A gentle introduction to Microswimming:

geometry, physics, analysis

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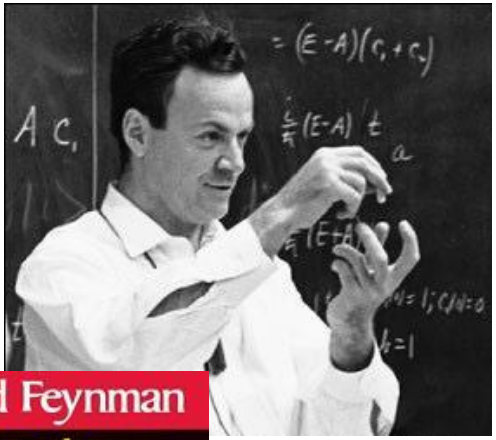
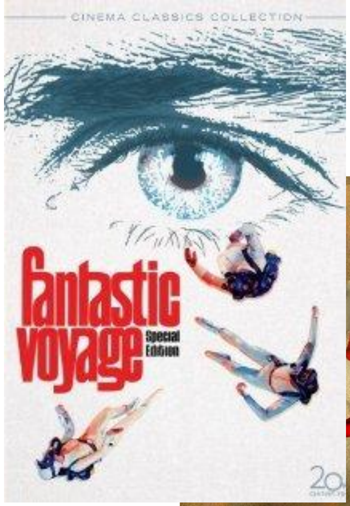
Marsden legacy July 2012 @

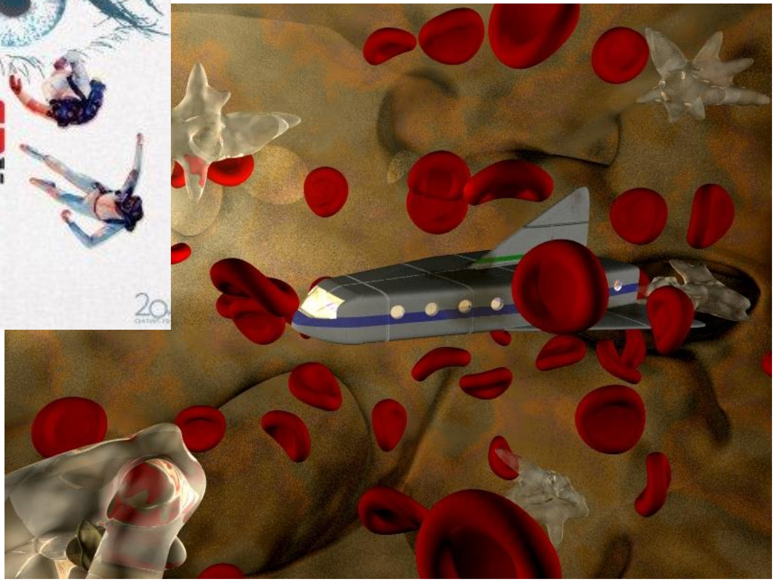
Why study microswimming?

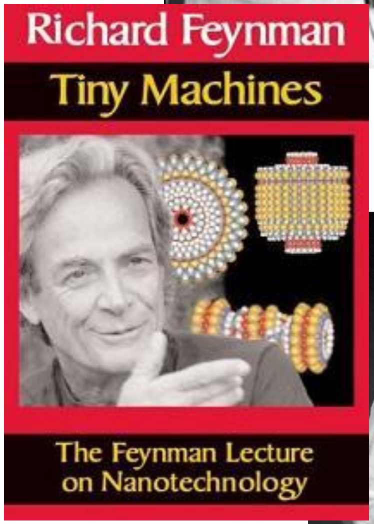
Feynman: there is plenty of room in the bottom!

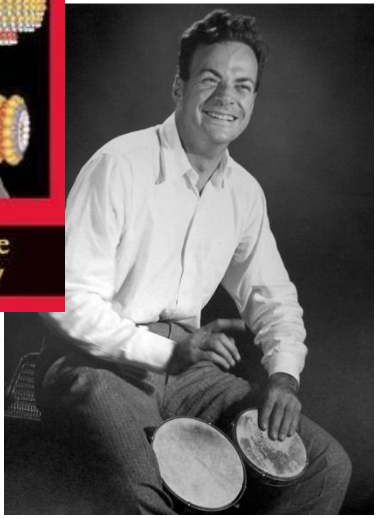
(and plenty of grant money, so it seems)











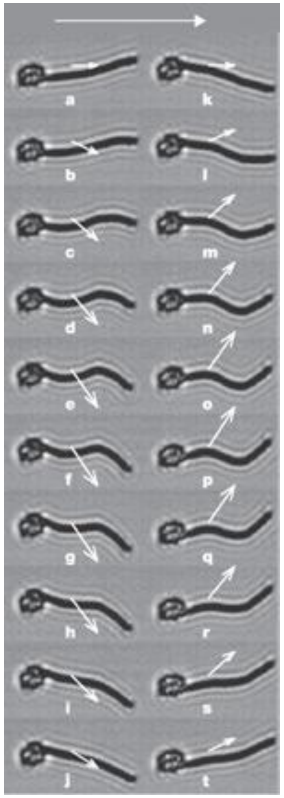
In the last 15 years:

New tools for particle visualization, in vivo cell manipulation, biochemical structure, genomics and function.

These developments are bringing new challenges and opportunities for the applied mathematician to do collaborative work with biologists and engineers.

One example:

Recent experiments in R.Goldstein, DAMPT



Dreyfus et al., Microscopic artificial swimmers, Nature 437, 862-865, 2005

Collaborators

Kurt Ehlers and Richard Montgomery

Joaquim Delgado

Marco Raupp, Alexandre Cherman, Gerusa Araujo, Fernando Duda

Advice/ suggestions

Howard Berg, Theodore Wu, Moyses Nussenzweig

Greg Huber, Scott Kelly, John Bush, Lisa Fauci, Peko Hosoi, …

Encouragement:

