Center manifolds for holomorphic vector fields in dimension three

joint with M. Mcquillan

Singular 1-dimensional foliations $\mathcal{F}$

Singular 1-dimensional foliations $\mathcal{F}$

- An analytic (real or complex manifold) M

Singular 1-dimensional foliations $\mathcal{F}$

- An analytic (real or complex manifold) M
- An open covering $\{U_i\}_{i \in I}$ of M

Singular 1-dimensional foliations $\mathcal{F}$

- An analytic (real or complex manifold) M
- An open covering $\{U_i\}_{i\in I}$ of M
- For each $i \in I$, a vector field $\partial_i \in \mathrm{Der}\big(\mathcal{O}(U_i)\big)$ such that

$$\partial_i = g_{ij}\,\partial_j, \text{ on } U_i \cap U_j$$

for some $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$.

Singular 1-dimensional foliations $\mathcal{F}$

- An analytic (real or complex manifold) M
- An open covering $\{U_i\}_{i \in I}$ of M
- For each $i \in I$, a vector field $\partial_i \in \mathrm{Der}\big(\mathcal{O}(U_i)\big)$ such that

$$\partial_i = g_{ij}\, \partial_j, \text{ on } U_i \cap U_j$$

for some $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$.

Each $\partial \in \{\partial_i\}_{i \in I}$ is a local generator of the foliation,

$$\partial = a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n}$$

Singular 1-dimensional foliations $\mathcal{F}$

- An analytic (real or complex manifold) M
- An open covering $\{U_i\}_{i\in I}$ of M
- For each $i \in I$, a vector field $\partial_i \in \mathrm{Der}\big(\mathcal{O}(U_i)\big)$ such that

$$\partial_i = g_{ij}\,\partial_j, \text{ on } U_i \cap U_j$$

for some $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$.

Each $\partial \in \{\partial_i\}_{i\in I}$ is a local generator of the foliation,

$$\partial = a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n}$$

We say that the foliation is saturated if $(a_1, \ldots, a_n) = 1$.

Singularities of a 1-dimensional foliation

Singularities of a 1-dimensional foliation

$$\mathrm{Sing}(\mathcal{F}) = \{a_1 = \cdots = a_n = 0\}$$

Singularities of a 1-dimensional foliation

$$\mathrm{Sing}(\mathcal{F}) = \{a_1 = \cdots = a_n = 0\}$$

A point $p \in \mathrm{Sing}(\mathcal{F})$ is elementary if the induced linear map

$$\partial : \mathbf{m}/\mathbf{m}^2 \to \mathbf{m}/\mathbf{m}^2$$

has at least one non-zero eigenvalue.

Singularities of a 1-dimensional foliation

$$\mathrm{Sing}(\mathcal{F}) = \{a_1 = \cdots = a_n = 0\}$$

A point $p \in \mathrm{Sing}(\mathcal{F})$ is elementary if the induced linear map

$$\partial : \mathbf{m}/\mathbf{m}^2 \to \mathbf{m}/\mathbf{m}^2$$

has at least one non-zero eigenvalue. That is, the Jacobian matrix

$$\left[\frac{\partial a_i}{\partial x_j}(p)\right]_{i,j=1,\ldots,n}$$

is non-nilpotent.

Resolution of singularities

Let $(M, \mathcal{F})$ be a saturated 1-dimensional singular foliation. Then, there exists a finite sequence of blowing-ups

$$(M, \mathcal{F}) = (M_0, \mathcal{F}_0) \leftarrow \cdots \leftarrow (M_n, \mathcal{F}_n) = (\widetilde{M}, \widetilde{\mathcal{F}})$$

such that all the singularities of $(\widetilde{M}, \widetilde{\mathcal{F}})$ are elementary.

Resolution of singularities

Let $(M, \mathcal{F})$ be a saturated 1-dimensional singular foliation. Then, there exists a finite sequence of blowing-ups

$$(M, \mathcal{F}) = (M_0, \mathcal{F}_0) \leftarrow \cdots \leftarrow (M_n, \mathcal{F}_n) = (\widetilde{M}, \widetilde{\mathcal{F}})$$

such that all the singularities of $(\widetilde{M}, \widetilde{\mathcal{F}})$ are elementary.

- $\dim M = 2$, Bendixson and Seidenberg.

Resolution of singularities

Let $(M, \mathcal{F})$ be a saturated 1-dimensional singular foliation. Then, there exists a finite sequence of blowing-ups

$$(M, \mathcal{F}) = (M_0, \mathcal{F}_0) \leftarrow \cdots \leftarrow (M_n, \mathcal{F}_n) = (\widetilde{M}, \widetilde{\mathcal{F}})$$

such that all the singularities of $(\widetilde{M}, \widetilde{\mathcal{F}})$ are elementary.

- $\dim M = 2$, Bendixson and Seidenberg.
- $\dim M = 3$, M analytic $/\mathbb{R}$ (2007).

Resolution of singularities

Let $(M, \mathcal{F})$ be a saturated 1-dimensional singular foliation. Then, there exists a finite sequence of blowing-ups

$$(M, \mathcal{F}) = (M_0, \mathcal{F}_0) \leftarrow \cdots \leftarrow (M_n, \mathcal{F}_n) = (\widetilde{M}, \widetilde{\mathcal{F}})$$

such that all the singularities of $(\widetilde{M}, \widetilde{\mathcal{F}})$ are elementary.

- $\dim M = 2$, Bendixson and Seidenberg.
- $\dim M = 3$, M analytic $/\mathbb{R}$ (2007).
- $\dim M = 3$, M projective $/\mathbb{C}$ (joint with Mcquillan 2009).

What are the final formal local models?

What are the final formal local models?

dim $M = 2$: There exists formal local coordinates $x, y \in \widehat{\mathcal{O}}_p$ such that a local generator of $\mathcal{F}_p$ is in:

What are the final formal local models?

dim $M = 2$: There exists formal local coordinates $x, y \in \widehat{\mathcal{O}}_p$ such that a local generator of $\mathcal{F}_p$ is in:

- Linearisable case:

$$\partial = \lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

What are the final formal local models?

dim $M = 2$: There exists formal local coordinates $x, y \in \widehat{\mathcal{O}}_p$ such that a local generator of $\mathcal{F}_p$ is in:

- Linearisable case:

$$\partial = \lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

- Positive resonance case:

$$\partial = (nx + \nu y^n) \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

What are the final formal local models?

dim $M = 2$: There exists formal local coordinates $x, y \in \widehat{\mathcal{O}}_p$ such that a local generator of $\mathcal{F}_p$ is in:

- Linearisable case:

$$\partial = \lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

- Positive resonance case:

$$\partial = (nx + \nu y^n) \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

- Resonant saddle case:

$$\partial = -qx \frac{\partial}{\partial x} + py \frac{\partial}{\partial y} + \frac{(x^p y^q)^r}{1 + \nu (x^p y^q)^r} x \frac{\partial}{\partial x}$$

What are the final formal local models?

dim $M = 2$: There exists formal local coordinates $x, y \in \widehat{\mathcal{O}}_p$ such that a local generator of $\mathcal{F}_p$ is in:

- Linearisable case:

$$\partial = \lambda x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

- Positive resonance case:

$$\partial = (nx + \nu y^n) \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

- Resonant saddle case:

$$\partial = -qx \frac{\partial}{\partial x} + py \frac{\partial}{\partial y} + \frac{(x^p y^q)^r}{1 + \nu (x^p y^q)^r} x \frac{\partial}{\partial x}$$

- Saddle-node case:

$$\partial = y \frac{\partial}{\partial y} + \frac{x^r}{1 + \nu x^r} x \frac{\partial}{\partial x}$$

What are the final analytic local models?

What are the final analytic local models?

$\dim M = 2$: Many interesting phenomena can appear for the analytic classification:

What are the final analytic local models?

dim $M = 2$: Many interesting phenomena can appear for the analytic classification:

- Linearisable case: (small denominators)

What are the final analytic local models?

$\dim M = 2$: Many interesting phenomena can appear for the analytic classification:

- Linearisable case: (small denominators)
- Positive resonance case: (always convergent)

What are the final analytic local models?

dim $M = 2$: Many interesting phenomena can appear for the analytic classification:

- Linearisable case: (small denominators)
- Positive resonance case: (always convergent)
- Resonant saddle case: (non-linear Stoke's phenomena - Ecalle, Ilyashenko, Martinet-Ramis, ...)

What are the final analytic local models?

dim $M = 2$: Many interesting phenomena can appear for the analytic classification:

- Linearisable case: (small denominators)
- Positive resonance case: (always convergent)
- Resonant saddle case: (non-linear Stoke's phenomena - Ecalle, Ilyashenko, Martinet-Ramis, ...)
- Saddle-node case: (non-linear Stoke's phenomena - Ecalle, Ilyashenko, Martinet-Ramis, ...)

What are the final analytic local models?

dim $M = 2$: Many interesting phenomena can appear for the analytic classification:

- Linearisable case: (small denominators)
- Positive resonance case: (always convergent)
- Resonant saddle case: (non-linear Stoke's phenomena - Ecalle, Ilyashenko, Martinet-Ramis, ...)
- Saddle-node case: (non-linear Stoke's phenomena - Ecalle, Ilyashenko, Martinet-Ramis, ...)

