)$, it is clear that if one is interested in knowing all automorphic representations, it is enough to know only those which appear discretely in $L^2(\Gamma\backslash G)$ (and the analogous spaces for Levi subgroups).

Now, an automorphic representation (as in the third, standard definition, hence irreducible) is a restricted tensor product over all places:

$$\pi = \otimes'_v \pi_v,$$

where π_v is an irreducible representation (in the same category as described in the second definition) of $G(k_v)$, which is unramified for almost all (finite) v, i.e.: $\pi_v^{G(\mathfrak{o}_v)} \neq 0$ for almost all v – say for $v \notin S$.

We will discuss how to attach invariants to this representation, more precisely:

- Satake parameters $(t_v)_{v\notin S}$;
- infinitesimal character $(\lambda_v)_{v|\infty}$.

2 L-group

Recall: the category of tori over a field k is (contravariantly) equivalent to the category of free, finitely generated $\mathbb{Z}$-modules with $\mathrm{Gal}(\bar{k}/k)$-action (by passing from a torus to its character group).

Given a torus A over k, the *dual torus* (over, say, an algebraically closed field) is defined (because of this equivalence of categories) by the condition $\mathcal{X}^*(\check{A})_{\bar{k}} = \mathrm{Hom}_{\bar{k}}(\check{A}, \mathbb{G}_m) = \mathrm{Hom}_{\bar{k}}(\mathbb{G}_m, A) = \mathcal{X}_*(A)_{\bar{k}}$. We would like to think of $\check{A}$ as a split torus over $\mathbb{Z}$ (or the field over which our representations are defined in each particular case), hence $\check{A} = \mathrm{spec}\,\mathbb{Z}[\mathcal{X}_*(A)_{\bar{k}}]$. It carries a natural action of $\mathrm{Gal}(\bar{k}/k)$, induced from its action on the character group of A. The L-group of A is the semidirect product:

$${}^L A = \check{A} \rtimes \mathrm{Gal}(\bar{k}/k).$$

Let G be a connected reductive group over k, and assume that it is quasisplit (i.e. has a Borel subgroup B over k). Let N be the unipotent radical of B, and $A = B/N$, a torus. (This torus doesn't really depend on the choice of B, in the sense that for any two choices there is a canonical isomorphism of the corresponding A's, using the fact that all Borels are conjugate, self-normalizing, and A is abelian; this torus A is called the "universal Cartan" of G.)

The adjoint representation of G gives rise to "root data", which can be thought of as an embellishment of the character and cocharacter groups of A by dual sets of roots (Φ) and coroots ($\check{\Phi}$). There is an obvious symmetry in the quadruple ($\mathcal{X}^*(A) \supset \Phi, \mathcal{X}_*(A) \supset \check{\Phi}$), and by switching the two abelian groups we get the data for a split dual group $\check{G}$ (over, say, $\mathbb{Z}$), with maximal torus $\check{A}$ and Galois action (not totally obvious) extending the Galois action on $\check{A}$. The semidirect product:

$${}^L G = \check{G} \rtimes \mathrm{Gal}(\bar{k}/k)$$

is the *L-group* of G.

The construction $G \rightsquigarrow {}^L G$ is not functorial, but it is functorial with respect to homomorphisms which induce the identity on adjoint groups. We discuss the following example (where we can forget about the action of Galois, because it is trivial):

- Consider the sequence:

  $$\mathrm{SL}_2 \to \mathrm{GL}_2 \to \mathrm{PGL}_2\,.$$

  We can identify the universal Cartan A_{GL} of GL_2 with $\mathbb{G}_m^2$ by $\begin{pmatrix} a & * \\ & b \end{pmatrix} \mapsto (a,b)$, then $\check{A}_{\mathrm{GL}} = \mathbb{G}_m^2$. On the other hand, the characters of a Borel of PGL_2 are those characters of A_{GL} which vanish on the diagonal, i.e. we can identify $\check{A}_{\mathrm{PGL}}$ with the subgroup (χ, χ^{-1}) of $\mathbb{G}_m^2$. Finally, the characters of A_{SL} extend non-uniquely to characters of A_{GL}, and the kernel of the restriction map is all characters of the form (χ,χ). The dual tori $\check{A}$ are maximal tori for the following sequence of dual groups:

  $$\mathrm{PGL}_2 \leftarrow \mathrm{GL}_2 \leftarrow \mathrm{SL}_2\,.$$

3 Satake isomorphism

Here v denotes a finite place, $\mathfrak{o}_v$ = the integers of k_v, $G_v = G(k_v)$.