Collaborators

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Advice/ suggestions

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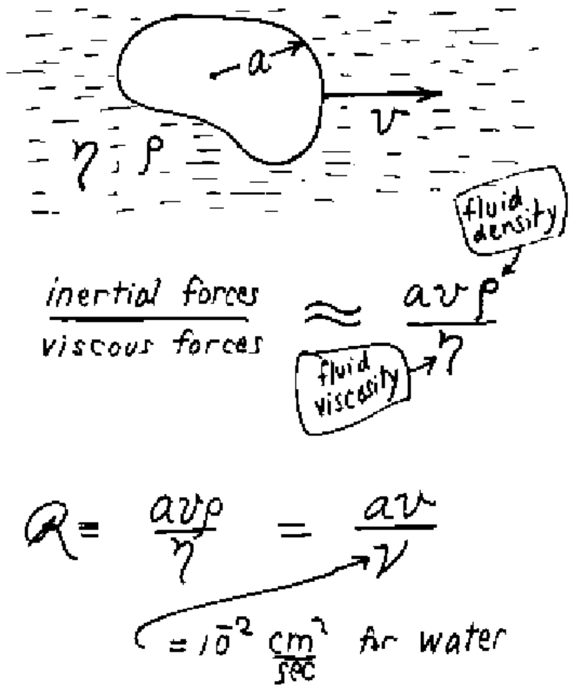
Greg Huber, Scott Kelly,

Encouragement: **Jerry Marsden** **[JK, KE, RM Problems and Progress]**

Microswimming is governed by Stokes equations on an incompressible fluid

Taylor (movie)



Ambient: R^2 (life at interface) or R^3

"Molasses Laplacian"

$$0 = -\ \mathrm{grad}\ p + \mu\ \Delta\ u$$

$$\mathrm{div}\ u = 0$$

Purcell Life at low Re

(drop the inertial term from Navier Stokes)

Reynolds = $O(10^{-5})$

Stress tensor T = -pI + σ , σ = ??

$$\nabla \mathbf{u} = \begin{bmatrix} \partial_x u & \partial_y u & \partial_z u \\ \partial_x v & \partial_y v & \partial_z v \\ \partial_x w & \partial_y w & \partial_z w \end{bmatrix}$$

denote the Jacobian matrix of **u**. By Taylor's theorem,

$$\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{x}) + \nabla \mathbf{u}(\mathbf{x}) \cdot \mathbf{h} + O(h^2), \qquad (1.2.2)$$

where $\nabla \mathbf{u}(\mathbf{x}) \cdot \mathbf{h}$ is a matrix multiplication, with $\mathbf{h}$ regarded as a column vector. Let

$$\mathbf{D} = \tfrac{1}{2}\left[\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right].$$

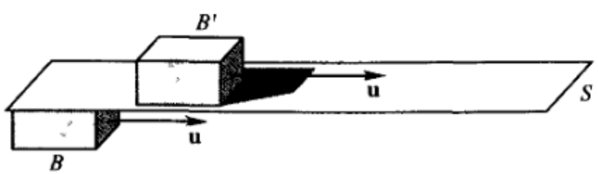


FIGURE 1.3.1. Faster molecules in B' can diffuse across S and impart momentum to B.

where $\mathbf{n}$ is the normal to S, we now assume that

$$\text{force on } S \text{ per unit area} = -p(\mathbf{x},t)\mathbf{n} + \boldsymbol{\sigma}(\mathbf{x},t)\cdot \mathbf{n}, \qquad (1.3.1)$$

where $\boldsymbol{\sigma}$ is a *matrix* called the ***stress tensor***, about which some assumptions will have to be made. The new feature is that $\boldsymbol{\sigma}\cdot\mathbf{n}$ need not be parallel to $\mathbf{n}$. The separation of the forces into pressure and other forces in (1.3.1) is somewhat ambiguous because $\boldsymbol{\sigma}\cdot\mathbf{n}$ may contain a component parallel to $\mathbf{n}$. This issue will be resolved later when we give a more definite functional form to $\boldsymbol{\sigma}$.

This is reasonable, because when a fluid undergoes a rigid body rotation, there should be no diffusion of momentum.

3. $\boldsymbol{\sigma}$ *is symmetric*. This property can be deduced as a consequence of balance of angular momentum.[7]

Since $\boldsymbol{\sigma}$ is symmetric, if follows from properties 1 and 2 that $\boldsymbol{\sigma}$ can depend only on the symmetric part of $\nabla \mathbf{u}$; that is, on the deformation $\mathbf{D}$. Because $\boldsymbol{\sigma}$ is a linear function of $\mathbf{D}$, $\boldsymbol{\sigma}$ and $\mathbf{D}$ commute and so can be simultaneously diagonalized. Thus, the eigenvalues of $\boldsymbol{\sigma}$ are linear functions of those of $\mathbf{D}$. By property 2, they must also be symmetric because we can choose $\mathbf{U}$ to permute two eigenvalues of $\mathbf{D}$ (by rotating through an angle $\pi/2$ about an eigenvector), and this must permute the corresponding eigenvalues of $\boldsymbol{\sigma}$. The only linear functions that are symmetric in this sense are of the form

$$\sigma_i = \lambda(d_1 + d_2 + d_3) + 2\mu d_i, \qquad i = 1,2,3,$$

where σ_i are the eigenvalues of $\boldsymbol{\sigma}$, and d_i are those of $\mathbf{D}$. This defines the constants λ and μ. Recalling that $d_1 + d_2 + d_3 = \operatorname{div}\mathbf{u}$, we can use property 2 to transform σ_i back to the usual basis and deduce that

$$\boldsymbol{\sigma} = \lambda(\operatorname{div}\mathbf{u})\mathbf{I} + 2\mu\,\mathbf{D}, \qquad (1.3.2)$$

where $\mathbf{I}$ is the identity. We can rewrite this by putting all the trace in one term:

$$\boldsymbol{\sigma} = 2\mu[\mathbf{D} - \tfrac{1}{3}(\operatorname{div}\mathbf{u})\mathbf{I}] + \zeta(\operatorname{div}\mathbf{u})\mathbf{I} \qquad (1.3.2)'$$

where μ is the ***first coefficient of viscosity***, and $\zeta = \lambda + \frac{2}{3}\mu$ is the ***second coefficient of viscosity***.

source: Marsden/Chorin

An organism/robot is a deforming boundary immersed in the ambient.

There are physical requirements for self propulsion.

What are them? (Wait a couple slides.)

For now: $T = -p I + 2 \mu D$

$\operatorname{div} T = 0$ and $\operatorname{div} u = 0$

$u \to 0$ at infinity ,

no slip condition imposed on all boundaries (moving or fixed)

Ambient: R^2 (life at interface) or R^3

but ... everything has boundaries !!!!

Common wisdom:

Boundaries affect motion substantially only

when organisms are close to them.

Geometry and Physics of Microswimming *

(Purcell & Shapere-Wilczek)

It 's a Gauge theory !

Key words: shape space, principal bundle, connection!

And a subriemannian geometry!