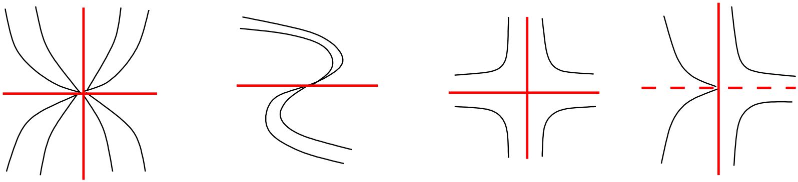
However, concerning the existence of invariant analytic curves (separatrices), the problem is completely understood:

What are the final analytic local models?

dim $M = 2$: Many interesting phenomena can appear for the analytic classification:

- Linearisable case: (small denominators)
- Positive resonance case: (always convergent)
- Resonant saddle case: (non-linear Stoke's phenomena - Ecalle, Ilyashenko, Martinet-Ramis, ...)
- Saddle-node case: (non-linear Stoke's phenomena - Ecalle, Ilyashenko, Martinet-Ramis, ...)

However, concerning the existence of invariant analytic curves (separatrices), the problem is completely understood:



What are the final analytic local models?

dim $M = 2$: Many interesting phenomena can appear for the analytic classification:

- Linearisable case: (small denominators)
- Positive resonance case: (always convergent)
- Resonant saddle case: (non-linear Stoke's phenomena - Ecalle, Ilyashenko, Martinet-Ramis, ...)
- Saddle-node case: (non-linear Stoke's phenomena - Ecalle, Ilyashenko, Martinet-Ramis, ...)

However, concerning the existence of invariant analytic curves (separatrices), the problem is completely understood:

formal existence $\Rightarrow$ analytic existence

What are the final analytic local models?

dim $M = 2$: Many interesting phenomena can appear for the analytic classification:

- Linearisable case: (small denominators)
- Positive resonance case: (always convergent)
- Resonant saddle case: (non-linear Stoke's phenomena - Ecalle, Ilyashenko, Martinet-Ramis, ...)
- Saddle-node case: (non-linear Stoke's phenomena - Ecalle, Ilyashenko, Martinet-Ramis, ...)

However, concerning the existence of invariant analytic curves (separatrices), the problem is completely understood:

$$\boxed{\text{formal existence} \Rightarrow \text{analytic existence}}$$

Except for the center manifold of the saddle-node

The center manifold for a saddle-node

The center manifold for a saddle-node

Consider a singularity of the form

$$\partial = a(x,y)\frac{\partial}{\partial x} + \big(y + b(x,y)\big)\frac{\partial}{\partial y} \qquad a, b \in \mathbb{C}\{x,y\} \cap \mathbf{m}^2$$

The center manifold for a saddle-node

Consider a singularity of the form

$$\partial = a(x,y)\frac{\partial}{\partial x} + \big(y + b(x,y)\big)\frac{\partial}{\partial y} \qquad a, b \in \mathbb{C}\{x,y\} \cap \mathbf{m}^2$$

we can assume ∂ in *Dulac normal form*

$$x^{r+1}\frac{\partial}{\partial x} + \Big(y(1+\varepsilon) - g\Big)\frac{\partial}{\partial y},$$

$r \geq 1$,

The center manifold for a saddle-node

Consider a singularity of the form

$$\partial = a(x,y)\frac{\partial}{\partial x} + \big(y + b(x,y)\big)\frac{\partial}{\partial y} \qquad a, b \in \mathbb{C}\{x,y\} \cap \mathbf{m}^2$$

we can assume ∂ in *Dulac normal form*

$$x^{r+1}\frac{\partial}{\partial x} + \Big(y(1+\varepsilon) - g\Big)\frac{\partial}{\partial y},$$

$r \geq 1, \;\; g \in \mathbb{C}\{x\} \cap \mathbf{m}^2,$

The center manifold for a saddle-node

Consider a singularity of the form

$$\partial = a(x,y)\frac{\partial}{\partial x} + \big(y + b(x,y)\big)\frac{\partial}{\partial y} \qquad a, b \in \mathbb{C}\{x,y\} \cap \mathbf{m}^2$$

we can assume ∂ in *Dulac normal form*

$$x^{r+1}\frac{\partial}{\partial x} + \Big(y(1+\varepsilon) - g\Big)\frac{\partial}{\partial y},$$

$r \geq 1,\ \ g \in \mathbb{C}\{x\} \cap \mathbf{m}^2,\ \ \varepsilon \in \mathbb{C}\{x,y\} \cap \mathbf{m}$

The center manifold for a saddle-node

The center manifold for a saddle-node

$$\partial = x^{r+1}\frac{\partial}{\partial x} + (y(1+\varepsilon) - g)\frac{\partial}{\partial y}$$

The center manifold for a saddle-node

$$\partial = x^{r+1}\frac{\partial}{\partial x} + (y(1+\varepsilon) - g)\frac{\partial}{\partial y}$$

We are are looking for an invariant curve of the form

$$S = \{y = f(x)\},$$

The center manifold for a saddle-node

$$\partial = x^{r+1}\frac{\partial}{\partial x} + (y(1+\varepsilon) - g)\,\frac{\partial}{\partial y}$$

We are are looking for an invariant curve of the form

$$S = \{y = f(x)\}, \qquad f \in \mathbb{C}[[x]]$$

The center manifold for a saddle-node

$$\partial = x^{r+1}\frac{\partial}{\partial x} + (y(1+\varepsilon) - g)\frac{\partial}{\partial y}$$

We are are looking for an invariant curve of the form

$$S = \{y = f(x)\}, \qquad f \in \mathbb{C}[[x]]$$

If we consider the ideal $I = \langle y - f(x)\rangle \subset \mathbb{C}[[x,y]]$ then

$$S \text{ is invariant} \quad \Longleftrightarrow \quad \partial(I) \subset I$$

The center manifold for a saddle-node

$$\partial = x^{r+1}\frac{\partial}{\partial x} + (y(1+\varepsilon) - g)\,\frac{\partial}{\partial y}$$

We are are looking for an invariant curve of the form

$$S = \{y = f(x)\}, \qquad f \in \mathbb{C}[[x]]$$

If we consider the ideal $I = \langle y - f(x)\rangle \subset \mathbb{C}[[x, y]]$ then

$$S \text{ is invariant} \quad \Longleftrightarrow \quad \partial(I) \subset I$$

which gives the ODE

$$\left\{(\mathbb{1} + \varepsilon) - x^{r+1}\frac{d}{dx}\right\}(f) = g$$

The center manifold for a saddle-node

$$\partial = x^{r+1}\frac{\partial}{\partial x} + (y(1+\varepsilon) - g)\frac{\partial}{\partial y}$$

We are are looking for an invariant curve of the form

$$S = \{y = f(x)\}, \qquad f \in \mathbb{C}[[x]]$$

If we consider the ideal $I = \langle y - f(x)\rangle \subset \mathbb{C}[[x, y]]$ then

$$S \text{ is invariant} \iff \partial(I) \subset I$$

which gives the ODE

$$\left\{(\mathbb{1} + \varepsilon) - x^{r+1}\frac{d}{dx}\right\}(f) = g$$

This is a nonlinear ODE, as ε usually depends on the unknown f.

The center manifold for a saddle-node

Let us suppose $r = 1$, and let $\xi = 1/x$. Then, we get the ODE

$$\left\{(1+\varepsilon)\mathbb{1} + \frac{d}{d\xi}\right\}(f) = g$$

For $\varepsilon = 0$, this is a linear ODE.

The center manifold for a saddle-node

Let us suppose $r = 1$, and let $\xi = 1/x$. Then, we get the ODE

$$\left\{(1+\varepsilon)\mathbb{1} + \frac{d}{d\xi}\right\}(f) = g$$

For $\varepsilon = 0$, this is a linear ODE.
We solve by formal series ($f, g \in \mathbb{C}[[1/\xi]] \cap \mathbf{m}$)

$$f = \left(\mathbb{1} + \frac{d}{d\xi}\right)^{-1} g$$

The center manifold for a saddle-node

Let us suppose $r = 1$, and let $\xi = 1/x$. Then, we get the ODE

$$\left\{(1+\varepsilon)\mathbb{1} + \frac{d}{d\xi}\right\}(f) = g$$

For $\varepsilon = 0$, this is a linear ODE.
We solve by formal series ($f, g \in \mathbb{C}[[1/\xi]] \cap \mathbf{m}$)

$$f = \left(\mathbb{1} + \frac{d}{d\xi}\right)^{-1} g = \sum_{n \geq 0} (-1)^n \frac{d^n}{d\xi^n} g$$

The center manifold for a saddle-node

Let us suppose $r = 1$, and let $\xi = 1/x$. Then, we get the ODE

$$\left\{(1+\varepsilon)\mathbb{1} + \frac{d}{d\xi}\right\}(f) = g$$

For $\varepsilon = 0$, this is a linear ODE.
We solve by formal series ($f, g \in \mathbb{C}[[1/\xi]] \cap \mathbf{m}$)

$$f = \left(\mathbb{1} + \frac{d}{d\xi}\right)^{-1} g = \sum_{n \geq 0} (-1)^n \frac{d^n}{d\xi^n} g$$

and e.g. for $g = 1/\xi$

The center manifold for a saddle-node

Let us suppose $r = 1$, and let $\xi = 1/x$. Then, we get the ODE

$$\left\{(1+\varepsilon)\mathbb{1} + \frac{d}{d\xi}\right\}(f) = g$$

For $\varepsilon = 0$, this is a linear ODE.
We solve by formal series ($f, g \in \mathbb{C}[[1/\xi]] \cap \mathbf{m}$)

$$f = \left(\mathbb{1} + \frac{d}{d\xi}\right)^{-1} g = \sum_{n\geq 0} (-1)^n \frac{d^n}{d\xi^n} g$$

and e.g. for $g = 1/\xi$, *one gets*

$$f(\xi) = \sum_{n\geq 0} (n!) \frac{1}{\xi^{n+1}}$$

The center manifold for a saddle-node

Let us suppose $r = 1$, and let $\xi = 1/x$. Then, we get the ODE

$$\left\{(1+\varepsilon)\mathbb{1} + \frac{d}{d\xi}\right\}(f) = g$$

For $\varepsilon = 0$, this is a linear ODE.
We solve by formal series ($f, g \in \mathbb{C}[[1/\xi]] \cap \mathbf{m}$)

$$f = \left(\mathbb{1} + \frac{d}{d\xi}\right)^{-1} g = \sum_{n \geq 0} (-1)^n \frac{d^n}{d\xi^n} g$$

and e.g. for $g = 1/\xi$, *one gets*

$$f(\xi) = \sum_{n \geq 0} (n!) \frac{1}{\xi^{n+1}} \qquad \textit{(Divergent)}$$

The center manifold for a saddle-node

Let us suppose $r = 1$, and let $\xi = 1/x$. Then, we get the ODE

$$\left\{(1+\varepsilon)\mathbb{1} + \frac{d}{d\xi}\right\}(f) = g$$

For $\varepsilon = 0$, this is a linear ODE.
We solve by formal series ($f, g \in \mathbb{C}[[1/\xi]] \cap \mathbf{m}$)

$$f = \left(\mathbb{1} + \frac{d}{d\xi}\right)^{-1} g = \sum_{n \geq 0} (-1)^n \frac{d^n}{d\xi^n} g$$

and e.g. for $g = 1/\xi$, one gets

$$f(\xi) = \sum_{n \geq 0} (n!) \frac{1}{\xi^{n+1}} \qquad \textit{(Divergent)}$$

There is no holomorphic center manifold in a neigh. of ∞.

