Geometric classification of graph C^*-algebras

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Program

1. Graphs & moves
2. Classification
3. Finite graphs

Outline

1 Graphs & moves

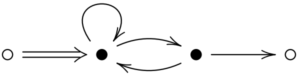
2 Classification

3 Finite graphs

Graph algebras

Any countable graph $E = (E^0, E^1)$ defines a C^*-algebra $C^*(E)$ given as a universal C^*-algebra by **projections** $\{p_v : v \in E^0\}$ and **partial isometries** $\{s_e : e \in E^1\}$ subject to the *Cuntz-Krieger relations*:

1. $p_v p_w = 0$ when $v \neq w$
2. $(s_e s_e^*)(s_f s_f^*) = 0$ when $e \neq f$
3. $s_e^* s_e = p_{r(e)}$ and $s_e s_e^* \leq p_{s(e)}$
4. $p_v = \sum_{s(e)=v} s_e s_e^*$ for every v with $0 < |\{e \mid s(e) = v\}| < \infty$.



Graph algebra need-to-know

There is a huge body of knowledge about graph algebras. Of prime importance here is

Observation

$$\gamma_z(p_v) = p_v \qquad \gamma_z(s_e) = zs_e$$

induces a **gauge action** $\mathbb{T} \mapsto \mathrm{Aut}(C^*(E))$

Theorem

Gauge invariant ideals are induced by **hereditary** *and* **saturated** *sets of vertices V:*

- $s(e) \in V \implies r(e) \in V$
- $r(s^{-1}(v)) \subseteq V \implies$ *[$v \in V$ or v is singular ($\circ$)]*

and when there are no **breaking vertices**, *all arise this way.*

The gauge simple case

Theorem

If a graph C^*-algebra has no non-trivial gauge invariant ideals, it is either

1. *an AF algebra;*
2. *a Kirchberg algebra; or*
3. $C(\mathbb{T}) \otimes \mathbb{K}(H)$ *for some Hilbert space* H.

It is easy to tell from the graph which case occurs. The first case occurs when the graph has no cycles; the second when one vertex supports several cycles.

The unital case

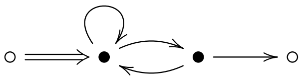
Observation

$C^*(E)$ is unital $\iff$ E_0 is finite

In this case we get a finite presentation, e.g.

$$\mathsf{A}_E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \infty & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathsf{A}_E^{\bullet} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathsf{A}_E^{\circ} = \begin{bmatrix} 0 & 0 \\ \infty & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

for



Moves

Move **(S)**

Remove a regular source, as

Move **(R)**

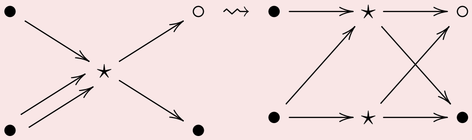
Reduce a configuration with a transitional regular vertex, as

or

Moves

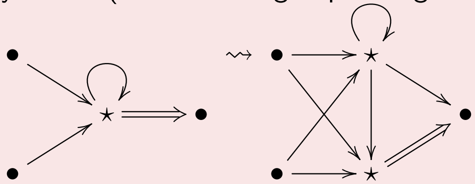
Move **(I)**

Insplit at regular vertex



Move **(O)**

Outsplit at any vertex (at most one group of edges infinite)



Definition

$E \sim_{C^*} F$ when $C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

Definition

$E \sim_m F$ when there is a finite sequence of moves of type

(S),**(R)**,**(O)**,**(I)**,

and their inverses, leading from E to F.

Proposition

$$E \sim_m F \Longrightarrow E \sim_{C^*} F$$

Our goal will be to try to reverse this implication.

Observation

$\dim C^*(E) < \infty$ precisely when E is a finite graph with no cycles

Let $\#_{\mathrm{sink}}(E)$ denote the number of sinks in the graph E.

Theorem

Assume that E, F are both finite graphs with no cycles. The following are equivalent

1. $E \sim_{C^*} F$
2. $E \sim_m F$ *via moves* **(O)** *and* **(R)**
3. $\#_{\mathrm{sink}}(E) = \#_{\mathrm{sink}}(F)$

Amplified graphs

Move (T)

Emit infinitely to any vertex reachable by a path starting with an edge with infinitely many parallels:

Lemma (E-Ruiz-Sørensen)

Move **(T)** *is generated by moves* **(R)**, **(I)**, *and* **(O)**.