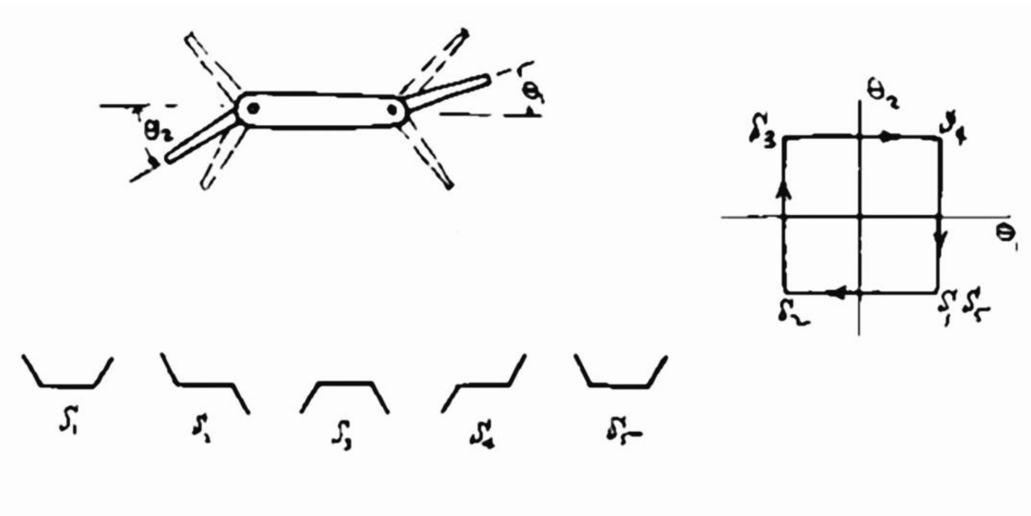
Metric is the hydrodynamical power efficiency notions

(For collective swimming: Ehresmann connection)

Recife notes
www.impa.br/~jair

* Taylor and Lighthill already knew in the 1950's what it was all about.
Later on, analysts occasionally make blunders (see O.P. (2.14))

Microswiming is a gauge theory!!



Purcell's 3 linked swimmer

(only recently studied)

toroidal animal (Taylor, Purcell)

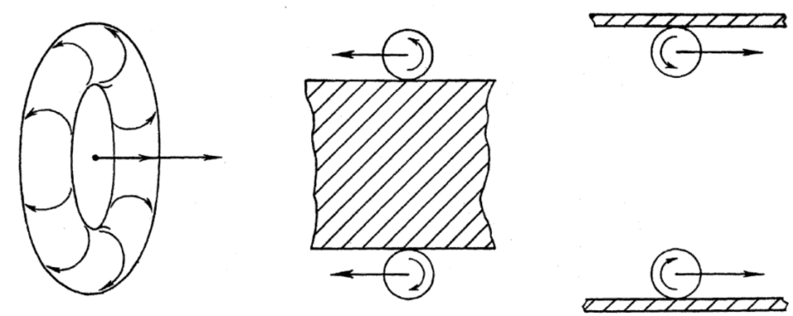


FIGURE 1a. Hypothetical ring-shaped animal capable of rotating its body in the direction indicated. b. Direction of motion when the ring rolls on the outside of a cylinder. c. Direction of motion when the ring rolls on the inside of a cylindrical tube.

What is the metric? Hydrodynamical power expenditure *

U = vectorfield along the boundary

u = solution of exterior Stokes equations (analogous to Dirichlet problem for Laplacian)

σ = stress tensor associated to u $F = \sigma . n$ along the boundary

integrate F . U on S , call it **<<U,U>>**

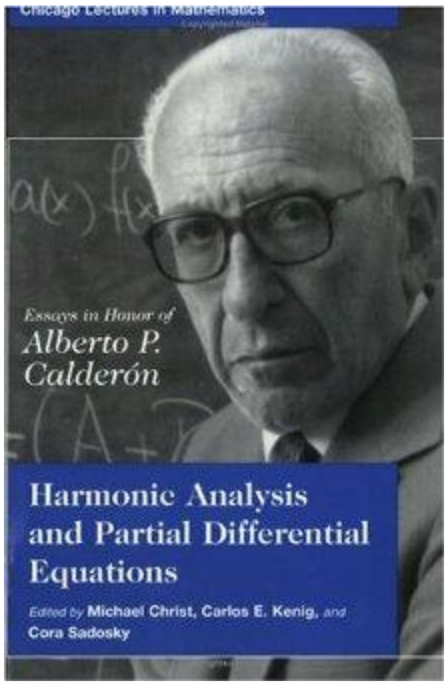
$P: U \longrightarrow F$ **Resistance operator**

symmetric (Lorenz reciprocity)

* Discuss the envelope approximation

A “wet” Calderon problem?

P : U ⟶ F analogous to “Dirichlet to Neumann”



It is well known (see, e.g., [21]) that, given $f \in C^{2,\alpha}(\Omega)$, there exists a unique solution of the boundary-value problem

$$\begin{cases} \nabla \cdot (\gamma(x,u)\nabla u) = 0 & \text{in } \Omega, \\ u\big|_{\partial\Omega} = f. \end{cases} \qquad (19.164)$$

We define the Dirichlet to Neumann map $\Lambda_\gamma : C^{2,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega)$ as the map given by

$$\Lambda_\gamma : f \to \nu \cdot \gamma(x,f)\nabla u\Big|_{\partial\Omega}, \qquad (19.165)$$

where u is the solution of (19.164) and ν denotes the unit outer normal of $\partial\Omega$.

Physically, $\gamma(x,u)$ represents the (anisotropic, quasilinear) conductivity of Ω and $\Lambda_\gamma(f)$ the current flux at the boundary induced by the voltage f.

We study the inverse boundary-value problem associated to (19.164): how much information about the coefficient matrix γ can be obtained from knowledge of the Dirichlet to Neumann map Λ_γ?

[Marsden on Calderon]

What is the Connection?

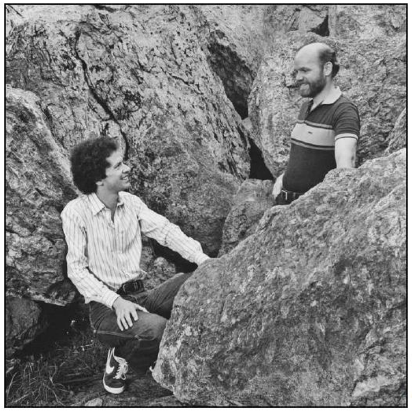


Photo by Margo Weinstein.

Horizontal spaces:

physically allowed motions for self propulsion

Total force = 0 Total torque = 0

Vertical spaces: rigid motions (with frozen shape)

A key fact: **Vertical $\perp$ Horizontal**

Proof.

