Kirchberg X-algebras with real rank zero and intermediate cancellation

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Motivation: Classification of Kirchberg algebras

Theorem (Kirchberg–Phillips 2000)

A KK*-equivalence between two stable Kirchberg algebras lifts to a* **-isomorphism.*

(Kirchberg algebra = nuclear, separable, purely infinite, simple)

Theorem (Rosenberg–Schochet 1987)

Let A and B be separable C^*-algebras. If A belongs to the bootstrap class $\mathcal{B}$, then there is a short exact sequence of $\mathbb{Z}/2$-graded Abelian groups

$$\mathrm{Ext}^1\big(\mathrm{K}_{*+1}(A), \mathrm{K}_*(B)\big) \rightarrowtail \mathrm{KK}_*(A, B) \twoheadrightarrow \mathrm{Hom}\big(\mathrm{K}_*(A), \mathrm{K}_*(B)\big).$$

Corollary

K_* *strongly classifies stable Kirchberg algebras in* $\mathcal{B}$.

Throughout this talk, X denotes an arbitrary finite T_0-space.

Definition

A Kirchberg X-algebra is a separable, nuclear, purely infinite C^*-algebra with a fixed homeomorphism $\mathrm{Prim}(A) \approx X$.

- An open subset $U \subseteq X$ gives an ideal $A(U)$ of A.
- A *-homomorphism $f\colon A \to B$ is over X if $f(A(U)) \subseteq B(U)$ for all $U \in \mathbb{O}(X)$.

We have bijections:

$\{T_0\text{-topologies on } X\}$

Alexandrov topology ↑ ↓ specialisation preorder

$\{\text{partial orders on } X\}$

reachability relation ↑ ↓ Hasse diagram

$\{\text{transitively reduced directed acyclic graphs with vertex set } X\}$

Kirchberg's theorem

Kirchberg constructed an X-equivariant version $\mathrm{KK}(X)$ of Kasparov's KK-theory.

Theorem (Kirchberg)

A $\mathrm{KK}(X)$-equivalence between two stable Kirchberg X-algebras lifts to a *-isomorphism over X.

Goal: Find an invariant that detects $\mathrm{KK}(X)$-equivalence!
(under appropriate bootstrap class assumptions)!

This problem depends heavily on the combinatorics of X.
Solutions have been established (and are expected to exist) only for certain classes of spaces.

Today, we make two extra assumptions that resolve these intricacies: real rank zero and intermediate cancellation.

Lemma (Brown–Pedersen, Lin–Rørdam)

A Kirchberg X-algebra A has real rank zero iff the boundary map $\mathrm{K}_0(J/I) \to \mathrm{K}_1(I)$ *vanishes for all ideals* $I \lhd J \lhd A$.

Definition (due to Lawrence G. Brown)

We say that A has intermediate cancellation if the following holds: if p and q are projections in A which generate the same ideal and which give rise to the same element in $\mathrm{K}_0(A)$, then $p \sim q$.

Lemma

A Kirchberg X-algebra A has intermediate cancellation iff $\mathrm{K}_1(J/I) \to \mathrm{K}_0(I)$ *vanishes for all ideals* $I \lhd J \lhd A$.

If both conditions hold, we say "A has vanishing boundary maps." This is automatic in the simple case.

Proof of Lemma

Lemma

A Kirchberg X-algebra A has intermediate cancellation iff $\mathrm{K}_1(J/I) \to \mathrm{K}_0(I)$ *vanishes for all ideals* $I \lhd J \lhd A$.

Proof

Let I be an ideal in A.
By Pasnicu–Rørdam, I contains a full projection.
By Rørdam (Cuntz),

$$\mathrm{K}_0(I) = \{[p] \mid p \text{ is a full projection in } I\}$$

and, if p and q are full projections in I with $[p] = [q]$ in $\mathrm{K}_0(I)$, then $p \sim q$.
The claim becomes apparent (from the six-term exact sequence).

Write U_x for the minimal open neighborhood of $x \in X$.

Definition

$\mathrm{XK}(A) = \big(\mathrm{K}_*(A(U_x))\big)_{x\in X}$ + maps induced by inclusions.

$\mathrm{XK}(A)$ is a module over the integral incidence algebra $\mathbb{Z}X$ (a "representation of X in $\mathbb{Z}/2$-graded Abelian groups").

Key Lemma

If A has vanishing boundary maps, the $\mathrm{proj.dim.}\big(\mathrm{XK}(A)\big) \leq 1$.