Theorem (E-Ruiz-Sørensen)

When E and F both have finitely many vertices and both have the property that if there is an edge between two vertices, there are infinitely many (i.e., they are **amplified***), then*

$$E \sim_{C^*} F \Longrightarrow E \sim_m F$$

through move **(T)** *only*

Let E be a finite graph with no sinks or sources. Then

$$\mathsf{X}_E = \{(e_n) \in (E^1)^{\mathbb{Z}} \mid r(e_n) = s(e_{n+1})\}$$

is a shift space; in fact an SFT.

We say that two shift spaces X and Y are *flow equivalent* and write $\mathsf{X} \sim_{FE} \mathsf{Y}$ when their suspension flows

$$S\mathsf{X} = \mathsf{X} \otimes \mathbb{R}/\langle (x,t) \sim (\sigma(x), t+1)\rangle$$

are homeomorphic in a way preserving the direction of the flow lines. The relevance of this notion comes from

Theorem (Parry-Sullivan)

When E, F are both finite graphs with no sinks or sources, then

$$\mathsf{X}_E \sim_{FE} \mathsf{X}_F \Longleftrightarrow E \sim_m F$$

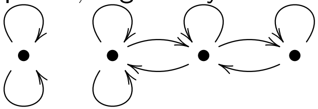
Theorem (Franks)

Let E and F be strongly connected finite graphs, not a single cycle. The following are equivalent

1. $\mathbb{Z}^n / \operatorname{im}(I - A_E) \simeq \mathbb{Z}^m / \operatorname{im}(I - A_F)$ *and* $\det(I - A_E) = \det(I - A_F)$
2. $X_E \sim_{FE} X_F$

$\mathcal{O}_2$ versus $\mathcal{O}_2^-$

Consider the two graphs E, F given by



We have

$$\mathbb{Z}/\operatorname{im}(I - \mathsf{A}_E) = 0 = \mathbb{Z}^2/\operatorname{im}(I - \mathsf{A}_F)$$

but

$$\det(I - \mathsf{A}_E) = -1 \neq 1 = \det(I - \mathsf{A}_F).$$

Theorem (Rørdam)

$C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$

Hence $E \sim_{C^*} F$ but $E \not\sim_m F$.

Move **(C)**

"Cuntz splice" on a vertex supporting two cycles

Theorem (Rørdam, Cuntz)

Let E, F be strongly connected finite graphs, not a single cycle. When $E \sim_{C^*} F$, then $\mathbb{Z}^n/(I - A_E)\mathbb{Z}^n \simeq \mathbb{Z}^m/(I - A_F)\mathbb{Z}^m$. When $\mathbb{Z}^n/(I - A_E)\mathbb{Z}^n \simeq \mathbb{Z}^m/(I - A_F)\mathbb{Z}^m$, then $E \sim_M F$ through moves **(R)**, **(I)**, **(O)**, *and* **(C)**.

Definition

$E \sim_M F$ when there is a finite sequence of moves of type

(S),**(R)**,**(O)**,**(I)**,**(C)**

and their inverses, leading from E to F.

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Filtered K-theory

Definition

Let $\mathfrak{A}$ be a C^*-algebra having only finitely many ideals. The collection of all sequences

$$\begin{array}{ccccc} K_0(\mathfrak{J}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{J}) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{K}/\mathfrak{J}) & \longleftarrow & K_1(\mathfrak{K}/\mathfrak{I}) & \longleftarrow & K_1(\mathfrak{J}/\mathfrak{I}) \end{array}$$

with $\mathfrak{I} \lhd \mathfrak{J} \lhd \mathfrak{K} \lhd \mathfrak{A}$ gauge invariant ideals is called the *filtered K-theory* of $\mathfrak{A}$ and denoted $FK(\mathfrak{A})$. Equipping all K_0-groups with order we arrive at the *ordered,filtered K-theory* $FK^+(\mathfrak{A})$.

Question

Suppose $\mathcal{C}[X]$ is a family of C^*-algebras with real rank zero and primitive ideal space X, so that it is known that $(K_*(-), K_*(-)_+)$ is a complete invariant for all simple subquotients of $\mathfrak{A} \in \mathcal{C}$. When can we conclude that $FK^+(-)$ is a complete invariant for the $\mathfrak{A}$'s themselves?