For any $f \in M_c^\infty(G_v)$ (compactly supported, smooth measures on G_v) we can define a convolution operator on $C^\infty(G(k)\backslash G(\mathbb{A}))$ (or any smooth representation, for that matter):

$$f \star \Phi(x) = \int_G \Phi(xg)f(g).$$

Notice that for a compact open subgroup $\prod_{v<\infty} K_v$ of the finite adeles of G, the *Hecke algebra* $\mathcal{H}(G_v, K_v) := M_c^\infty(G_v)^{K_v \times K_v}$ preserves the subspace $C^\infty(G(k)\backslash G(\mathbb{A})/\prod_v K_v)$. Now we assume that G has a *reductive* integral model away from a finite number S of places, and $v \notin S$; this implies, in particular, that G_v is unramified (=it is quasisplit, and splits over an unramified extension); and by enlarging S let us assume also that $K_v = G(\mathfrak{o}_v)$ for $v \notin S$. We will discuss the Hecke algebra $\mathcal{H}(G_v, K_v)$ for such a v.

Theorem (Satake). *Let G be unramified over k_v; in particular, the decomposition group at v acts on $\check{G}$ via its unramified quotient $\langle \mathrm{Frob}_v \rangle$. Then there is a canonical isomorphism:*

$$\mathcal{H}(G_v, K_v) = \mathbb{C}[\check{G} /\!\!/ \mathrm{Frob}_v\text{-}\textit{twisted conjugacy}],$$

where $/\!\!/$ denotes the quotient in the category of affine varieties (i.e. by definition $\mathbb{C}[\check{G} /\!\!/ \mathrm{Frob}_v\text{-twisted conjugacy}] = \mathbb{C}[\check{G}]^{\mathrm{Frob}_v\text{-twisted conjugacy}}$).

If G_v is split, this is the same as $\mathbb{C}[\check{A}/W]$, where $\check{A}$ is the dual of the abstract Cartan and W is the Weyl group. This (Chevalley isomorphism) is also the same as $\mathbb{C}[\check{G}]^{\check{G}\text{-conj}}$, which is the same as $\mathrm{Rep}(\check{G}) \otimes \mathbb{C}$ (by associating to each – finite dimensional, algebraic – representation its character). In the general case, replace $\check{G}$-conjugacy by Frobenius-twisted $\check{G}$-conjugacy.

Given π_v: an irreducible smooth representation with $\pi_v^{K_v} \neq 0$, the $\mathcal{H}(G_v, K_v)$-module $\pi_v^{K_v}$ is irreducible, and since $\mathcal{H}(G_v, K_v)$ is abelian this means that it is one-dimensional, and $\mathcal{H}(G_v, K_v)$ acts by a scalar. In other words, from π_v we get a $\mathbb{C}$-valued point in $\operatorname{spec} \mathcal{H}(G_v, K_v) = \check{G} /\!\!/ \mathrm{Frob}_v$-twisted conjugacy, i.e. a semisimple (=closed) (Frob_v-twisted) conjugacy class t_p in $\check{G}(\mathbb{C})$. *This is the Satake parameter of* π_v. Equivalently, it is the image of Frob_v under any unramified homomorphism

$$\rho_v : \mathcal{W}_{k_v} \to {}^L G$$

over $\mathrm{Gal}(\bar{k}_v/k_v)$ with semisimple image. (Here $\mathcal{W}_{k_v}$ denotes the Weil group of k_v, and "unramified" means that the projection to $\check{G}$ factors through the map $\mathcal{W}_{k_v} \to \langle \mathrm{Frob}_v \rangle$.)

