A new uniqueness theorem for k-graph C*-algebras

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Cuntz Algebra (1977): $\mathcal{O}_n$, generated by n partial isometries S_i satisfying $\forall i,\ S_i^* S_i = \sum_{j=1}^{n} S_j S_j^*$.

Cuntz-Krieger Algebras (1980): $\mathcal{O}_A$, generated by partial isometries $S_1, \ldots S_n$, with relations $S_i^* S_i = \sum_{j=1}^{n} A_{ij} S_j S_j^*$ for an $n \times n$ matrix A over $\{0, 1\}$, i.e., the adjacency matrix of a finite directed graph with no multiple edges.

Graph algebras: generalization to arbitrary directed graphs.

Generalizations and related constructions: Exel crossed product algebras, Leavitt path algebras (Abrams, Ruiz, Tomforde), topological graph algebras (Katsura), Ruelle algebras (Putnam, Spielberg), Exel-Laca algebras, ultragraphs (Tomforde), Cuntz-Pimsner algebras, higher-rank Cuntz-Krieger algebras (Robertson-Steger), etc.

k-graph algebras (Kumjian and Pask, 2000)

• developed to generalize graph algebras and higher-rank Cuntz-Krieger algebras,
• whether simple, purely infinite, or AF can be determined from properties of the graph (Kumjian-Pask, Evans-Sims),
• can be described from a k-colored directed graph—a "skeleton"—along with a collection of "commuting squares" (Fowler, Sims, Hazlewood, Raeburn, Webster),
• are groupoid C*-algebras,
• include examples of algebras that are simple but neither AF nor purely infinite, and hence not graph algebras (Pask-Raeburn-Rørdam-Sims),
• include examples that can be constructed from shift spaces (Pask-Raeburn-Weaver),
• can be used to construct any Kirchberg algebra (Spielberg).

Let $k \in \mathbb{N}^+$. We regard $\mathbb{N}^k$ as a category with a single object, 0, and with composition of morphisms given by addition.

A ***k*-graph** is a countable category Λ along with a degree functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *unique factorization property*:

For all $\lambda \in \Lambda$, and $m, n \in \mathbb{N}^k$, if $d(\lambda) = m + n$ then there are unique $\mu \in d^{-1}(m)$ and $\nu \in d^{-1}(n)$ such that $\lambda = \mu\nu$.

- Denote the range and source maps $r, s : \Lambda \to \Lambda$.
- Refer to the objects of Λ as *vertices* and the morphisms of Λ as *paths*.
- Unique factorization implies that $d(\lambda) = 0$ iff λ a vertex.

Illustration of unique factorization in $k = 2$ case.

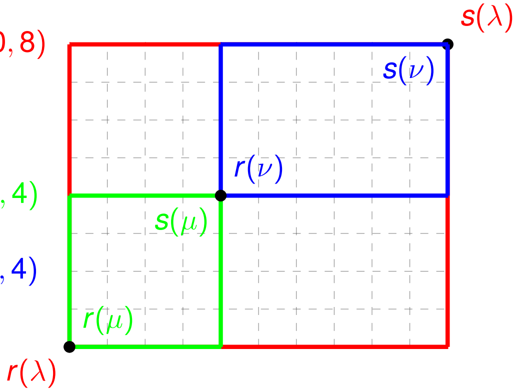
$\lambda \in \Lambda$

$d(\lambda) = (10, 8)$

$\lambda = \mu\nu$

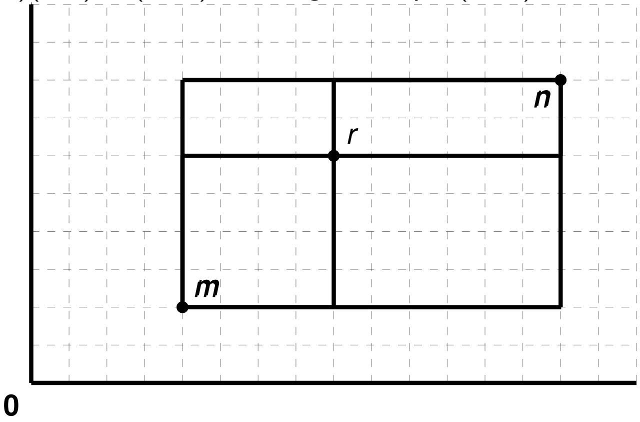
$d(\mu) = (4, 4)$

$d(\nu) = (6, 4)$

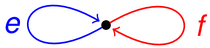


1. The set E^*, where (E^0, E^1, r, s) is a directed graph. Set $d(\lambda) = d$ iff λ has length d.

2. Let $\Omega_k := \{(m,n) \in \mathbb{N}^k \times \mathbb{N}^k \mid m \leq n\}$ with composition $(m,r)(r,n) = (m,n)$ and degree map $d(m,n) = n - m$.