The center manifold for a saddle-node

We consider again the ODE

$$\left\{(1+\varepsilon)\mathbb{1} + \frac{d}{d\xi}\right\}(f) = g$$

For $\varepsilon = 0$, this is a linear ODE.

The center manifold for a saddle-node

We consider again the ODE

$$\left\{(1+\varepsilon)\mathbb{1}+\frac{d}{d\xi}\right\}(f)=g$$

For $\varepsilon = 0$, this is a linear ODE.
We solve by variation of constants (with initial condition $f(*)=0$),

$$f(\xi)=e^{-\xi}\int_{*}^{\xi}e^{t}g(t)dv$$

The center manifold for a saddle-node

We consider again the ODE

$$\left\{(1+\varepsilon)\mathbb{1} + \frac{d}{d\xi}\right\}(f) = g$$

For $\varepsilon = 0$, this is a linear ODE.
We solve by variation of constants (with initial condition $f(*) = 0$),

$$f(\xi) = e^{-\xi}\int_*^{\xi} e^t g(t)dv = K(g)(\xi)$$

K is the *right inverse* of the operator $\left\{\mathbb{1} + \frac{d}{d\xi}\right\}$.

The center manifold for a saddle-node

We consider again the ODE

$$\left\{(1+\varepsilon)\mathbb{1} + \frac{d}{d\xi}\right\}(f) = g$$

For $\varepsilon = 0$, this is a linear ODE.
We solve by variation of constants (with initial condition $f(*) = 0$),

$$f(\xi) = e^{-\xi}\int_{*}^{\xi} e^{t} g(t) dv = K(g)(\xi)$$

K is the *right inverse* of the operator $\left\{\mathbb{1} + \frac{d}{d\xi}\right\}$. But we want to have a *bounded* inverse on some functional space.

The center manifold for a saddle-node

We consider again the ODE

$$\left\{(1+\varepsilon)\mathbb{1} + \frac{d}{d\xi}\right\}(f) = g$$

For $\varepsilon = 0$, this is a linear ODE.
We solve by variation of constants (with initial condition $f(*) = 0$),

$$f(\xi) = e^{-\xi} \int_{*}^{\xi} e^{t} g(t) dv = K(g)(\xi)$$

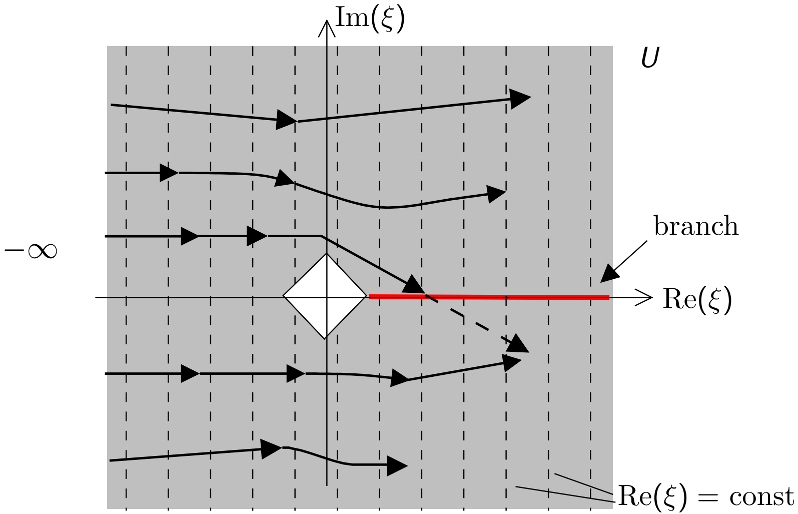
K is the *right inverse* of the operator $\left\{\mathbb{1} + \frac{d}{d\xi}\right\}$. But we want to have a *bounded* inverse on some functional space.
(remember that we want to solve the nonlinear ODE with $\varepsilon \neq 0$.)

The center manifold

We will choose $* = -\infty$ and the *paths of integration* as follows:

The center manifold

We will choose $* = -\infty$ and the *paths of integration* as follows:



The center manifold

For all $\xi \in U$, we can choose a path $\gamma_\xi(t)$, $t \in \mathbb{R}_{<0}$ such that

$$\gamma_\xi(0) = \xi$$

The center manifold

For all $\xi \in U$, we can choose a path $\gamma_\xi(t)$, $t \in \mathbb{R}_{<0}$ such that

$$\gamma_\xi(0) = \xi \quad ; \quad \lim_{t \to -\infty} \operatorname{Re} \gamma_\xi(t) = -\infty$$

and

The center manifold

For all $\xi \in U$, we can choose a path $\gamma_\xi(t)$, $t \in \mathbb{R}_{<0}$ such that

$$\gamma_\xi(0) = \xi \quad ; \quad \lim_{t \to -\infty} \operatorname{Re} \gamma_\xi(t) = -\infty$$

and

$$\|\dot{\gamma}_\xi(t)\| \leq C \frac{d}{dt} \operatorname{Re}(\gamma_\xi)(t)$$

The center manifold

For all $\xi \in U$, we can choose a path $\gamma_\xi(t)$, $t \in \mathbb{R}_{<0}$ such that

$$\gamma_\xi(0) = \xi \quad ; \quad \lim_{t \to -\infty} \operatorname{Re} \gamma_\xi(t) = -\infty$$

and

$$\|\dot{\gamma}_\xi(t)\| \leq C \frac{d}{dt} \operatorname{Re}(\gamma_\xi)(t)$$

Then

$$\Big\| K(g) \Big\|_U \leq C \Big\| g \Big\|_U$$

Indeed,

$$\begin{aligned} \|K(g)\| \leq & e^{-\operatorname{Re}\xi} \int_{-\infty}^{0} e^{\operatorname{Re}\gamma(t)} \|\dot{\gamma}(t)\| dt \\ \leq & C\, e^{-\operatorname{Re}\xi} \int_{-\infty}^{0} e^{\operatorname{Re}\gamma(t)} \frac{d}{dt} \operatorname{Re}\gamma(t) dt = C\|g\| \end{aligned}$$

The center manifold

For all $\xi \in U$, we can choose a path $\gamma_\xi(t)$, $t \in \mathbb{R}_{<0}$ such that

$$\gamma_\xi(0) = \xi \quad ; \quad \lim_{t \to -\infty} \operatorname{Re} \gamma_\xi(t) = -\infty$$

and

$$\|\dot{\gamma}_\xi(t)\| \leq C \frac{d}{dt} \operatorname{Re}(\gamma_\xi)(t)$$

Then

$$\Big\| K(g) \Big\|_U \leq C \Big\| g \Big\|_U$$

Indeed,

$$\begin{aligned} \|K(g)\| \leq & e^{-\operatorname{Re}\xi} \int_{-\infty}^{0} e^{\operatorname{Re}\gamma(t)} \|\dot{\gamma}(t)\| dt \\ \leq & C\, e^{-\operatorname{Re}\xi} \int_{-\infty}^{0} e^{\operatorname{Re}\gamma(t)} \frac{d}{dt} \operatorname{Re}\gamma(t) dt = C\|g\| \end{aligned}$$

The center manifold

Now, we can solve the nonlinear case $\varepsilon \neq 0$ by writing

$$\left\{(1+\varepsilon)\mathbb{1} + \frac{d}{d\xi}\right\} K = \mathbb{1} + Q, \qquad \text{(where } Q := \varepsilon K \text{ is small)}$$

The center manifold

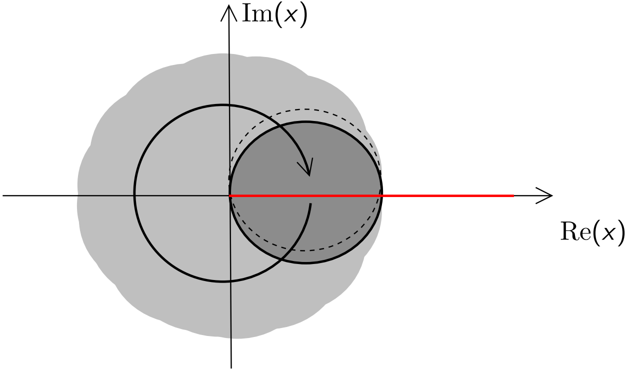
Now, we can solve the nonlinear case $\varepsilon \neq 0$ by writing

$$\left\{(1+\varepsilon)\mathbb{1} + \frac{d}{d\xi}\right\} K = \mathbb{1} + Q, \qquad \text{(where } Q := \varepsilon K \text{ is small)}$$

and applying the iterative scheme $f_0 = 0$, $f_{n+1} = g - Q(f_n)$.