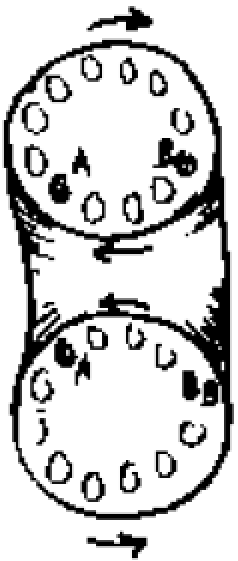
"Lorenz reciprocity" (Green like identities for the "gooey " Laplacian)

Answer: Its the mechanical connection

The connection 1 form solves the self rotating torus

(Taylor, Purcell)

Nano: DNA ring



Hold the shape in place,

SR boundary conditions: solve Stokes equations, compute total force F and total torque T (most likely T = 0).

RB counterflow: solve Stokes equations with unit velocity. Compute total force and adjust to have minus the force calculated previously.

You can invent other animals that have no trouble swimming. We had better be able to invent them, since we know they exist. One you might think of first as a physicist, is a torus. I don't know whether there is a toroidal animal, but whatever other physiological problems it might face, it clearly could swim at low Reynolds number

Purcell, Life at low Reynolds

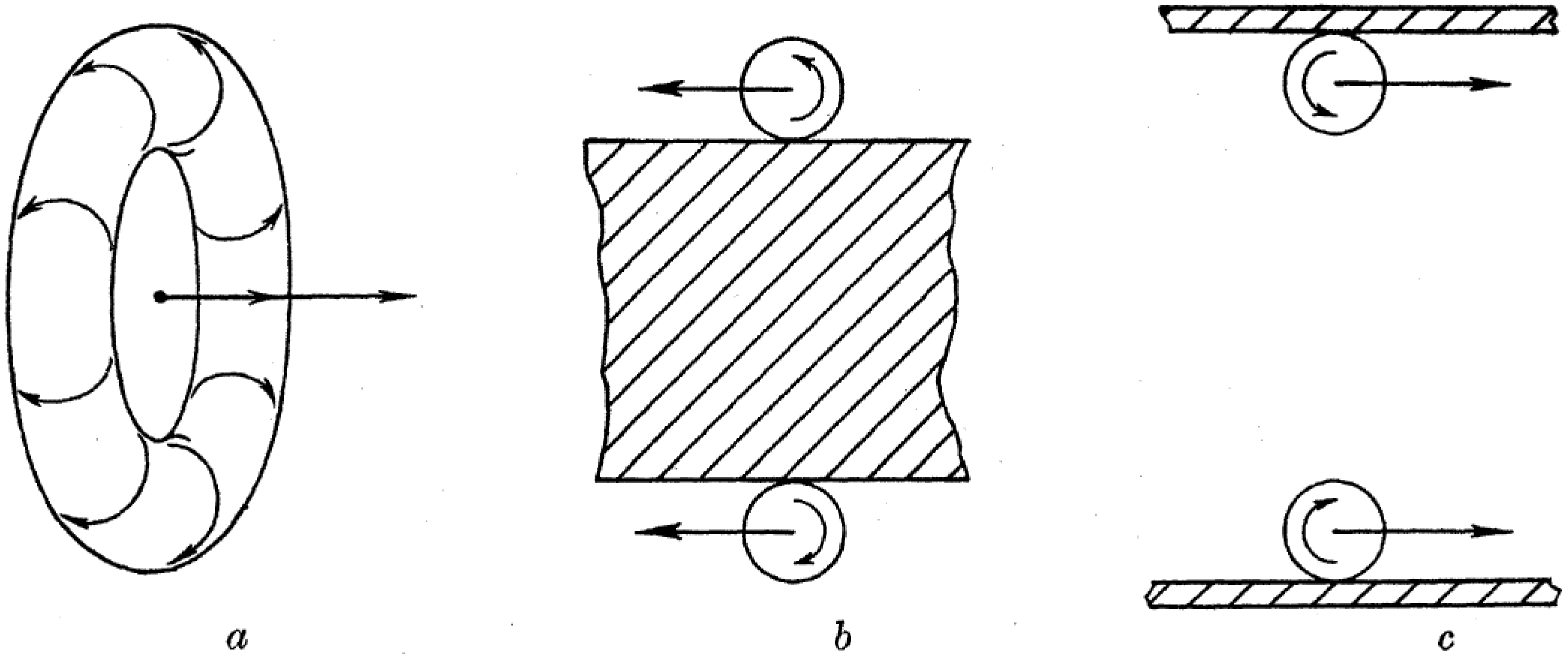


FIGURE 1*a*. Hypothetical ring-shaped animal capable of rotating its body in the direction indicated. *b*. Direction of motion when the ring rolls on the outside of a cylinder. *c*. Direction of motion when the ring rolls on the inside of a cylindrical tube.

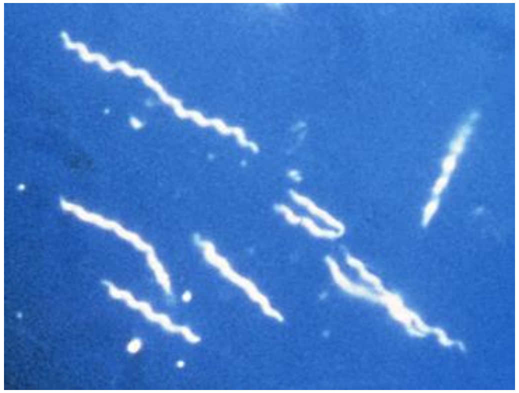
The action of waving cylindrical tails in propelling microscopic organisms

BY SIR GEOFFREY TAYLOR, F.R.S.

(*Received* 3 *October* 1951)

Spirochetes

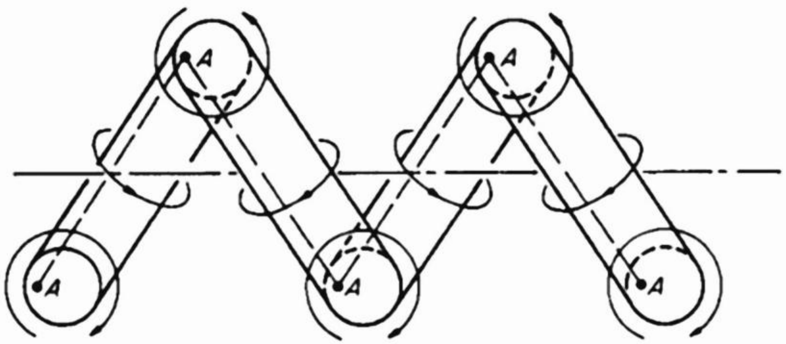
http://www.annualreviews.org/doi/full/10.1146/annurev.genet.36.041602.134359



Self rotation induced flow (depicted in the figure)

Rigid Body counterflow with the opposite total torque

(note that total force vanishes)



Schematic figure showing the mechanism for the self rotation about a local body axis.