3. We can define a 2-graph from the directed colored graph $E = (E^0, E^1, r, s)$ with color map $c : E^1 \to \{1, 2\}$ as follows.



Endow E^* with the degree functor given by

$$d(e_1 e_2 \dots e_n) = (m_1, m_2), \text{ where } m_i = |c^{-1}(i)|.$$

Since $(0,1) + (1,0) = (1,0) + (0,1)$ and the only paths of degrees $(1,0)$ and $(0,1)$ are, respectively, e and f, to define a 2-graph from E^* we must declare $ef = fe$. In fact, any two paths of equal degree must be equal.

The 2-graph we obtain is the semigroup $\mathbb{N}^2$ with degree map the identity.

Notation:

- For $n \in \mathbb{N}^k$, we denote $\Lambda^n = \{\lambda \in \Lambda \mid d(\lambda) = n\}$.
- For $v \in \Lambda^0$ denote $v\Lambda^n = \{\lambda \in \Lambda^n \mid r(\lambda) = v\}$.

A k-graph Λ is **row-finite** and has **no sources** if

$$\forall v \in \Lambda^0, \forall n \in \mathbb{N}^k,\ 0 < |v\Lambda^n| < \infty.$$

Assume all k-graphs are row-finite and have no sources.

A **Cuntz-Krieger Λ-family** in a C*-algebra A is a set $\{T_\lambda, \lambda \in \Lambda\}$ of partial isometries in A satisfying

(i) $\{T_v \mid v \in \Lambda^0\}$ is a family of mutually orthogonal projections,

(ii) $T_{\lambda\mu} = T_\lambda T_\mu$ for all $\lambda, \mu \in \Lambda$ s.t. $s(\lambda) = r(\mu)$,

(iii) $T_\lambda^* T_\lambda = T_{s(\lambda)}$ for all $\lambda \in \Lambda$, and

(iv) for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, $\;T_v = \sum_{\lambda \in v\Lambda^n} T_\lambda T_\lambda^*$.

For $\lambda \in \Lambda$, denote $Q_\lambda := T_\lambda T_\lambda^*$.

$C^*(\Lambda)$ will denote the C*-algebra generated by a universal Cuntz-Krieger Λ-family, $(S_\lambda, \lambda \in \Lambda)$, with $P_\lambda = S_\lambda S_\lambda^*$.

Q: When is a *-homomorphism $\Phi : C^*(\Lambda) \to A$ injective?

Necessary: Φ is **nondegenerate**, i.e., it is injective on the diagonal subalgebra $\mathscr{D} := C^*(\{P_\mu \mid \mu \in \Lambda\})$.

Our new uniqueness theorem proves the sufficiency of injectivity on a (usually) larger subalgebra, $\mathscr{M} \supseteq \mathscr{D}$, and generalizes our theorem for directed graphs, where $\mathscr{M}$ is called the Abelian Core of $C^*(\Lambda)$.

[NR1] G. Nagy and S. Reznikoff, *Abelian core of graph algebras*, J. Lond. Math. Soc. (2) **85** (2012), no. 3, 889–908.

[NR2] G. Nagy and S. Reznikoff, *Pseudo-diagonals and uniqueness theorems*, (2013), to appear in Proc. AMS.

[S] W. Szymański, *General Cuntz-Krieger uniqueness theorem*, Internat. J. Math. **13** (2002) 549–555.