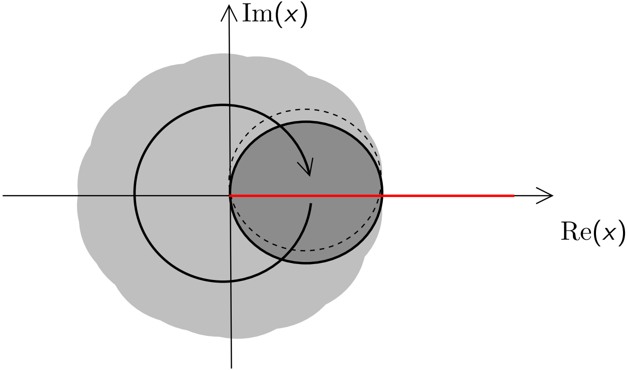
The center manifold

Going back to the x-variable, one gets existence of solution in a sectorial region U of opening 3π



The center manifold

Going back to the x-variable, one gets existence of solution in a sectorial region U of opening 3π



Notice however that the solution is multivalued on $U \cap \{\mathrm{Re}(x) \geq 0\}$.

Going to the normal form...

Once one knows how to invert $\left\{\mathbb{1} - x^{r+1}\frac{d}{dx}\right\}$ in a bounded way, we can obtain a conjugation from

$$\partial = x^{r+1}\frac{\partial}{\partial x} + \left(y(1+\varepsilon) - g\right)\frac{\partial}{\partial y}$$

Going to the normal form...

Once one knows how to invert $\left\{\mathbb{1} - x^{r+1}\frac{d}{dx}\right\}$ in a bounded way, we can obtain a conjugation from

$$\partial = x^{r+1}\frac{\partial}{\partial x} + (y(1+\varepsilon) - g)\frac{\partial}{\partial y}$$

to its normal form

$$\frac{x^r}{1+\nu x^r} x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

Going to the normal form...

Once one knows how to invert $\left\{\mathbb{1} - x^{r+1}\frac{d}{dx}\right\}$ in a bounded way, we can obtain a conjugation from

$$\partial = x^{r+1}\frac{\partial}{\partial x} + \left(y(1+\varepsilon) - g\right)\frac{\partial}{\partial y}$$

to its normal form

$$\frac{x^r}{1+\nu x^r}x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

By successively inverting operators of type

$$\left\{\chi\mathbb{1} - x^{r+1}\frac{d}{dx}\right\}$$

Going to the normal form...

Once one knows how to invert $\left\{\mathbb{1} - x^{r+1}\frac{d}{dx}\right\}$ in a bounded way, we can obtain a conjugation from

$$\partial = x^{r+1}\frac{\partial}{\partial x} + \left(y(1+\varepsilon) - g\right)\frac{\partial}{\partial y}$$

to its normal form

$$\frac{x^r}{1+\nu x^r} x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

By successively inverting operators of type

$$\left\{\chi\mathbb{1} - x^{r+1}\frac{d}{dx}\right\} \text{ for } \chi \in \mathbb{Z}^*_{\leq 1}$$

Going to the normal form...

Once one knows how to invert $\left\{\mathbb{1} - x^{r+1}\frac{d}{dx}\right\}$ in a bounded way, we can obtain a conjugation from

$$\partial = x^{r+1}\frac{\partial}{\partial x} + (y(1+\varepsilon) - g)\frac{\partial}{\partial y}$$

to its normal form

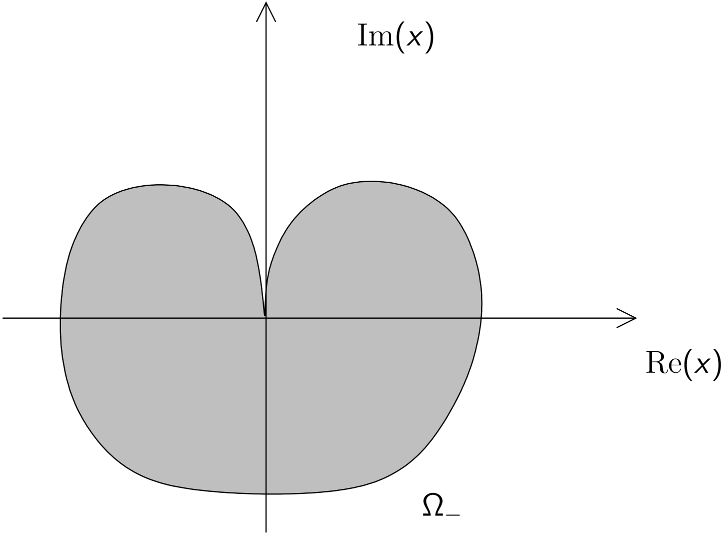
$$\frac{x^r}{1+\nu x^r} x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

By successively inverting operators of type

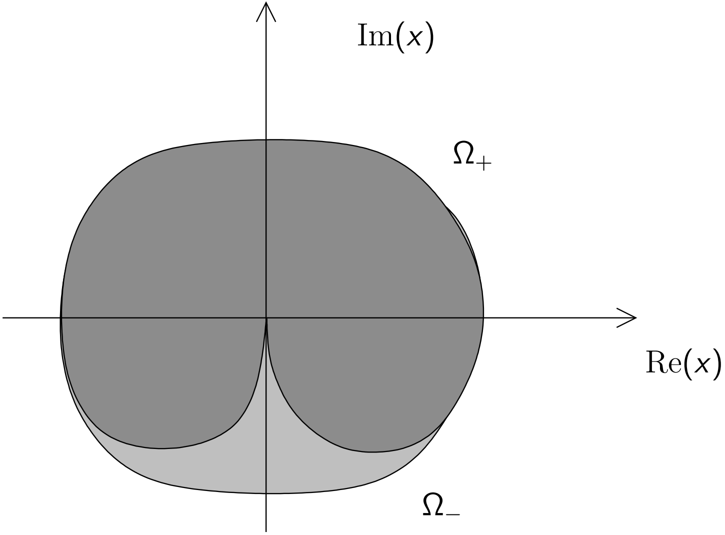
$$\left\{\chi\mathbb{1} - x^{r+1}\frac{d}{dx}\right\} \text{ for } \chi \in \mathbb{Z}^*_{\leq 1}$$

Notice that the branch *changes side* for $\chi < 0$.

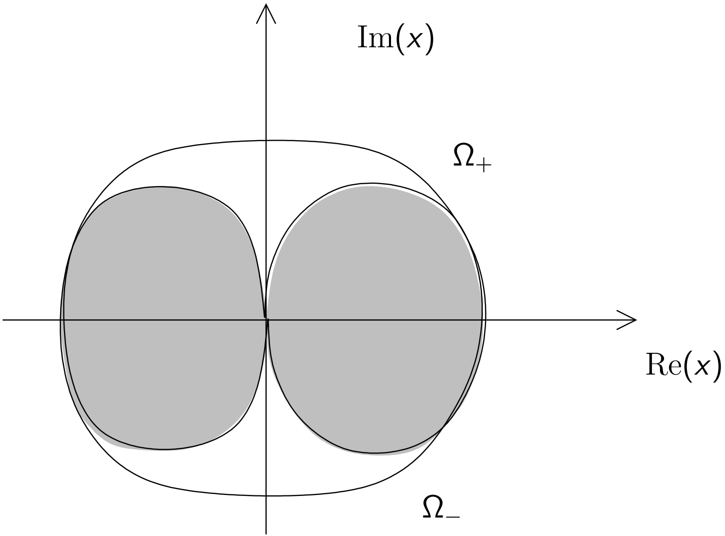
Domains of existence of the normal form



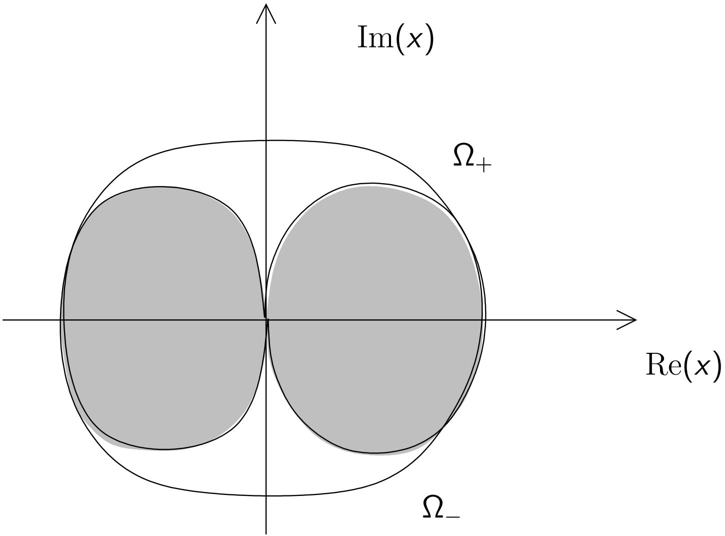
Domains of existence of the normal form



Domains of existence of the normal form



Domains of existence of the normal form



General philosophy: The invariant curves are the organizing centers of the dynamics (Thom).

Center manifolds in dimension three

Center manifolds in dimension three

We consider the following typical situation

$$\partial = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \big(z(1+\varepsilon) - g\big)\frac{\partial}{\partial z}, \qquad a, b \in \mathbb{C}\{x, y, z\} \cap \mathbf{m}^2,$$

$$g \in \mathbb{C}\{x, y\} \cap \mathbf{m}^2$$

Center manifolds in dimension three

We consider the following typical situation

$$\partial = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \big(z(1+\varepsilon) - g\big)\frac{\partial}{\partial z}, \qquad a, b \in \mathbb{C}\{x, y, z\} \cap \mathbf{m}^2,$$

$$g \in \mathbb{C}\{x, y\} \cap \mathbf{m}^2$$

and look for a center manifold $S = \{z = f(x, y)\}$.