Theorem (Elliott, Kirchberg-Phillips)

When $C^*(E)$ and $C^*(F)$ are simple, then

$$K_*(C^*(E)) \simeq K_*(C^*(F)) \Longleftrightarrow C^*(E) \otimes \mathbb{K} \simeq C^*(F) \otimes \mathbb{K}$$

Status quo

$FK^+(-)$ is known to be a complete invariant for graph C^*-algebras over X when

- $|X| = 2$ [E-Tomforde]
- $|X| = 3$ and all K-groups are finitely generated [E-Restorff-Ruiz]

and in roughly $2/3$ of the possible cases with $|X| = 4$ [Arklint-Bentmann-E-Katsura-Köhler-Restorff-Ruiz]. No counterexamples are known.

Theorem (Sørensen)

Let E and F be graphs so that $C^*(E)$ and $C^*(F)$ are unital and gauge simple. The following are equivalent

1. $K_*(C^*(E)) \simeq K_*(C^*(F))$
2. $E \sim_M F$
3. $E \sim_{C^*} F$

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Gauge filtered K-theory

Definition

Let $(\mathfrak{A}, \gamma)$ be a C^*-algebra with a gauge action, having only finitely many gauge-invariant ideals. The collection of all sequences

$$\begin{array}{ccccc} K_0(\mathfrak{J}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{I}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{J}) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{K}/\mathfrak{J}) & \longleftarrow & K_1(\mathfrak{K}/\mathfrak{I}) & \longleftarrow & K_1(\mathfrak{J}/\mathfrak{I}) \end{array}$$

with $\mathfrak{I} \lhd \mathfrak{J} \lhd \mathfrak{K} \lhd \mathfrak{A}$ gauge invariant ideals is called the *gauge filtered K-theory* of $\mathfrak{A}$ and denoted $FK^{\gamma}(\mathfrak{A})$. Equipping all K_0-groups with order we arrive at the *ordered, gauge filtered K-theory* $FK^{\gamma,+}(\mathfrak{A})$.

Definition

The **reduced** ordered, gauge filtered K-theory $FK^{\gamma,+,red}(\mathfrak{A})$ consists of

$$\begin{array}{ccccc} K_0(\mathfrak{J}) & \longrightarrow & K_0(\mathfrak{K}) & \longrightarrow & K_0(\mathfrak{K}/\mathfrak{J}) \\ \uparrow & & & & \\ K_1(\mathfrak{K}/\mathfrak{J}) & & & & \end{array}$$

with $\mathfrak{J}$ a maximal gauge invariant ideal inside a gauge prime ideal $\mathfrak{K}$, along with

$$K_0(\mathfrak{J}_n) \to K_0(\mathfrak{J})$$

whenever $\mathfrak{J}, \mathfrak{J}_n$ are gauge prime with $\mathfrak{J} = \cup \mathfrak{J}_n$.

Definition

The **signature** of $\mathfrak{A}$ with finitely many gauge invariant ideals is a map

$$\tau : \mathsf{Prim}^{\gamma}(\mathfrak{A}) \to \mathbb{Z}$$

given by

$$\tau(\mathfrak{I}) = \begin{cases} -2 & \mathfrak{I}/\mathfrak{I}_0 \text{ is not simple} \\ -1 & \mathfrak{I}/\mathfrak{I}_0 \text{ is simple and } AF \\ \operatorname{rank} K_0(\mathfrak{I}/\mathfrak{I}_0) - \operatorname{rank} K_1(\mathfrak{I}/\mathfrak{I}_0) & \text{otherwise} \end{cases}$$

when $\mathfrak{I}_0$ is the maximal gauge invariant proper ideal of $\mathfrak{I}$.

Theorem (E-Ruiz-Sørensen)

Let E and F be finite graphs. Then the following are equivalent

(1) $E \sim_M F$

(2) $E \sim_{C^*} F$

(3) $FK^{\gamma,+}(C^*(E)) \simeq FK^{\gamma,+}(C^*(F))$

(4) $\tau_E = \tau_F$ and $FK^{\gamma,+,red}(C^*(E)) \simeq FK^{\gamma,+,red}(C^*(F))$

Restorff proved (2) $\iff$ (3) $\iff$ (4) for E, F with no sinks and sources, and every vertex reaching a vertex supporting two cycles (condition (II)).