Integrable Systems, gradient flows and dissipation

Anthony Bloch
(Recent work with Morrison and Ratiu)

- **Toda and gradient flows**
- **Normal and Kahler metrics**
- **PDE's on S^1 and Loop groups**
- **Metriplectic Flows**
- **Double bracket dissipation (w. Jerry, Krishna, Tudor)**

Toda Flow:

$$\dot{X} = [X, \Pi_S X]$$

Double Bracket Flow:

$$\dot{X} = [X, [X, N]]$$

– gradient but special case yields Toda. (with Brockett and Ratiu)

$$\dot{P} = [P, [P, \Lambda]$$

(Bloch, Bloch, Flashcka and Ratiu, Total Least Squares).

Heat equation

$$u_t = u_{xx}$$

Kahler flow:

$$u_t = (-\Delta)^{1/2} u$$

(with Morrison and Ratiu)

Dispersionless Toda flow

$$\dot{x} = \{x, \{x, z\}\}$$

(Bloch, Flaschka, Ratiu)

An important and beautiful mechanical system that describes the interaction of particles on the line (i.e., in one dimension) is the Toda lattice. We shall describe the nonperiodic finite Toda lattice following the treatment of Moser.

This is a key example in integrable systems theory.

The model consists of n particles moving freely on the x-axis and interacting under an exponential potential. Denoting the position of the kth particle by x_k, the Hamiltonian is given by

$$H(x, y) = \frac{1}{2} \sum_{k=1}^{n} y_k^2 + \sum_{k=1}^{n-1} e^{(x_k - x_{k+1})}.$$

The associated Hamiltonian equations are

$$\dot{x}_k = \frac{\partial H}{\partial y_k} = y_k \,, \tag{0.1}$$

$$\dot{y}_k = -\frac{\partial H}{\partial x_k} = e^{x_{k-1}-x_k} - e^{x_k - x_{k+1}} \,, \tag{0.2}$$

where we use the convention $e^{x_0 - x_1} = e^{x_n - x_{n+1}} = 0$**, which corresponds to formally setting** $x_0 = -\infty$ **and** $x_{n+1} = +\infty$**.**

This system of equations has an extraordinarily rich structure. Part of this is revealed by Flaschka's (Flaschka 1974) change of variables given by

$$a_k = \frac{1}{2} e^{(x_k - x_{k+1})/2} \quad \textbf{and} \quad b_k = -\frac{1}{2} y_k \,. \tag{0.3}$$

In these new variables, the equations of motion then become

$$\dot{a}_k = a_k(b_{k+1} - b_k)\,, \qquad k = 1, \ldots, n-1\,, \tag{0.4}$$
$$\dot{b}_k = 2(a_k^2 - a_{k-1}^2)\,, \qquad k = 1, \ldots, n\,, \tag{0.5}$$

with the boundary conditions $a_0 = a_n = 0$. This system may be written in the following Lax pair representation:

$$\frac{d}{dt}L = [B, L] = BL - LB, \tag{0.6}$$

where

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ & & \ddots & & \\ & & & b_{n-1} & a_{n-1} \\ 0 & & & a_{n-1} & b_n \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & 0 \\ & & \ddots & & \\ & & & 0 & a_{n-1} \\ 0 & & & -a_{n-1} & 0 \end{pmatrix}.$$

Can show system is integrable.

More structure in this example. For instance, if N is the matrix $\mathrm{diag}[1, 2, \ldots, n]$, the Toda flow (0.6) may be written in the following double bracket form:

$$\dot{L} = [L, [L, N]] \,. \tag{0.7}$$

See Bloch [1990], Bloch, Brockett and Ratiu [1990], and Bloch, Flaschka and Ratiu [1990]. This double bracket equation restricted to a level set of the integrals is in fact the gradient flow of the function $\mathrm{Tr} LN$ with respect to the so-called normal metric.

From this observation it is easy to show that the flow tends asymptotically to a diagonal matrix with the eigenvalues of $L(0)$ on the diagonal and ordered according to magnitude, recovering the observation of Moser, Symes.

• **Four-Dimensional Toda. Here we simulate the Toda lattice in four dimensions. The Hamiltonian is**

$$H(a,b) = a_1^2 + a_2^2 + b_1^2 + b_2^2 + b_1 b_2 \,. \tag{0.8}$$

and one has the equations of motion

$$\begin{aligned} \dot a_1 &= -a_1(b_1 - b_2) & \dot b_1 &= 2a_1^2 \,, \\ \dot a_2 &= -a_2(b_1 + 2b_2) & \dot b_2 &= -2(a_1^2 - a_2^2) \,. \end{aligned} \tag{0.9}$$

(setting $b_1 + b_2 + b_3 = 0$, for convenience, which we may do since the trace is preserved along the flow). In particular, $\mathrm{Trace}\, LN$ is, in this case, equal to b_2 and can be checked to decrease along the flow.

Figure 0.1 exhibits the asymptotic behavior of the Toda flow.