Center manifolds in dimension three

We consider the following typical situation

$$\partial = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \big(z(1+\varepsilon) - g\big)\frac{\partial}{\partial z}, \qquad a, b \in \mathbb{C}\{x, y, z\} \cap \mathbf{m}^2,$$
$$g \in \mathbb{C}\{x, y\} \cap \mathbf{m}^2$$

and look for a center manifold $S = \{z = f(x, y)\}$.
This gives the PDE

$$\Big\{(1+\varepsilon)\mathbb{1} - \big(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\big)\Big\}(f) = g$$

Center manifolds in dimension three

We consider the following typical situation

$$\partial = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \big(z(1+\varepsilon) - g\big)\frac{\partial}{\partial z}, \qquad a, b \in \mathbb{C}\{x, y, z\} \cap \mathbf{m}^2,$$

$$g \in \mathbb{C}\{x, y\} \cap \mathbf{m}^2$$

and look for a center manifold $S = \{z = f(x, y)\}$.
This gives the PDE

$$\left\{(1+\varepsilon)\mathbb{1} - \big(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\big)\right\}(f) = g$$

notice that this is highly nonlinear, as ε usually depend on the unknown f.

Center manifolds in dimension three

We consider the following typical situation

$$\partial = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \big(z(1+\varepsilon) - g\big)\frac{\partial}{\partial z}, \qquad a, b \in \mathbb{C}\{x, y, z\} \cap \mathbf{m}^2,$$

$$g \in \mathbb{C}\{x, y\} \cap \mathbf{m}^2$$

and look for a center manifold $S = \{z = f(x, y)\}$.
This gives the PDE

$$\Big\{(1+\varepsilon)\mathbb{1} - \big(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\big)\Big\}(f) = g$$

notice that this is highly nonlinear, as ε, and a, b usually depend on the unknown f.

New difficulties

New difficulties

- Find a right inverse K for the *linearized PDE* operator

$$\left\{(1+\varepsilon(x,y,0))\mathbb{1} - \big(a(x,y,0)\frac{\partial}{\partial x} + b(x,y,0)\frac{\partial}{\partial y}\big)\right\}(f) = g$$

integrating *along paths*

New difficulties

- Find a right inverse K for the *linearized PDE* operator

$$\left\{(1+\varepsilon(x,y,0))\mathbb{1} - \left(a(x,y,0)\frac{\partial}{\partial x} + b(x,y,0)\frac{\partial}{\partial y}\right)\right\}(f) = g$$

integrating *along paths* (*method of characteristics*).

New difficulties

- Find a right inverse K for the *linearized PDE* operator

$$\left\{(1+\varepsilon(x,y,0))\mathbb{1} - \left(a(x,y,0)\frac{\partial}{\partial x} + b(x,y,0)\frac{\partial}{\partial y}\right)\right\}(f) = g$$

  integrating *along paths* (*method of characteristics*).
- Apply an iterative procedure to solve the nonlinear case.

New difficulties

- Find a right inverse K for the *linearized PDE* operator

$$\left\{(1+\varepsilon(x,y,0))\mathbb{1} - \left(a(x,y,0)\frac{\partial}{\partial x} + b(x,y,0)\frac{\partial}{\partial y}\right)\right\}(f) = g$$

  integrating *along paths* (*method of characteristics*).
- Apply an iterative procedure to solve the nonlinear case. Writing

$$\left\{(1+\varepsilon)\mathbb{1} - \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\right\} K = \mathbb{1} + Q$$

New difficulties

- Find a right inverse K for the *linearized PDE* operator

$$\left\{(1+\varepsilon(x,y,0))\mathbb{1} - \left(a(x,y,0)\frac{\partial}{\partial x} + b(x,y,0)\frac{\partial}{\partial y}\right)\right\}(f) = g$$

  integrating *along paths* (*method of characteristics*).

- Apply an iterative procedure to solve the nonlinear case. Writing

$$\left\{(1+\varepsilon)\mathbb{1} - \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\right\}K = \mathbb{1} + Q$$

  Notice that Q is now a *differential operator*

New difficulties

- Find a right inverse K for the *linearized PDE* operator

$$\left\{(1+\varepsilon(x,y,0))\mathbb{1} - \left(a(x,y,0)\frac{\partial}{\partial x} + b(x,y,0)\frac{\partial}{\partial y}\right)\right\}(f) = g$$

  integrating *along paths* (*method of characteristics*).
- Apply an iterative procedure to solve the nonlinear case. Writing

$$\left\{(1+\varepsilon)\mathbb{1} - \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\right\}K = \mathbb{1} + Q$$

  Notice that Q is now a *differential operator*

  how can Q be small?

New difficulties

- Find a right inverse K for the *linearized PDE* operator

$$\left\{(1+\varepsilon(x,y,0))\mathbb{1} - \left(a(x,y,0)\frac{\partial}{\partial x} + b(x,y,0)\frac{\partial}{\partial y}\right)\right\}(f) = g$$

  integrating *along paths* (*method of characteristics*).
- Apply an iterative procedure to solve the nonlinear case. Writing

$$\left\{(1+\varepsilon)\mathbb{1} - \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\right\}K = \mathbb{1} + Q$$

  Notice that Q is now a *differential operator*

  how can Q be small?

  Can we apply the iterative procedure $f_{n+1} = g - Q(f_n)$?

New difficulties

- Find a right inverse K for the *linearized PDE* operator

$$\left\{(1+\varepsilon(x,y,0))\mathbb{1} - \big(a(x,y,0)\frac{\partial}{\partial x} + b(x,y,0)\frac{\partial}{\partial y}\big)\right\}(f) = g$$

integrating *along paths* (*method of characteristics*).

- Apply an iterative procedure to solve the nonlinear case. Writing

$$\left\{(1+\varepsilon)\mathbb{1} - \big(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\big)\right\}K = \mathbb{1} + Q$$

Notice that Q is now a *differential operator*

how can Q be small?

Can we apply the iterative procedure $f_{n+1} = g - Q(f_n)$?

An implicit function theorem (Mcquillan)

An implicit function theorem (Mcquillan)

- $U = P_1 \times \cdots \times P_n$ product domain in $\mathbb{C}^n$

An implicit function theorem (Mcquillan)

- $U = P_1 \times \cdots \times P_n$ product domain in $\mathbb{C}^n$
- (basic estimate) For $\partial_i = \frac{\partial}{\partial x_i}$,

An implicit function theorem (Mcquillan)

- $U = P_1 \times \cdots \times P_n$ product domain in $\mathbb{C}^n$
- (basic estimate) For $\partial_i = \frac{\partial}{\partial x_i}$,

$$\left|\partial_i f(p)\right| \leq \frac{C}{\mathrm{dist}(p_i, \partial P_i)} \|f\|_U$$

for some constant C depending only on U.

An implicit function theorem (Mcquillan)

- $U = P_1 \times \cdots \times P_n$ product domain in $\mathbb{C}^n$
- (basic estimate) For $\partial_i = \frac{\partial}{\partial x_i}$,

$$\left|\partial_i f(p)\right| \leq \frac{C}{\mathrm{dist}(p_i, \partial P_i)} \|f\|_U$$

  for some constant C depending only on U.
- More generally, for $a = (a_1, \ldots, a_n)$,

An implicit function theorem (Mcquillan)

- $U = P_1 \times \cdots \times P_n$ product domain in $\mathbb{C}^n$
- (basic estimate) For $\partial_i = \frac{\partial}{\partial x_i}$,

$$\left|\partial_i f(p)\right| \leq \frac{C}{\mathrm{dist}(p_i, \partial P_i)} \|f\|_U$$

for some constant C depending only on U.

- More generally, for $a = (a_1, \ldots, a_n)$,

$$\left\|\partial_1^{a_1} \ldots \partial_n^{a_n} f\right\|_{U(\underline{d})} \leq \frac{a_1! \cdots a_n!}{d_1^{a_1} \cdots d_n^{a_n}} C^{|a|} \|f\|_U$$

for some constant C depending only on U and n.

An implicit function theorem (Mcquillan)

- $U = P_1 \times \cdots \times P_n$ product domain in $\mathbb{C}^n$
- (basic estimate) For $\partial_i = \frac{\partial}{\partial x_i}$,

$$\left|\partial_i f(p)\right| \leq \frac{C}{\mathrm{dist}(p_i, \partial P_i)} \|f\|_U$$

for some constant C depending only on U.

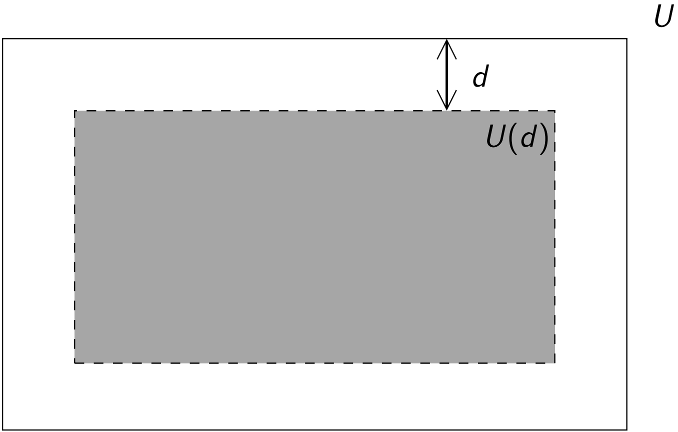
- More generally, for $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$,

$$\left\|\partial_1^{a_1} \ldots \partial_n^{a_n} f\right\|_{U(\underline{d})} \leq \frac{a_1! \cdots a_n!}{d_1^{a_1} \cdots d_n^{a_n}} C^{|a|} \|f\|_U$$

for some constant C depending only on U and n.