Lighthill's analysis

More later in this talk

* **Different trick by spiroplasma (Greg Huber)**

How to compute the curvature?

(Some <u>tricks</u> of the trade that we learned in the 1990's)

Is it really needed to solve Stokes equations for any deformation?

Answer: yes and no.

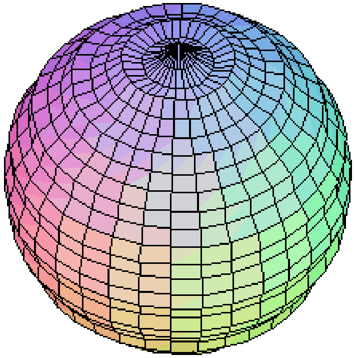
Yes, we need to extend vectorfields defined on the boundary of the shape as the external Stokes flows; we need to Lie bracket them.

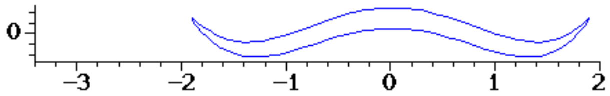
No, if we need only the connection 1 form.

Shapere/Wilzek recipie and more explainded (see Recife lectures)

[Taylor's swimming sheet]

With curvature form, get Propulsion operator F (antisymmetric)





F(U,V) = infinitesimal displacement generated by U,V,-U,-V

An element of sE(3)

Let $\{v_n\}$ be a basis for the vectorfields on the surface of a given located shape q. Define $\mathcal{F}_{mn}$ to be the infinitesimal Euclidean motion given by the coupling of the modes v_m and v_n (e.g., Fourier modes). The $\mathcal{F}_{mn}$ are nothing more than the components of the curvature two-form $\mathcal{F}$ of the connection form A, evaluated at the shape q contracted with the vectors v_m and v_n. A formula for $\mathcal{F}_{mn}$ is

$$\mathcal{F}_{mn} = A([v_n^h, v_m^h]),$$

where

$$[v_n^h, v_m^h] = (v_n^h \cdot \nabla)\hat{v}^h_{m_{|\text{shape}}} - (v_m^h \cdot \nabla)\hat{v}^h_{n_{|\text{shape}}}$$

is the Lie bracket. The hat indicates the Stokes' extension of the boundary condition to the fluid; the fluid response in a neighborhood of the boundary is necessary in order to compute the derivatives. The superscript h denotes "horizontal projection"—which in practice means subtracting $A_q(v)$ from the input boundary conditions so that their Stokes' extensions lead to no net force or torque on the fluid.

Once we have the components of the curvature calculated at a particular shape we can approximate the connection form in a neighborhood of that shape. Let a_m be the coordinates associated to the v_m. A boundary condition can then be written $v = \sum a_m v_m$, and the a_m are to be thought of as the amplitudes. Then,

$$\begin{aligned}\mathcal{F}_{|S} &= \sum_{m<n} \mathcal{F}_{mn} da_m \wedge da_n \\ &= \sum_{m<n} d(a_m \mathcal{F}_{mn} da_n).\end{aligned}$$

Hence $A \cong \sum a_m \mathcal{F}_{mn} da_n$ + exact. So if a swimming motion is gauge-parameterized by

$$s(t) = q + \sum a_n(t) v_n,$$

where q is a given located shape, then substituting the approximation for A into formula (9), we obtain an approximation for the net motion associated to the periodic swimming stroke:

$$\bar{P} \exp \int_0^1 A(t)\, dt = I + \int \sum_{m<n} \mathcal{F}_{mn} a_m \dot{a}_n\, dt + O(|a|^3).$$

The "Curvature Approximation Formula"

(Shapere/Wilczek)

Concerns small deformations of an ``average shape''

(or envelope model)

- **Geometric thinking organizes the Stokes flows calculations.**

(Lie brackets "lurk" in the papers by the founding fathers)

Optimization

Min P , subject to prescribed F

For small deformations of amplitude ε

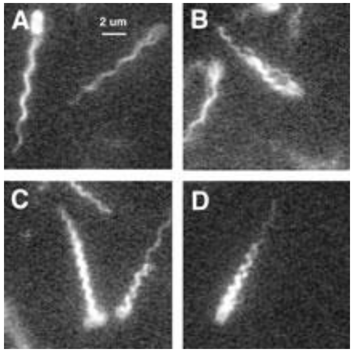
of an ``average shape'' s_o

We get a linear algebra problem (in infinite dimension)

Some efficiency concepts were given in Shapere/Wilczek (others recently)

See discussion in JK/J.Delgado on "Pareto optimization".

Purcellian Mechanics



1. Cell & flagellum Bacterial motor (Berg)

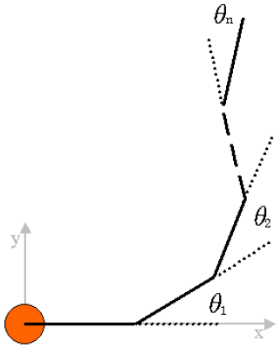
2. Two linked swimmer

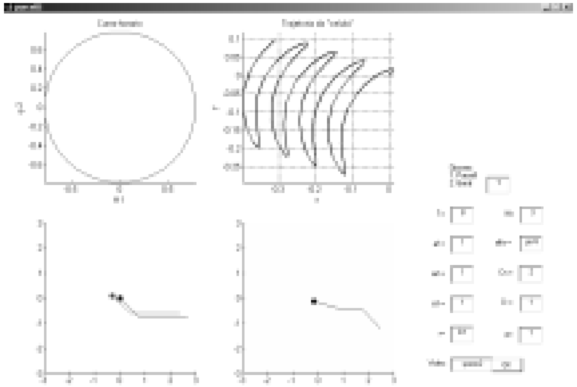
(groups of Peko Hosoi , Howard Stone , Greg Huber ...)