Gauge Actions

The universal C*-algebra of a k-graph Λ has a *gauge action* $\alpha : \mathbb{T}^k \to \text{Aut}\, C^*(\Lambda)$ given by

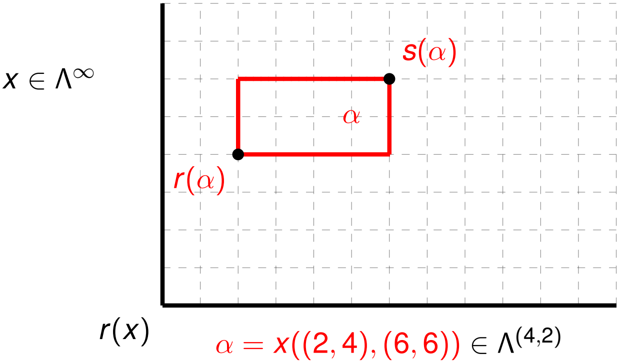
$$\alpha_t(S_\lambda) = t^{d(\lambda)} S_\lambda = t_1^{d_1} t_2^{d_2} \dots t_k^{d_k} S_\lambda,$$

where $t = (t_1, t_2, \dots t_k)$ and $d(\lambda) = (d_1, d_2, \dots d_k)$.

Gauge-Invariant Uniqueness Theorem (Kumjian-Pask):
If $\Phi : C^*(\Lambda) \to A$ is a nondegenerate $*$-representation and intertwines a gauge action $\beta : \mathbb{T}^k \to \text{Aut}(A)$ with α, then Φ is injective.

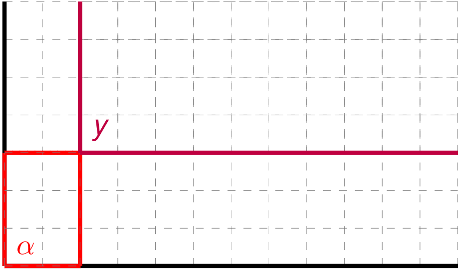
Recall $\Omega_k := \{(m,n) \in \mathbb{N}^k \times \mathbb{N}^k \mid m \le n\}$, with degree map $d(m,n) = n - m$ and composition $(m,n)(n,r) = (m,r)$.

The **infinite path space** Λ^∞ is the set of all degree-preserving covariant functors $x : \Omega_k \to \Lambda$.



For $\alpha \in \Lambda$ and $y \in \Lambda^\infty$, if $s(\alpha) = r(y)$ then αy is the unique $x \in \Lambda^\infty$ s.t. $x(0, N) = \alpha y(d(\alpha), N)$ for all $N \geq d(\alpha)$.

$x = \alpha y \in \Lambda^\infty$



Using the topology generated by the cylinder sets

$$\begin{aligned} Z(\alpha) &= \{x \in \Lambda^\infty \mid x(0, d(\alpha)) = \alpha\} \\ &= \{x \in \Lambda^\infty \mid \exists y \in \Lambda^\infty \text{ s.t. } x = \alpha y\}, \end{aligned}$$

Λ^∞ is a locally compact Hausdorff space.

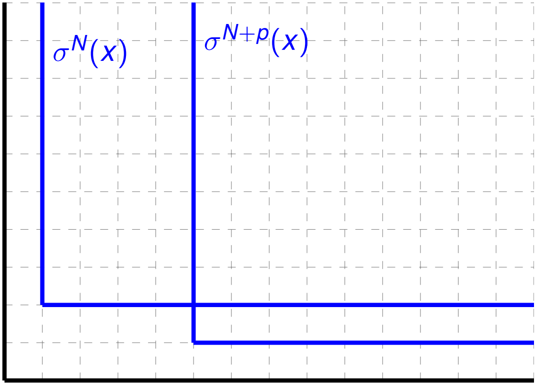
The shift map: For $x \in \Lambda^\infty$ and $N \in \mathbb{N}^k$, $\sigma^N(x)$ is defined to be the element of Λ^∞ given by $\sigma^N(x)(m,n) = x(m+N, n+N)$.

$x \in \Lambda^\infty$ is *eventually periodic* if there is an $N \in \mathbb{N}^k$ and an $p \in \mathbb{Z}^k$ such that $\sigma^N(x) = \sigma^{N+p}(x)$; otherwise x is *aperiodic*.

$x \in \Lambda^\infty$

$N = (1,2)$

$p = (4,-1)$



Theorem (Kumjian-Pask) If Λ satisfies

(A) for every $v \in \Lambda^0$ there is an aperiodic path $x \in v\Lambda^\infty$,

then any nondegenerate representation of $C^*(\Lambda)$ is injective.

Theorem (Raeburn, Sims, Yeend)
If Λ satisfies

(B) For each $v \in \Lambda^0$ there is an $x \in v\Lambda^\infty$ s.t.
$$\forall \alpha, \beta \in \Lambda \quad (\alpha \neq \beta \Rightarrow \alpha x \neq \beta x)$$

then any nondegenerate representation of $C^*(\Lambda)$ is injective.