An implicit function theorem (Mcquillan)

An implicit function theorem (Mcquillan)



An implicit function theorem (Mcquillan)

We consider the Fréchet space $\mathcal{O}(U)$ with the family of seminorms

$$d \geq 0 \;\Rightarrow\; \left\|f\right\|_{U(d)} = \sup_{x \in U(d)} |f(x)|,$$

An implicit function theorem (Mcquillan)

We consider the Fréchet space $\mathcal{O}(U)$ with the family of seminorms

$$d \geq 0 \;\Rightarrow\; \left\| f \right\|_{U(d)} = \sup_{x \in U(d)} |f(x)|,$$

An implicit function theorem (Mcquillan)

We consider the Fréchet space $\mathcal{O}(U)$ with the family of seminorms

$$d \geq 0 \;\Rightarrow\; \left\| f \right\|_{U(d)} = \sup_{x \in U(d)} |f(x)|,$$

$$\left\| f \right\|_{U(d)} \leq \left\| f \right\|_{U(e)}, \quad \text{for } d \geq e$$

An implicit function theorem (Mcquillan)

We consider the Fréchet space $\mathcal{O}(U)$ with the family of seminorms

$$d \geq 0 \ \Rightarrow \ \left\|f\right\|_{U(d)} = \sup_{x \in U(d)} |f(x)|,$$

$$\left\|f\right\|_{U(d)} \leq \left\|f\right\|_{U(e)}, \quad \text{for } d \geq e$$

We are going to need a IFT in scales of seminorms

An implicit function theorem (Mcquillan)

We consider the Fréchet space $\mathcal{O}(U)$ with the family of seminorms

$$d \geq 0 \;\Rightarrow\; \left\|f\right\|_{U(d)} = \sup_{x \in U(d)} |f(x)|,$$

$$\left\|f\right\|_{U(d)} \leq \left\|f\right\|_{U(e)}\,, \quad \text{for } d \geq e$$

We are going to need a IFT in scales of seminorms

(similar to *Nash-Moser's).*

An implicit function theorem (Mcquillan)

Coming back to our differential operator:

$$P = \left\{(1+\varepsilon)\mathbb{1} - \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\right\}$$

An implicit function theorem (Mcquillan)

Coming back to our differential operator:

$$P = \left\{ (1+\varepsilon)\mathbb{1} - \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) \right\} = P'(0) + \text{ higher order terms}$$

An implicit function theorem (Mcquillan)

Coming back to our differential operator:

$$P = \left\{ (1+\varepsilon)\mathbb{1} - \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \right) \right\} = P'(0) + \text{ higher order terms}$$

An implicit function theorem (Mcquillan)

Coming back to our differential operator:

$$P = \left\{ (1+\varepsilon)\mathbb{1} - \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) \right\} = P'(0) + \text{ higher order terms}$$

- We construct (see later!) an operator K such that $P'(0)K = \mathbb{1}$ and

An implicit function theorem (Mcquillan)

Coming back to our differential operator:

$$P = \left\{ (1+\varepsilon)\mathbb{1} - \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) \right\} = P'(0) + \text{ higher order terms}$$

- We construct (see later!) an operator K such that $P'(0)K = \mathbb{1}$ and

$$\left\| Kh \right\|_{U(d)} \leq \left\| h \right\|_{U(e)} \psi(d-e), \qquad \text{for } d \geq e$$

for $\psi : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>1}$ decreasing.

An implicit function theorem (Mcquillan)

Coming back to our differential operator:

$$P = \left\{(1+\varepsilon)\mathbb{1} - \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\right\} = P'(0) + \text{ higher order terms}$$

- We construct (see later!) an operator K such that $P'(0)K = \mathbb{1}$ and

$$\left\|Kh\right\|_{U(d)} \leq \left\|h\right\|_{U(e)} \psi(d-e), \qquad \text{for } d \geq e$$

  for $\psi : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>1}$ decreasing.
- It can be shown that

$$\left\|Ph - P'(0)h\right\|_{U(d)} \leq \left\|h\right\|^{1+\alpha}_{U(e)} \phi(d-e), \quad \text{for } d \geq e$$

  for $\alpha > 0$ and $\phi : \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>1}$ decreasing.

An implicit function theorem (Mcquillan)

Put $PK = \mathbb{1} + Q$,

An implicit function theorem (Mcquillan)

Put $PK = \mathbb{1} + Q$, then for section $h \in \mathcal{O}(U)$,

$$\left\|Qh\right\|_{U(d)} \leq \left\|h\right\|_{U(e)}^{1+\alpha} \theta(d-e), \quad \text{for } d \geq e$$

we want to find a solution of $P(f) = g$ by the iteration

$$f_{n+1} = g - Q(f_n), \qquad f_0 = 0$$

An implicit function theorem (Mcquillan)

Put $PK = \mathbb{1} + Q$, then for section $h \in \mathcal{O}(U)$,

$$\left\|Qh\right\|_{U(d)} \leq \left\|h\right\|_{U(e)}^{1+\alpha}\, \theta(d-e), \quad \text{for } d \geq e$$

we want to find a solution of $P(f) = g$ by the iteration

$$f_{n+1} = g - Q(f_n), \qquad f_0 = 0$$

This is possible provided that $\|g\|_U$ is sufficiently small and ...

An implicit function theorem (Mcquillan)

Put $PK = \mathbb{1} + Q$, then for section $h \in \mathcal{O}(U)$,

$$\left\|Qh\right\|_{U(d)} \leq \left\|h\right\|_{U(e)}^{1+\alpha} \theta(d-e), \quad \text{for } d \geq e$$

we want to find a solution of $P(f) = g$ by the iteration

$$f_{n+1} = g - Q(f_n), \qquad f_0 = 0$$

This is possible provided that $\|g\|_U$ is sufficiently small and ...

An implicit function theorem (Mcquillan)

... the logarithm of the function

$$t \mapsto \theta(td) = \phi(td)\psi(td)^{1+\alpha}$$

is absolutely integrable at $t = 0$.

An implicit function theorem (Mcquillan)

... the logarithm of the function

$$t \mapsto \theta(td) = \phi(td)\psi(td)^{1+\alpha}$$

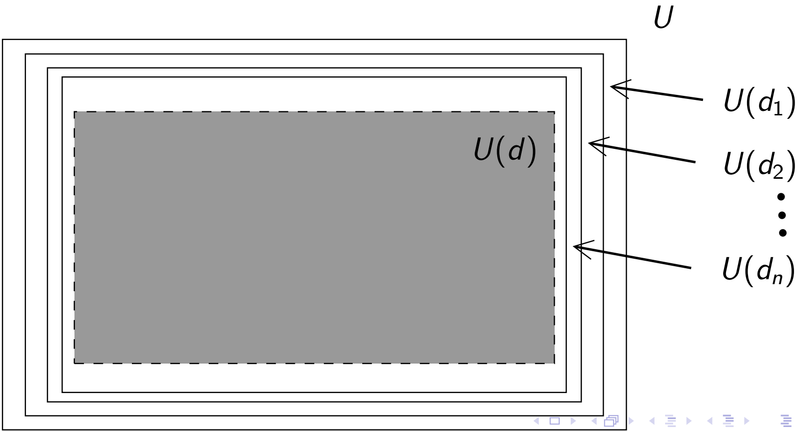
is absolutely integrable at $t = 0$. We estimate $\left\| Q^n f_0 \right\|_{U(d)}$ by *interpolating* n domains $U(d_1), \ldots, U(d_n)$.

An implicit function theorem (Mcquillan)

... the logarithm of the function

$$t \mapsto \theta(td) = \phi(td)\psi(td)^{1+\alpha}$$

is absolutely integrable at $t = 0$. We estimate $\left\| Q^n f_0 \right\|_{U(d)}$ by *interpolating* n domains $U(d_1), \ldots, U(d_n)$.



Solving the linear PDE $P'(0)$

Recall that

Solving the linear PDE $P'(0)$

Recall that

$$P'(0) = \left\{ (1+\varepsilon) - \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) \right\}, \qquad a, b, \varepsilon \in \mathbb{C}\{x, y\} \cap \mathbf{m}^2$$

and $D = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)$ is an arbitrary vector field in dimension two!

Solving the linear PDE $P'(0)$

Recall that

$$P'(0) = \left\{ (1+\varepsilon) - \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) \right\}, \qquad a, b, \varepsilon \in \mathbb{C}\{x, y\} \cap \mathbf{m}^2$$

and $D = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)$ is an arbitrary vector field in dimension two!

The paths which define K should be contained in the leafs of the associated foliation $\mathcal{L}_D$ in $\mathbb{C}^2$.

Solving the linear PDE $P'(0)$

Recall that

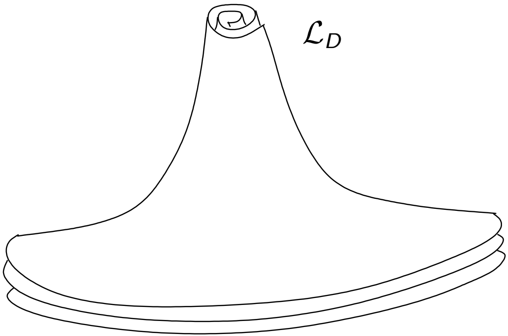
$$P'(0) = \left\{ (1+\varepsilon) - \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) \right\}, \qquad a, b, \varepsilon \in \mathbb{C}\{x, y\} \cap \mathbf{m}^2$$

and $D = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)$ is an arbitrary vector field in dimension two!