One of the fundamental axiom: for rods $F_\perp = 2 F_\parallel$

3. Axiomatization: resistance matrices add; equivariance

Geometric Mechanics of N-linked Swimmers (with Gerusa Araujo)





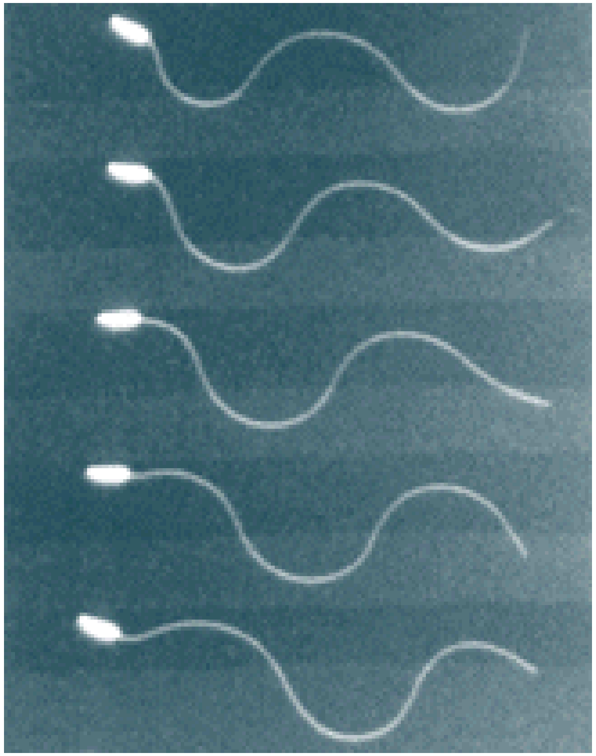
Problem 1 (hopefully not too hard).

Apply Pontryagin, what are the optimal patterns?

... and in the continuous limit?

do you get progressive waves

of arcs of circles from base to tip ?



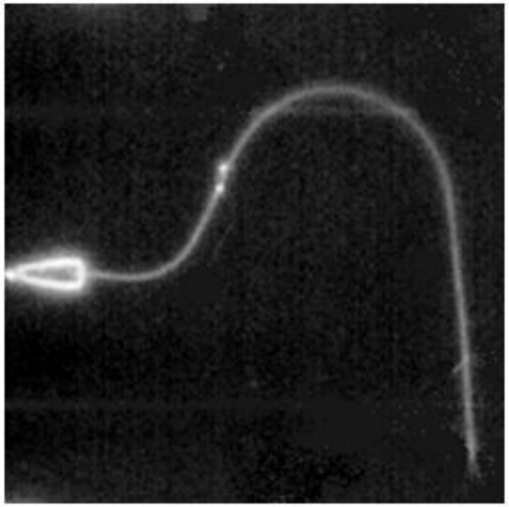
The holy grail: how molecular motors (dyneyn) act ?

Problem 2 (hard):   **Incorporate internal forces in the modeling.**

Use a Geometric Mechanics approach to organize the analysis.

Can you infer what are the internal forces from the movies?

Start with internal force fields with biological interpretation;   how to they relate with the stress tensor at the solid fluid contact?



C.Brokaw

H.Gadelha

Comment :   **immersed boundary method**

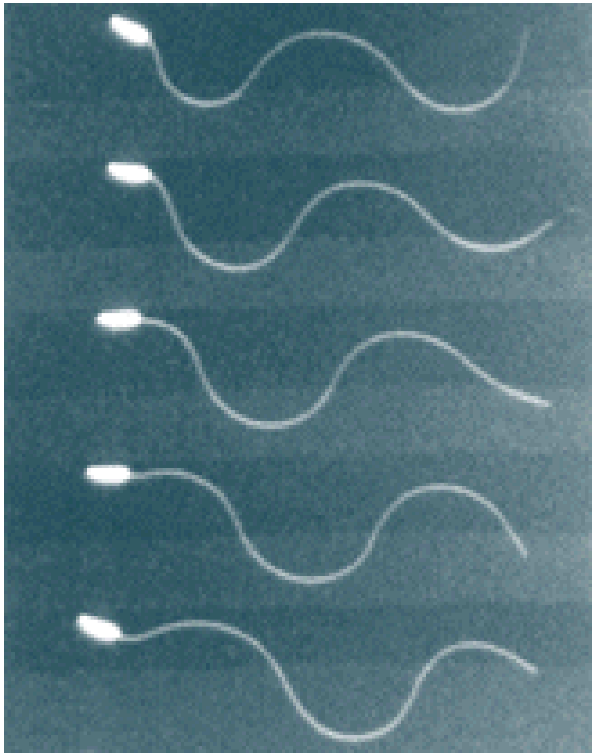
internal dissipation

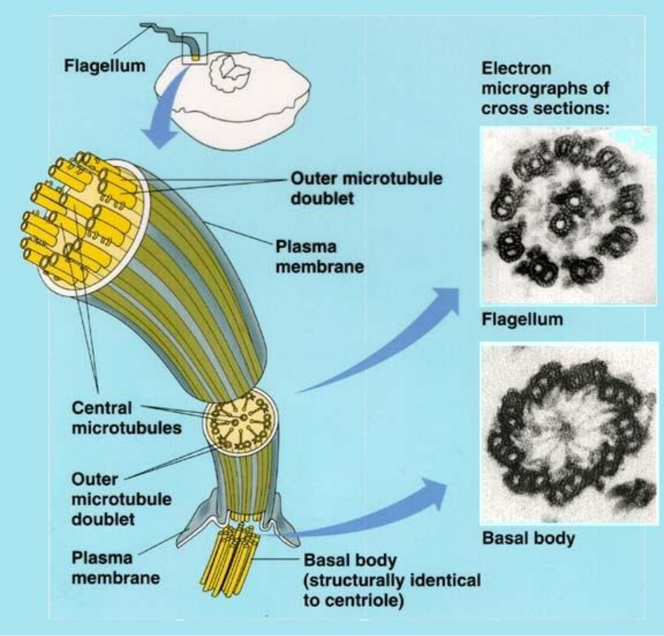
How to start this program?   **Lighthill**   **(J. Eng. Math. , 1996)**

Distributed molecular motors of the eukariote flagella

[CharlesBrokaw]() [Microtubule sliding]()

[Bending patterns]()

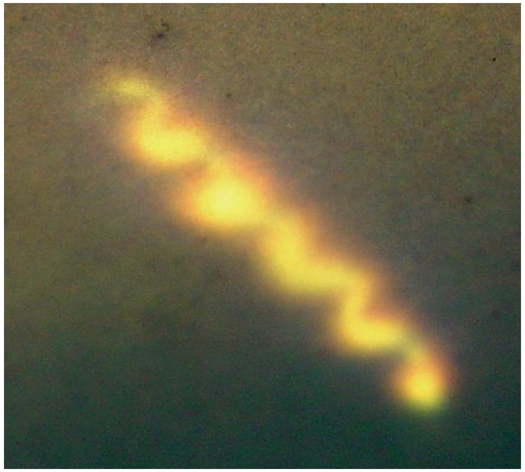




Question: are optimal patterns waves formed by arcs of circles?