Where does the Satake isomorphism come from? Consider the action of $\mathcal{H}(G, K)$ (we drop the indices v for now) on the module $M := C_c^\infty(N\backslash G)^K$ by convolution. The orbit map on 1_{NK} gives a linear map:

$$\mathcal{H}(G, K) \ni h \mapsto h \star 1_{NK} \in C_c^\infty(N\backslash G)^K. \quad (*)$$

On the other hand, we have a commuting action of A/A_0 (where A_0 is the maximal compact subgroup of A) "on the left", and the action map of $\mathcal{H}(A, A_0)$ on 1_{NK} identifies M with $\mathcal{H}(A, A_0)$ (by the Iwasawa decomposition: $G = BK$). Here we must *normalize* the action of A on the left so that it is unitary:

$$a \cdot f(Nx) = \delta(a)^{-\frac{1}{2}} f(Nax).$$

In combination with $(*)$ we get a map:

$$\mathcal{H}(G, K) \to C_c^\infty(N\backslash G)^K \to \mathcal{H}(A, A_0), \quad (**)$$

and the composition is a homomorphism of algebras (easy!).

If S is the maximal split subtorus of A, we have: $A/A_0 = \mathcal{X}_*(S)$, hence $\mathcal{H}(A, A_0) = \mathbb{C}[\mathcal{X}_*(S)] = \mathbb{C}[\check{S}]$, and it turns out that (**) is injective and its image lies in the invariants of the relative Weyl group W_k. Thus:

$$\mathcal{H}(G, K) = \mathbb{C}[\check{S}]^{W_k},$$

and it turns out again that this is the same as $\mathbb{C}[\check{G} /\!\!/ \mathrm{Frob}_v\text{-twisted conjugacy}]$. (All these facts, except for the statement about the image of (**), are very easily seen in the split case.)

Notice that both sides of the Satake isomorphism have natural $\mathbb{Z}$-structures: the Hecke algebra from identifying it with $M_c(K_v\backslash G_v)^{K_v}$ and considering measures which are integral on the discrete set $K_v\backslash G_v$, and the coordinate ring $\mathbb{C}[\check{S}]^{W_k}$ by the basis of $\mathbb{C}[\check{S}]$ by cocharacters of S (equivalently, in the split case, as the representation ring of $\check{G}$ over $\mathbb{Z}$.) These structures are not compatible, in general, not even over $\mathbb{Q}$, because of the necessary normalization of the action of A. In the general case, one needs to adjoin square roots of $q^{\pm 1}$ (the residual degree) to $\mathbb{Z}$. See the article of Gross "On the Satake isomorphism" for a good discussion of the issue.

4 Infinitesimal character

If G is a reductive group over $\mathbb{R}$ (if it is defined over $\mathbb{C}$ we consider it as a group over $\mathbb{R}$ by restriction of scalars) then we set $\mathfrak{z}(\mathfrak{g}) = \mathrm{Cent}(\mathrm{U}(\mathfrak{g}_{\mathbb{C}}))$, the center of the complexified[1] universal enveloping algebra.

In the case of $\mathrm{SL}_2(\mathbb{R})$ we mentioned in the previous lecture that $\mathfrak{z}(\mathfrak{g})$ is a polynomial algebra generated by the Casimir operator C. In general, Harish-Chandra proved:

Theorem 4.1 (Harish-Chandra isomorphism)**.** *There is a canonical isomorphism:* $\mathfrak{z}(\mathfrak{g}) \simeq \mathbb{C}[\check{\mathfrak{a}}]^W$.

Here $\check{\mathfrak{a}}$ is what you can imagine from the notation, namely the Lie algebra of the dual Cartan. Equivalently, $\check{\mathfrak{a}} = \mathcal{X}^*(A)_{\mathbb{C}} \otimes_{\mathbb{Z}} \mathbb{C}$. We can rewrite the quotient $\check{\mathfrak{a}}/W$ as $\check{\mathfrak{g}} /\!\!/ \check{G}$ (affine quotient under the adjoint action).