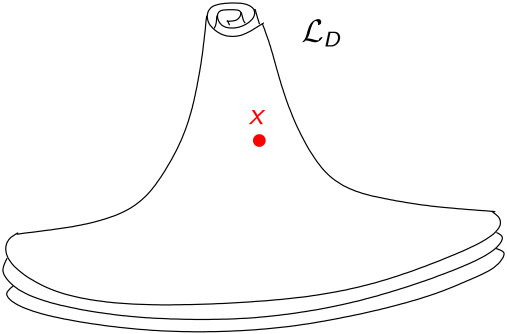
The paths which define K should be contained in the leafs of the associated foliation $\mathcal{L}_D$ in $\mathbb{C}^2$. Thus...

We must understand $\mathcal{L}_D$

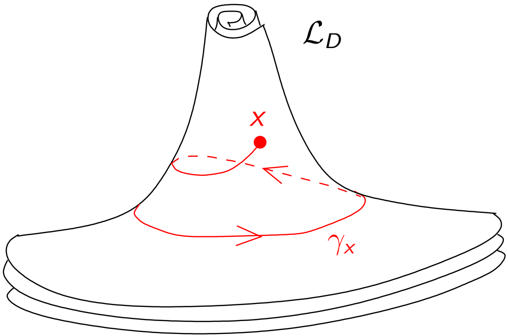
Solving the linear PDE $P'(0)$



Solving the linear PDE $P'(0)$



Solving the linear PDE $P'(0)$



Going further in resolution...

Up to an additional finite number of (quadratic) blowing-ups, we can suppose that

$$P'(0) = \{(1+\varepsilon) - D\},$$

is such that

Going further in resolution...

Up to an additional finite number of (quadratic) blowing-ups, we can suppose that

$$P'(0) = \{(1+\varepsilon) - D\} \, ,$$

is such that

$$D = x^p y^q \left(f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} \right)$$

Going further in resolution...

Up to an additional finite number of (quadratic) blowing-ups, we can suppose that

$$P'(0) = \{(1+\varepsilon) - D\},$$

is such that

$$D = x^p y^q \left(f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} \right)$$

Going further in resolution...

Up to a finite number of (quadratic) blowing-ups, we can suppose that

$$P'(0) = \{(1+\varepsilon)\mathbb{1} - D\},$$

is such that

$$D = \underbrace{x^p y^q}_{\text{monomial}} \underbrace{\left(f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} \right)}_{\text{Elementary}}$$

Going further in resolution...

Up to a finite number of (quadratic) blowing-ups, we can suppose that

$$P'(0) = \{(1+\varepsilon)\mathbb{1} - D\},$$

is such that

$$D = \underbrace{x^p y^q}_{\text{monomial}} \underbrace{\left(f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} \right)}_{\text{Elementary}}$$

Then we can hope to find the right paths to define the operator K.

Going further in resolution...

An example of integrable case:

$$P'(0) = \left\{ \mathbb{1} + x^p \, \frac{\partial}{\partial y} \right\},$$

Going further in resolution...

An example of integrable case:

$$P'(0) = \left\{ \mathbb{1} + x^p \, \frac{\partial}{\partial y} \right\},$$

We consider the new coordinates

$$s = x, \qquad \xi = y x^{-p}$$

Going further in resolution...

An example of integrable case:

$$P'(0) = \left\{ \mathbb{1} + x^p \, \frac{\partial}{\partial y} \right\},$$

We consider the new coordinates

$$s = x, \qquad \xi = yx^{-p}$$

In the variables (s, ξ), the PDE assumes the form

$$P'(0) = \left\{ \mathbb{1} + \frac{\partial}{\partial \xi} \right\}$$

An integrable case

$$\left\{ \mathbb{1} + \frac{\partial}{\partial \xi} \right\} (f) = g$$

An integrable case

$$\left\{ \mathbb{1} + \frac{\partial}{\partial \xi} \right\} (f) = g$$

and the right inverse is

$$f = (Kg)(s, \xi) = e^{-\xi} \int_{\text{base}}^{\xi} e^{\rho} g(s, \rho) d\rho$$

An integrable case

$$\left\{ \mathbb{1} + \frac{\partial}{\partial \xi} \right\} (f) = g$$

and the right inverse is

$$f = (Kg)(s, \xi) = e^{-\xi} \int_{\text{base}}^{\xi} e^{\rho} g(s, \rho) d\rho$$

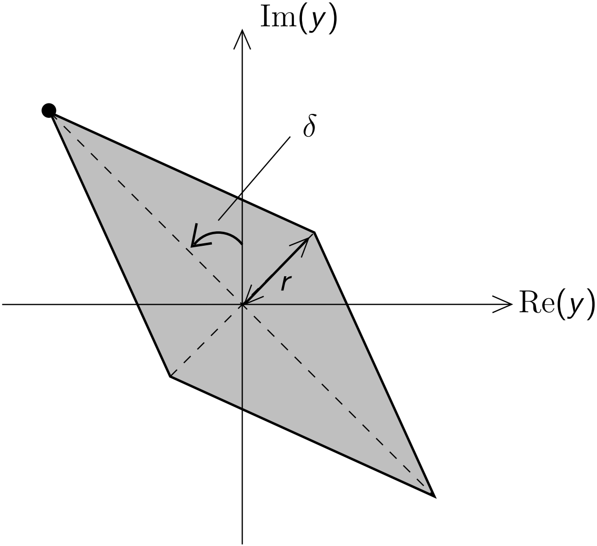
where the base point and the domain of existence is...

An integrable case

For $x \in S$ (sector of opening at most π/p), we consider the domain

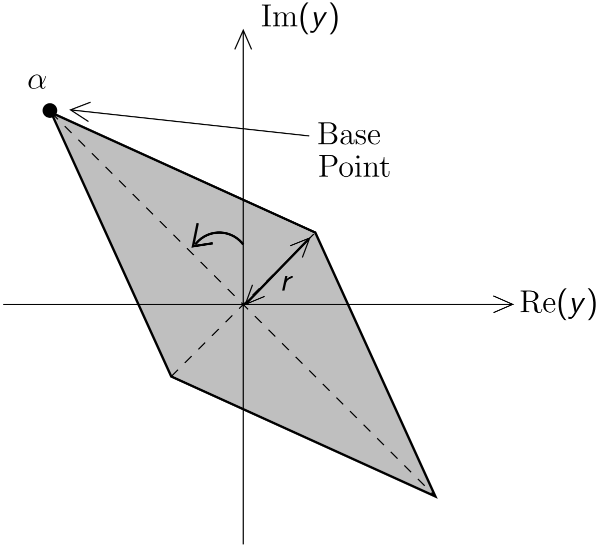
An integrable case

For $x \in S$ (sector of opening at most π/p), we consider the domain



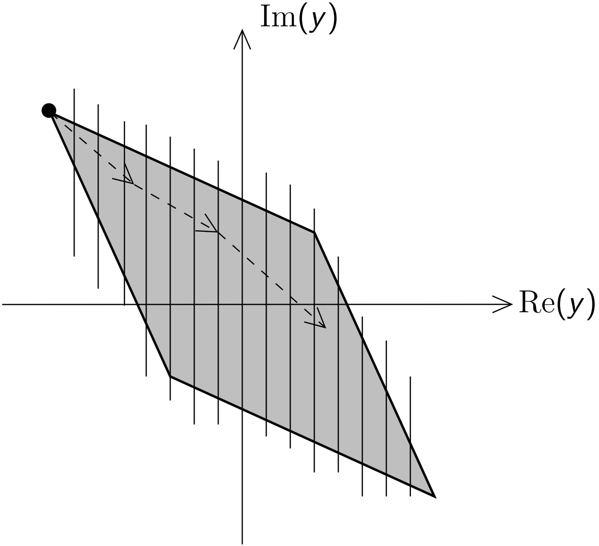
An integrable case

For $x \in S$ (sector of opening at most π/p), we consider the domain



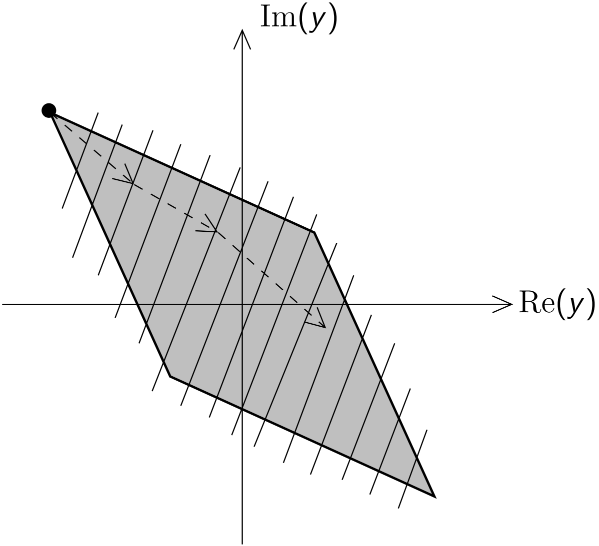
An integrable case

For $x \in S$ (sector of opening at most π/p), we consider the domain



An integrable case

For $x \in S$ (sector of opening at most π/p), we consider the domain



The *hardest case*

The *hardest case*

The saddle-node

$$P'(0) = x^p \left(y \frac{\partial}{\partial y} + x^{r+1} \frac{\partial}{\partial x} + \cdots \right), \quad r \geq 1$$

The *hardest case*

The saddle-node

$$P'(0) = x^p \left(y \frac{\partial}{\partial y} + x^{r+1} \frac{\partial}{\partial x} + \cdots \right), \quad r \geq 1$$

We can obtain the sectorial normal form

$$P'(0) = x^p R(x) y \frac{\partial}{\partial y} + \frac{x^{p+r+1}}{1 + \nu x^{p+r}} \frac{\partial}{\partial x},$$

$\nu \in \mathbb{C}$, $R(x)$ polynomial of degree smaller than r, $R(0) = 1$.