(or near to)

Spirochetes revisited

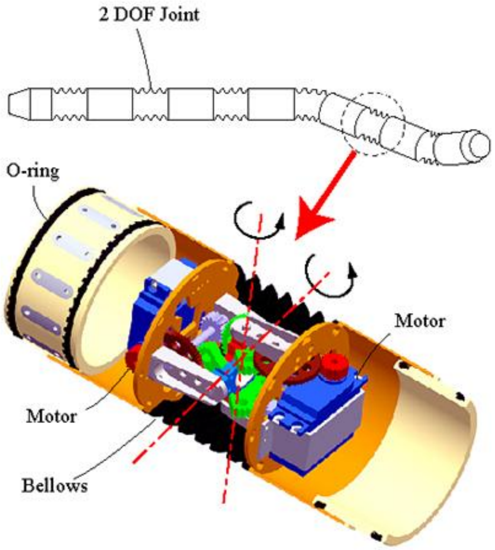


Problem 3 (defy the experts!)

A 6 pack of beer offered !!

Encapsulated propulsion mechanisms

Myxo



Internal helical flagellum

Spirochete on a box

Would you like to swallow this?

Hirose lab movie

Swimming in “Fatland” (S^n , $n = 2$ or 3)

Problem 4*: Topology matters?

*** Another six-pack of beer for the first answer - does not need to be correct.**

Swimming in "Fatland" (S^n , n = 2 or 3)

Topology matters? Does it prevent swimming?

[I hope not. **Rewrite the connection condition in terms of aMomentum map**

$$J: T^*Q \rightarrow g^*$$

Q = { embeddings of reference body B in S^n },

G = SO(n) acting in targets ∑ = q(B) by rigid motion

Identify T Q ≡ T*Q by Power metric

Microswimming is just like the cart flip: **J = 0 .]**

Final Remarks I. Analysis /numerics: tools for Stokes flows

biharmonic equation / Darboux representation (2d)

slender body approximations (Lighthill)

(multi)pole collocation methods (Wu, Weinbaum, ...)

regularized Stokeslets (Cortez)

immersed boundary method (Peskin)

boundary integral methods (Pozrikidis)

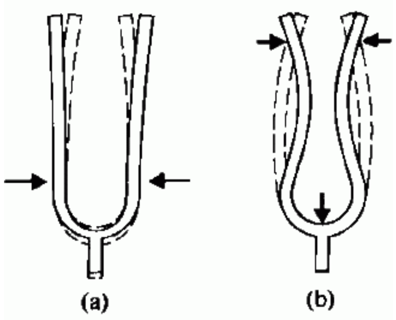
....

(we will discuss none of them!)

ESCAPE ROUTE: Taylor waving sheet + tangent plane approximation

Final Remark II. Can you one-up the Scallop Paradox?

Tuning fork in molasses



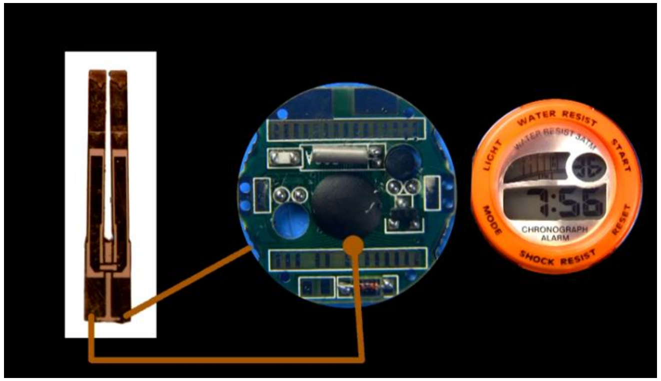
Acoustic streaming

Play a guitar under water:

For MEMS devices

Quartz tuning fork

(cost: $ 10)



just a cookie:

"Snapping shrimp"
(Detlef Lohse)

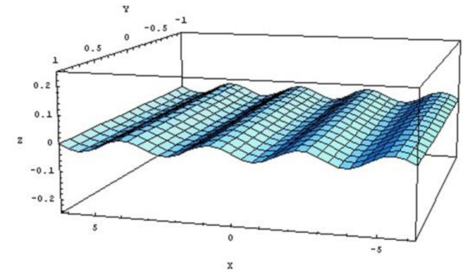
Final Remark III (final) Taylor's waving sheet and the tangent plane approximation

TPA: Analogous to the planar wave superposition for Laplacian operator.

Taylor's waving sheet

Taylor paper

Kurt Ehlers 8th order



Application: Synechococcus locomotion

Acoustic streaming? 2.5 more efficient

Taylor waving sheet revisited recently!!

Wu (not so recent, 1961)

Kozlov-Ramodanov (2002, potential flows + "recoil")

Kozlov-Onischenko (2004)

Childress IMAtalk SC-Spag-Tokieda (all Reynolds + "recoil")

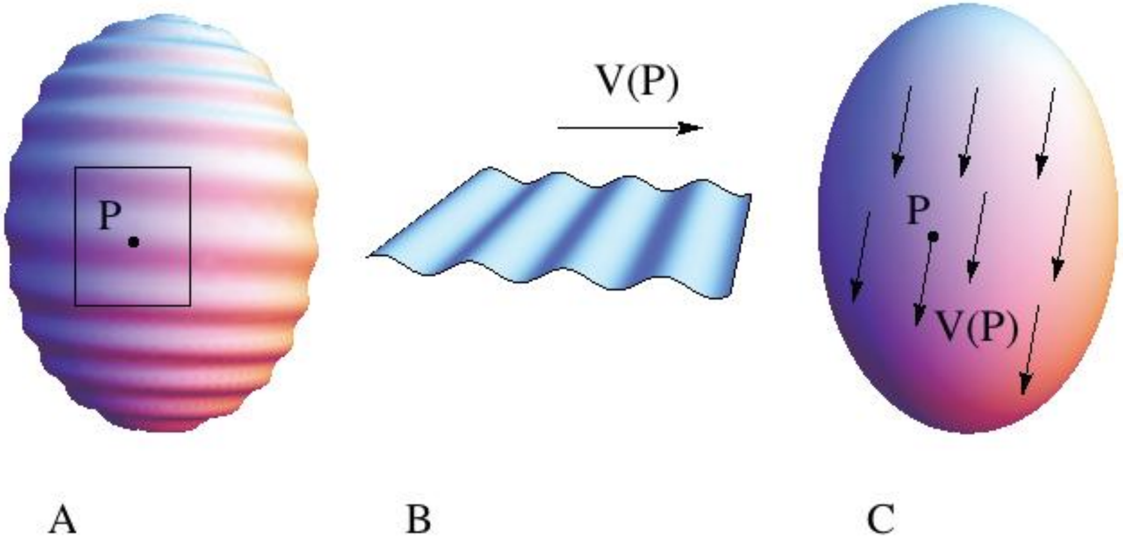
Eric Lauga (transient) phaselocking (cooperation) higher order

Kurt calculations Question: why only even terms are present?