A comment on "canonical": again, the isomorphism is defined by reference to the torus case. In the torus case, $\mathfrak{z}(\mathfrak{a}) = U(\mathfrak{a}_{\mathbb{C}}) = S^{\bullet}\mathfrak{a}_{\mathbb{C}} = \mathbb{C}[\check{\mathfrak{a}}]$. The general case is obtained by considering the natural filtration of $\mathfrak{z}(\mathfrak{g})$ (by degree), and relating the associated graded to $U(\mathfrak{a}_{\mathbb{C}})$.

Since (by a version of Schur's lemma) $\mathfrak{z}(\mathfrak{g})$ acts by scalars on irreducible admissible representations, every irreducible admissible representation π corresponds to a closed point in $\operatorname{spec} \mathfrak{z}(\mathfrak{g}) = \check{\mathfrak{a}}/W$. This is the *infinitesimal character* of π.

Again, if G is quasisplit we can understand Harish-Chandra parameters in terms of generalized principal series, since every irreducible representation is a subquotient of a generalized principal series. Namely, we let $I(\chi) = \mathrm{Ind}(\chi\delta^{\frac{1}{2}})$, where χ is a character of the Borel subgroup. The differential of χ defines a character of $\mathfrak{a}$, i.e. a point of $\check{\mathfrak{a}} = \mathfrak{a}^*_{\mathbb{C}}$. The Harish-Chandra parameter of $I(\chi)$ is the image of this point in $\check{\mathfrak{a}}/W$.

An infinitesimal character $\lambda \in \check{\mathfrak{a}}/W$ is called *integral* if it is in the image of $\mathcal{X}^*(A)_{\mathbb{C}} \subset \check{\mathfrak{a}}$.

Conjecture. *If π is an automorphic representation with $\lambda(\pi_v)$ integral for all $v|\infty$, then there is a Galois representation:*

$$\rho : \mathrm{Gal}(\bar{k}/k) \to {}^L G(\overline{\mathbb{Q}_l}).$$

associated to π.

Here l is a chosen prime, and "associated to π" means (at least) that at all places v not dividing l where G, π_v are unramified the restriction of ρ to the decomposition group at v is unramified and coincides with the representation $\langle \mathrm{Frob}_v \rangle \to {}^L G$ associated to π_v by the Satake isomorphism. See the article of Buzzard and Gee for more properties that ρ should have.

Example 4.2. The following example may seem a bit surprising. Consider a holomorphic modular form of weight 2 with trivial nebentypus and rational Fourier coefficients. We have seen that it corresponds to an automorphic representation π of GL_2 with trivial central character (hence, an automorphic representation of PGL_2, really) and infinity type equal to the discrete series representation of $\mathrm{PGL}_2(\mathbb{R})$ with restriction $\mathcal{D}_2$ to SL_2. This is a subquotient of the (normalized) induced character:

$$\chi : \begin{pmatrix} a & * \\ & b \end{pmatrix} \mapsto \left|\frac{a}{b}\right|^{\frac{1}{2}}.$$

This is an integral character when restricted to SL_2, but it is *not* integral for GL_2 (or PGL_2). So, why can we attach a Galois representation to it?

Notice that the Galois representation is obtained by the action of the Galois group on the Tate module of an elliptic curve, which has determinant equal to the cyclotomic character. Hence, it does not have image in SL_2 (the dual group of PGL_2), and cannot even be twisted by a character to have image in SL_2 (the cyclotomic character does not have a

[1] There is a choice of isomorphism $\bar{\mathbb{R}} \simeq \mathbb{C}$ implicit here, which for global purposes is better to avoid; see the paper of Buzzard and Gee "The conjectural connections ..." for better formulations, and a precise statement of the conjecture on algebraicity.

square root). The answer to this riddle is that the Galois representation is not really associated to π itself, but to the twist $\pi \otimes |\det|^{\frac{1}{2}}$ of π, which is not any more an automorphic representation of PGL_2, but of GL_2. You can check that this twist has integral infinitesimal character.