Relative homological algebra in $\mathrm{KK}(X)$ (à la Meyer–Nest) then yields an exact universal coefficient sequence

$$\mathrm{Ext}^1\big(\mathrm{XK}(A)[1], \mathrm{XK}(B)\big) \rightarrowtail \mathrm{KK}_*(X; A, B) \twoheadrightarrow \mathrm{Hom}\big(\mathrm{XK}(A), \mathrm{XK}(B)\big).$$

Corollary (via Kirchberg)

XK *strongly classifies stable Kirchberg X-algebras in $\mathcal{B}(X)$ with real rank zero and intermediate cancellation.*

Proof of Key Lemma in a special case

Key Lemma

If A has vanishing boundary maps, the $\mathrm{proj.dim.}(\mathsf{XK}(A)) \leq 1$*.*

Proof for $X = \bullet \to \bullet$

Let $(I \lhd A)$ be a Kirchberg X-algebra.
We have $\mathsf{XK}(A) = \big(\mathsf{K}_*(I) \xrightarrow{i} \mathsf{K}_*(A)\big)$.
If A has vanishing boundary maps, then i is injective.
We decompose

$$\Big(\mathsf{K}_*(I) \to \mathsf{K}_*(I)\Big) \rightarrowtail \Big(\mathsf{K}_*(I) \rightarrowtail \mathsf{K}_*(A)\Big) \twoheadrightarrow \Big(0 \to \mathsf{K}_*(A/I)\Big).$$

By the Horseshoe Lemma, it suffices to find length-one resolution for submodule and quotient.
These can be obtained from resolutions of the groups $\mathsf{K}_*(I)$ and $\mathsf{K}_*(A/I)$. The general argument uses induction.

Classification for unital algebras

Definition

$\mathbb{O}\mathsf{K}(A) = \big(\mathsf{K}_*(A(U))\big)_{U \in \mathbb{O}(X)} \;+\;$ maps induced by inclusions.

For A unital, set $\mathbb{O}\mathsf{K}^+(A) = \big(\mathbb{O}\mathsf{K}(A), [1_A] \in \mathsf{K}_0(A)\big)$.

- $\mathbb{O}\mathsf{K}(A)$ is a precosheaf on $\mathbb{O}(X)$.
- If A has vanishing boundary maps, $\mathbb{O}\mathsf{K}(A)$ is a flabby cosheaf and is naturally determined by $\mathsf{XK}(A)$.

Corollary (via Eilers–Restorff–Ruiz's meta-theorem)

$\mathbb{O}\mathsf{K}^+$ *strongly classifies unital Kirchberg* X*-algebras in* $\mathcal{B}(X)$ *with real rank zero and intermediate cancellation.*

Theorem (via Arklint–B–Katsura)

A flabby pointed cosheaf is isomorphic to $\mathbb{O}\mathrm{K}^+(\mathcal{O}_A)$ for some Cuntz–Krieger algebra $\mathcal{O}_A$ over X with intermediate cancellation iff it has "free quotients in odd degree and finite equal ranks."

Corollary

Let $I \rightarrowtail A \twoheadrightarrow B$ be an extension with A unital. Then A is a Cuntz–Krieger algebra with intermediate cancellation iff

- *the quotient B is a Cuntz–Krieger algebra with intermediate cancellation,*
- *the ideal I is stably isomorphic to a Cuntz–Krieger algebra with intermediate cancellation,*
- *the boundary map $\mathrm{K}_*(B) \to \mathrm{K}_{*+1}(I)$ vanishes.*

There is a version for unital purely infinite graph C^*-algebras and a "stable version" for purely infinite graph C^*-algebras.

Outlook on the general case

Universal coefficient theorems for $\mathrm{KK}(X)$ for certain spaces X have been found by Bonkat, Restorff, Meyer–Nest, B–Köhler.
The methods fail for other spaces.

Conjecture

There is a UCT-invariant represented by a finite family of objects in $\mathcal{B}(X)$ iff the poset X is derived equivalent to a Dynkin quiver.

Theorem (via Schwede–Shipley in stable homotopy theory)

$$\mathcal{B}(X) \otimes \mathbb{Q} \cong \mathrm{Der}^{\mathbb{Z}/2}_{\mathrm{countable}}(\mathbb{Q}X)$$

($\mathbb{Q}X$ is the rational incidence algebra of X.)