Remarks:

- When Λ has no sources, (A) $\Rightarrow$ (B).
- (B) $\Rightarrow$ (A) holds for 1-graphs.

The super-normal subalgebra

Observation: $C^*(\Lambda) = \overline{\text{span}}\{S_\mu S_\nu^* \,|, \mu, \nu \in \Lambda,\ s(\mu) = s(\nu)\}$.
Recall $P_\alpha := S_\alpha S_\alpha^*$.

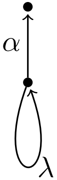
Defn. We call the element $S_\alpha S_\beta^*$ *super-normal* if it is normal and commutes with $\mathscr{D} := C^*(\{P_\mu\})$.

Prop. The following are equivalent for $\alpha \neq \beta$.

(i) $S_\alpha S_\beta^*$ is super-normal.

(ii) For all $\gamma \in \Lambda$, $P_{\alpha\gamma} = P_{\beta\gamma}$.

(iii) For all $\gamma \in s(\alpha)\Lambda$, the pair $(\alpha\gamma, \beta\gamma)$ is a generalized cycle without entry, in the sense of Evans and Sims.



Example ($k = 1$): Suppose λ is a cycle without entry, $r(\lambda) = s(\alpha)$, and $\beta = \lambda \circ \alpha$. Then it is easy to verify that for all $\gamma \in \Lambda$, $P_{\alpha\gamma} = P_{\beta\gamma}$, so $S_\alpha S_\beta^*$ is super-normal.

On the other hand:

Fact: If $s(\alpha) = s(\beta)$ but $\alpha \neq \beta$, and there exists an aperiodic $x \in s(\alpha)\Lambda^\infty$, then $S_\alpha S_\beta^*$ is not super-normal.

Therefore, if Λ satisfies Condition (A) then the only super-normal generators are the projections $P_\mu = S_\mu S_\mu^*$.

Let $\mathscr{M} = C^*(\{S_\alpha S_\beta^* \text{ super-normal}\})$.

Theorem (Brown-Nagy-R, 2013)
For a representation $\Phi : C^*(\Lambda) \to B$, TFAE:

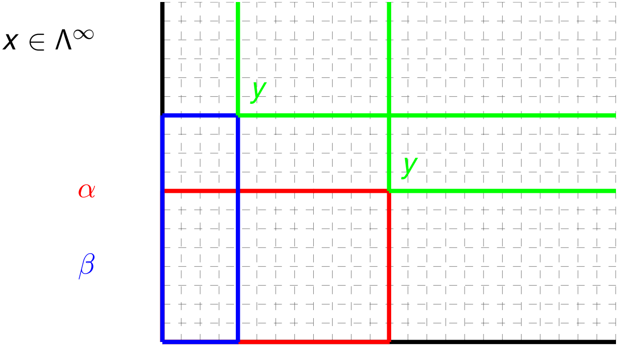
(i) Φ is injective

(ii) Φ is injective on $\mathscr{M}$.

Rmk: By the observation on the previous page, if Λ satisfies Condition (A) then $\mathscr{M} = \mathscr{D} := C^*(\{P_\mu\})$.

The proof involves examining a representation of $C^*(\Lambda)$ in $B(\ell^2(X))$, for $X \subset \Lambda^\infty$ the set of "regular paths" of Λ.

For $\alpha, \beta \in \Lambda$, let $F_{\alpha,\beta} := \{x \in \Lambda^\infty \mid \exists\ y \in \Lambda^\infty\ x = \alpha y = \beta y\}$.



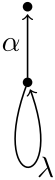
Facts:

- $x \in F_{\alpha,\beta}$ is eventually periodic of period $p = d(\beta) - d(\alpha)$.
- Any eventually periodic x is in some $F_{\alpha,\beta}$.
- $F_{\alpha,\beta}$ is closed, and if $\alpha = \beta$, then $F_{\alpha,\beta} = Z(\alpha)$.

The *regular paths* are the elements of

$$X := \Lambda^\infty \setminus \bigcup_{\alpha,\beta\in\Lambda} \partial F_{\alpha,\beta}.$$

- X is dense in Λ^∞ (uses Baire Category).
- X is closed under the shift.
- When $k = 1$,
  $X = \{$infinite "essentially aperiodic" paths$\}$.