Annete Hosoi-Wilkening Annete-Chan

The tangent plane approximation

Can be used on problems where a "local" wave can be identified .

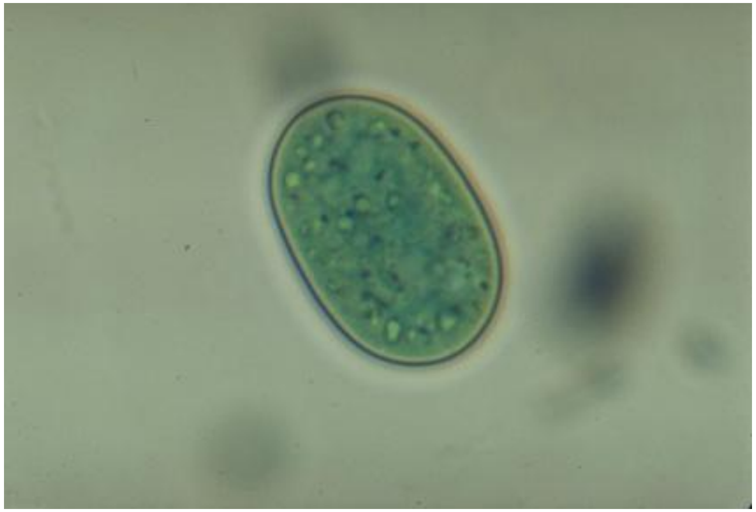


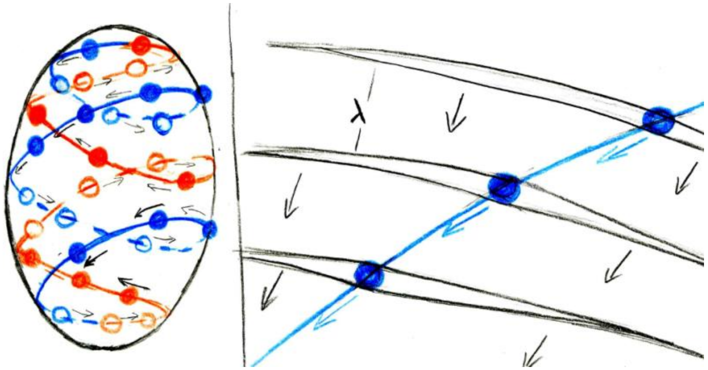
Helical suface waves may explain the mystery of Synechococcus swimming

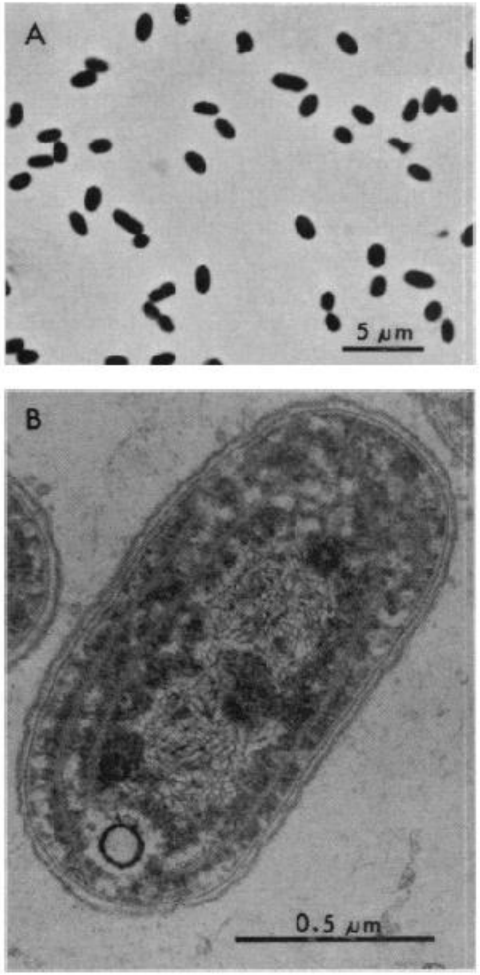
Our recent work

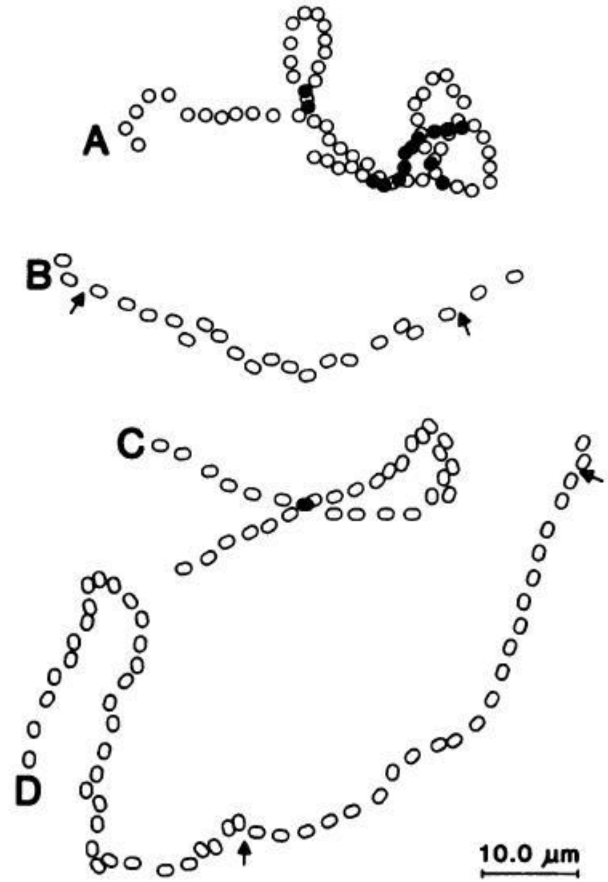
(JK, Kurt Ehlers) (KE, G.Oster)

spheroid.mov









The mysterious open sea swimmer Synechococcus

(Waterbury et al., 1985)

Food for thought.

Research on autonomous micro swimming devices is attracting great interest due to their potential for medical and industrial applications.

Most proposals are inspired by bacteria with external flagella.

Could micro-robots driven by internal mechanisms be competitive? **Synmov1** **(Berg)**

Photo by George Bergman.

Jerry Marsden, Berkeley, 1997.

Thank you, Jerry!