The *hardest case*

For $R = 1$, one gets

$$\left(\mathbb{1} - x^p \Big(\frac{x^{r+1}}{q}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Big)\right) f = x^p g \tag{1}$$

The *hardest case*

For $R = 1$, one gets

$$\left(\mathbb{1} - x^p\big(\frac{x^{r+1}}{q}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\big)\right) f = x^p g \qquad (1)$$

We put $\xi = x^{-r}$ and obtain

$$\left(\mathbb{1} + \xi^{-\alpha}\big(\frac{\partial}{\partial \xi} - y\frac{\partial}{\partial y}\big)\right) f = \xi^{-\alpha} g, \qquad \text{where } \alpha = p/r > 0$$

The *hardest case*

For $R = 1$, one gets

$$\left(\mathbb{1} - x^p\left(\frac{x^{r+1}}{q}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\right) f = x^p g \qquad (1)$$

We put $\xi = x^{-r}$ and obtain

$$\left(\mathbb{1} + \xi^{-\alpha}\left(\frac{\partial}{\partial \xi} - y\frac{\partial}{\partial y}\right)\right) f = \xi^{-\alpha} g, \qquad \text{where } \alpha = p/r > 0$$

We expand $f(\xi, y)$ and $g(\xi, y)$ in powers of y

$$f(\xi, y) = \sum f_k(\xi) y^k, \qquad g(\xi, y) = \sum g_k(\xi) y^k$$

and solve separately each differential equation for f_k.

The *hardest case*

For $R = 1$, one gets

$$\left(\mathbb{1} - x^p\left(\frac{x^{r+1}}{q}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\right) f = x^p g \tag{1}$$

We put $\xi = x^{-r}$ and obtain

$$\left(\mathbb{1} + \xi^{-\alpha}\left(\frac{\partial}{\partial \xi} - y\frac{\partial}{\partial y}\right)\right) f = \xi^{-\alpha} g, \qquad \text{where } \alpha = p/r > 0$$

Plugging into the differential equation, we obtain

$$\frac{d}{d\xi} f_k = (k - \xi^{\alpha}) f_k + g_k$$

whose general solution (vanishing at $\xi = \xi_0$) is

$$f_k(\xi) = \int_{\xi_0}^{\xi} \exp\left(k(\xi - s) - \frac{\xi^{\alpha+1} - s^{\alpha+1}}{\alpha + 1}\right) g_k(s) ds$$

$$\phi(x) = x^{\alpha+1}/(\alpha+1) - x$$

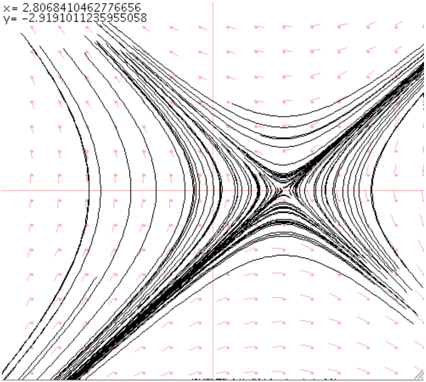


Figure: Level curves of $\mathrm{Re}\,\phi(x)$ for $\alpha = 1$ (the saddle is at $x = 1$)

$$\phi(x) = x^{\alpha+1}/(\alpha+1) - x$$

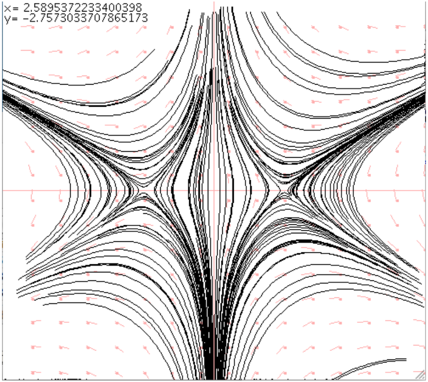


Figure: Level curves of $\mathrm{Re}\ \phi(x)$ for $\alpha = 2$ (there are two saddles saddles)

$$\phi(x) = x^{\alpha+1}/(\alpha+1) - x$$

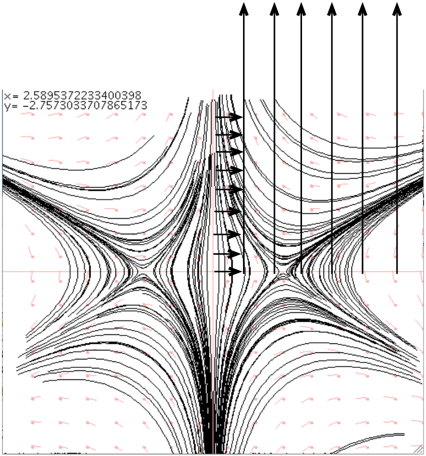


Figure: Integration paths for $\alpha = 2$

The main result

Theorem

Let ∂ be a germ of vector field of the form

$$\partial = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \big(z(1+\varepsilon) - g\big)\frac{\partial}{\partial z}$$

such that

The main result

Theorem

Let ∂ be a germ of vector field of the form

$$\partial = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \big(z(1+\varepsilon) - g\big)\frac{\partial}{\partial z}$$

such that

$$a(x,y,0)\frac{\partial}{\partial x} + b(x,y,0)\frac{\partial}{\partial y} = x^p y^q \Big(\text{ elementary sing. of dim 2 } \Big)$$

The main result

Theorem

Let ∂ be a germ of vector field of the form

$$\partial = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \big(z(1+\varepsilon) - g\big)\frac{\partial}{\partial z}$$

such that

$$a(x,y,0)\frac{\partial}{\partial x} + b(x,y,0)\frac{\partial}{\partial y} = x^p y^q \Big(\text{elementary sing. of dim 2} \Big)$$

Then there exists open sets U_i (for $i = 1, \dots, n$) which cover a full neighborhood of 0 in $\mathbb{C}^2 \setminus \{0\}$, and holomorphic functions

$$f_i \in \mathcal{O}(U_i)$$

The main result

Theorem

Let ∂ be a germ of vector field of the form

$$\partial = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \big(z(1+\varepsilon) - g\big)\frac{\partial}{\partial z}$$

such that

$$a(x,y,0)\frac{\partial}{\partial x} + b(x,y,0)\frac{\partial}{\partial y} = x^p y^q \Big(\text{ elementary sing. of dim 2 } \Big)$$

Then there exists open sets U_i (for $i = 1, \ldots, n$) which cover a full neighborhood of 0 in $\mathbb{C}^2 \setminus \{0\}$, and holomorphic functions

$$f_i \in \mathcal{O}(U_i) \quad \text{bounded at the origin}$$

such that

$$\partial\big(z - f_i(x,y)\big) \in \big\langle z - f_i(x,y)\big\rangle$$

Spiralling Sectors

At some situations, e.g.

$$P'(0) = x^p \left(x \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y} \right)$$

Spiralling Sectors

At some situations, e.g.

$$P'(0) = x^p \left(x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y} \right) \quad p \in \mathbb{N}, \mathrm{Re}(\lambda) < 0, \lambda \notin \mathbb{R}$$

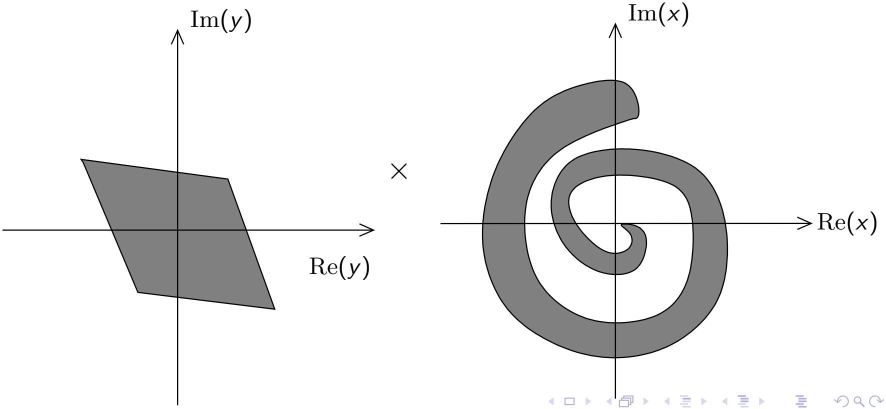
these regions U_i can be given by product domains like

Spiralling Sectors

At some situations, e.g.

$$P'(0) = x^p \left(x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y} \right) \quad p \in \mathbb{N}, \mathrm{Re}(\lambda) < 0, \lambda \notin \mathbb{R}$$

these regions U_i can be given by product domains like



The next step - Analytic NF in dim 3

Normal form for strongly elementary vector fields in dim. 3,

$$\partial = z\frac{\partial}{\partial z} + D$$

where

$$D = x^p y^q \partial_1$$

with ∂_1 an elementary vector field of dimension 2.

The next step - Analytic NF in dim 3

Normal form for strongly elementary vector fields in dim. 3,

$$\partial = z\frac{\partial}{\partial z} + D$$

where

$$D = x^p y^q \partial_1$$

with ∂_1 an elementary vector field of dimension 2.

This is a new problem, even in the formal setting!

The next step - Analytic NF in dim 3

Normal form for strongly elementary vector fields in dim. 3,

$$\partial = z\frac{\partial}{\partial z} + D$$

where

$$D = x^p y^q \partial_1$$

with ∂_1 an elementary vector field of dimension 2.

This is a new problem, even in the formal setting!

Apparently, we are always lead to solve PDE's of type

$$\left\{(1+\varepsilon)\mathbb{1} - \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)\right\}(f) = g$$