Aperiodic paths are essentially aperiodic.

If λ cycle with no entry, $\alpha \in \Lambda$, $r(\lambda) = s(\alpha)$, then $x = \alpha\lambda^\infty$ is essentially aperiodic.

There is a Cuntz-Krieger Λ family $(T_\alpha, \alpha \in \Lambda)$ in $B(\ell^2(X))$, given by

$$T_\alpha \delta_x = \begin{cases} \delta_{\alpha x} & \text{if } x \in s(\alpha)\Lambda^\infty \\ 0 & \text{otherwise.} \end{cases}$$

We define the *aperiodic representation*:

$$\pi_{\mathrm{ap}} : C^*(\Lambda) \to B(\ell^2(X))$$
$$S_\lambda \mapsto T_\lambda$$

We first prove that for representations of $\pi_{\mathrm{ap}}(C^*(\Lambda))$ injectivity on $\pi_{\mathrm{ap}}(\mathscr{M})$ lifts.

Abstract Uniqueness Theorem (Brown-Nagy-R, 2013)
Let A be a C*-algebra and $M \subset A$ an abelian C*-subalgebra. Suppose there is a set $\mathcal{S}$ of pure states on M satisfying

(i) each $\psi \in \mathcal{S}$ extends uniquely to a state $\tilde{\psi}$ on A, and
(ii) the collection $\tilde{S} := \{\tilde{\psi} \,|\, \psi \in \mathcal{S}\}$ is "jointly faithful" on A.

Then a $*$-homomorphism $\Phi : A \to B$ is injective iff $\Phi|_M$ is injective. Moreover, M' is a masa in A.

Corollary
A $*$-representation $\Phi : \pi_{\rm ap}(C^*(\Lambda)) \to B$ is injective iff it is injective on $\pi_{\rm ap}(\mathscr{M})$.

Proof: The hypotheses of the Abstract Uniqueness Theorem hold with $\mathcal{S}$ a set of "evaluation states". (See extra slides after biblio. for proof sketches.)

To handle representations of $C^*(\Lambda)$:
Define the "twisted aperiodic representation"

$$\Psi_{\mathrm{ap}} : C^*(\Lambda) \to B(\ell^2(X \times \mathbb{Z}^k)).$$

Now the gauge invariance theorem applies.
Adapt the previous argument to $\Psi_{\mathrm{ap}}(C^*(\Lambda))$. Pull back the jointly faithful set of uniquely extending states to $C^*(\Lambda)$ to prove:

Theorem (Brown-Nagy-R, 2013)
For a representation $\Phi : C^*(\Lambda) \to B$, TFAE:

(i) Φ is injective.
(ii) Φ is injective on $C^*(\mathscr{M})$.

(Renault, '80) A C*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$ is **Cartan** if

- $\mathcal{B}$ is a masa in $\mathcal{A}$,
- $\exists$ a faithful conditional expectation $\mathcal{A} \to \mathcal{B}$,
- The normalizer of $\mathcal{B}$ in $\mathcal{A}$ generates $\mathcal{A}$, and
- $\mathcal{B}$ contains an approximate unit of $\mathcal{A}$.

Theorem (Nagy-R, 2011)
If Λ is a 1-graph then $\mathscr{M} \subseteq C^*(\Lambda)$ is Cartan.

Defn. $\mathcal{B} \subseteq \mathcal{A}$ has the Unique Extension Property (UEP) if every pure state on $\mathcal{B}$ extends uniquely to a pure state on $\mathcal{A}$.

• A Cartan C*-subalgebra with the UEP is a **C*-Diagonal**.
• For $k = 1$, $\mathscr{M}$ is a **pseudo-diagonal**: densely many pure states extend uniquely and there is a faithful conditional exp.
• For arbitrary k, $\mathscr{M}'$ is a MASA. Is it a pseudo-diagonal?

K.R. Davidson, S.C. Power, and D. Yang, *Dilation theory for rank 2 graph algebras*, J. Operator Theory.

D. G. Evans and A. Sims, *When is the Cuntz-Krieger algebra of a higher-rank graph approximately finite-dimensional?*, J. Funct. Anal. **263** (2012), no. 1, 183–215.

A. Kumjian and D. Pask, *Higher rank graph C*-algebras*, New York J. Math. **6** (2000), 1–20.

A. Kumjian, D. Pask, and I. Raeburn, *Cuntz-Krieger algebras of directed graphs*, Pacific J. Math. **184** (1998) 161–174.

A. Kumjian, D. Pask, I. Raeburn, and J. Renault, *Graphs, groupoids and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997), 505–541

G. Nagy and S. Reznikoff, *Abelian core of graph algebras*, J. Lond. Math. Soc. (2) **85** (2012), no. 3, 889–908.

G. Nagy and S. Reznikoff, *Pseudo-diagonals and uniqueness theorems*, (2013), to appear in Proc. AMS.

D. Pask, I. Raeburn, M. Rørdam, A. Sims, *Rank-two graphs whose C*-algebras are direct limits of circle algebras*, J. Functional Anal. **144** (2006), 137–178.

I. Raeburn, A. Sims and T. Yeend, *Higher-rank graphs and their C*-algebras*, Proc. Edin. Math. Soc. **46** (2003) 99–115.

D. Robertson and A. Sims, *Simplicity of C*-algebras associated to higher-rank graphs.* Bull. Lond. Math. Soc. **39** (2007), no. 2, 337–344.

- G. Robertson and T. Steger, *Affine buildings, tiling systems and higher rank Cuntz-Krieger algebras*, J. Reine Angew. Math. **513** (1999), 115–144.
- A. Sims, *Gauge-invariant ideals in the C*-algebras of finitely aligned higher-rank graphs*, Canad. J. Math. **58** (2006), no. 6, 1268–1290.
- J. Spielberg, *Graph-based models for Kirchberg algebras*, J. Operator Theory **57** (2007), 347–374.
- W. Szymański, *General Cuntz-Krieger uniqueness theorem*, Internat. J. Math. **13** (2002) 549–555.

Thank you!

Sketch of corollary proof:

Let $A = \pi_{\text{ap}}(C^*(\Lambda))$, $M = \pi_{\text{ap}}(\mathscr{M})$, and $D = \pi_{\text{ap}}(\mathscr{D})$.

• Why M is abelian: Note that if $T \in D'$ then T commutes with all $p_x :=$SOT-$\lim_{n\to\infty} Q_{x(0,n)}$ so $T \in \ell^\infty(X)$. Thus D' is abelian, and $M \subseteq D'$ by definition.

• The states in S: For each $x \in X$ define $\text{ev}_x^D(Q_\alpha) = \chi_{Z(\alpha)}(x)$. Let ϕ be an extension of ev_x^D to A. We show that $\phi(T_\alpha T_\beta^*)$ depends only on x, α, and β. To do this, we extend α and β to μ and ν with $T_\nu = T_\mu$. Denote the unique extension ϕ_x and let $S = \{\phi_x|_M \,|\, x \in X\}$.

• Why the extensions ϕ_x are jointly faithful on A: Easy to see that $\phi_x(T) = \langle T\delta_x, \delta_x\rangle$ and so if $T = (T^{1/2})^2$ and $\phi_x(T) = 0$ for all x then $T^{1/2} = 0$ too.

Ideas in proof of Abstract Uniqueness Theorem:

We are assuming the states $\psi \in S$ on M extends uniquely to states $\tilde{\psi} \in \tilde{S}$ on A, and the collection of the extensions is jointly faithful on A.

• If $\ker \phi|_M \subseteq \ker \psi$ then $\ker \phi \subseteq \ker \pi_\psi$ (the GNS representation associated with ψ).

• If $\tilde{S}$ is jointly faithful then $\cap_{\psi \in S} \ker \pi_\psi = \{0\}$.

The conditional expectation when $k = 1$:

For $x \in X$, let $p_x =_{\text{SOT}}\text{-}\lim_{n\to\infty} Q_{x(0,n)} \in B(\ell^2(X))$.

- p_x is the projection onto $\mathrm{span}\{\delta_{x,m} \,|\, m \in \mathbb{Z}^k\}$
- $\phi_x(T_\alpha T_\beta^*)p_x = p_x T_\alpha T_\beta^* p_x$.

Define

$$E_{\mathrm{ap}} : B(\ell^2(X)) \to \{p_x \,|\, x \in X\}'$$
$$A \mapsto \sum_{x\in X} p_x A p_x$$

E_{ap} is a faithful conditional expectation; moreover Ψ_{ap} intertwines it with a faithful conditional expectation $E_\Lambda : C^*(\Lambda) \to \mathscr{M}$.