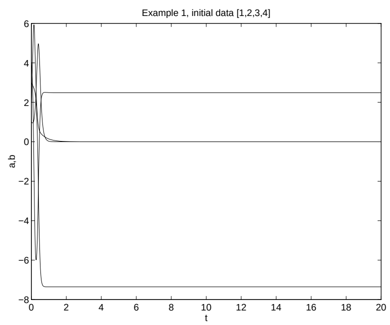

Figure 0.1: Asymptotic behavior of the solutions of the four-dimensional Toda lattice.

It is also of interest to note that the Toda flow may be written as a different double bracket flow on the space of rank one projection matrices. The idea is to represent the flow in the variables $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ where the λ_i are the (conserved) eigenvalues of L and r_i, $\sum_i r_i^2 = 1$ are the top components of the normalized eiqenvectors of L (see Moser). Then one can show (Bloch (1990)) that the flow may be written as

$$\dot{P} = [P, [P, \Lambda]] \tag{0.10}$$

where $P = \mathbf{r}\mathbf{r}^T$ and $\Lambda = \mathbf{diag}(\lambda)$.

This flow is a flow on a simplex The Toda flow in its original variables can also be mapped to a flow convex polytope (see Bloch, Brockett and Ratiu, Bloch, Flaschka and Ratiu).

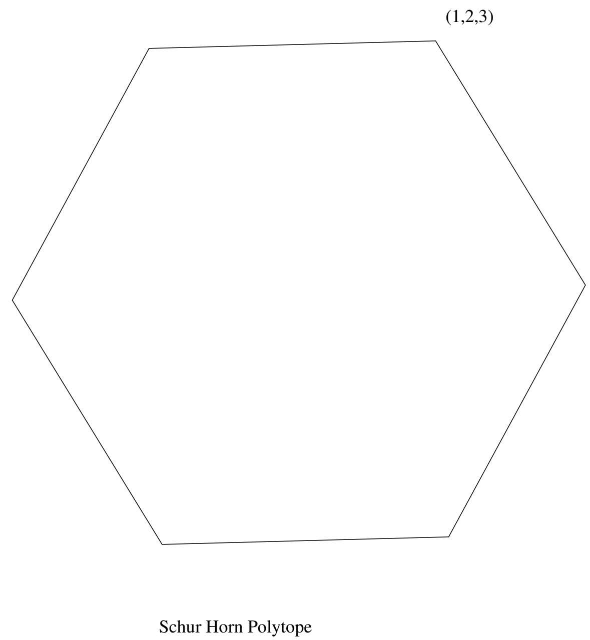

Figure 0.2: Image of Toda Flow

Metrics on finite-dimensional orbits

Let $\mathfrak{g}_u$ be the compact real form of a complex semisimple Lie algebra $\mathfrak{g}$ and consider the flow on an adjoint orbit of $\mathfrak{g}_u$ given by

$$\dot{L}(t) = [L(t), [L(t), N]] \,. \tag{0.11}$$

Consider the gradient flow with respect to the "normal" metric (see Atiyah). Explicitly this metric is given as follows.

Decompose orthogonally, relative to $-\kappa(\ ,\ ) = \langle\ ,\ \rangle$, $\mathfrak{g}_u = \mathfrak{g}_u^L \oplus \mathfrak{g}_{uL}$ where $\mathfrak{g}_{uL}$ is the centralizer of L and $\mathfrak{g}_u^L =$ Im $ad\, L$. For $X \in \mathfrak{g}_u$ denote by $X^L \in \mathfrak{g}_u^L$ the orthogonal projection of X on $\mathfrak{g}_u^L$. Then set the inner product of the tangent vectors $[L, X]$ and $[L, Y]$ to be equal to $\langle X^L, Y^L\rangle$. Denote this metric by $\langle\ ,\ \rangle_N$. Then we have

Proposition 0.1. *The flow (0.11) is the gradient vector field of $H(L) = \kappa(L, N)$, κ the Killing form, on the adjoint orbit $\mathcal{O}$ of $\mathfrak{g}_u$ containing the initial condition $L(0) = L_0$, with respect to the normal metric $\langle\ \ ,\ \ \rangle_N$ on $\mathcal{O}$.*

Proof. We have, by the definition of the gradient,

$$dH \cdot [L, \delta L] = \langle \operatorname{grad} H, [L, \delta L] \rangle_N \tag{0.12}$$

where $\cdot$ denotes the natural pairing between 1-forms and tangent vectors and $[L, \delta L]$ is a tangent vector at L. Set $\operatorname{grad} H = [L, X]$. Then (0.12) becomes

$$-\langle [L, \delta L], N \rangle = \langle [L, X], [L, \delta L] \rangle_N$$

or

$$\langle [L, N], \delta L \rangle = \langle X^L, \delta L^L \rangle \, .$$

Thus

$$X^L = ([L, N])^L = [L, N]$$

and

$$\operatorname{grad} H = [L, [L, N]]$$

as required. ■

For L and N as above obtain the Toda lattice flow. Full Toda may be also obtained with a modified metric.

Now in addition to the normal metric on an orbit there exist two other natural metrics, the induced and Kahler metrics.

• There is the natural metric b on G/T induced from the invariant metric on the Lie algebra –this is the induced metric.

• There is the normal metric described above which, following Atiyah we call b_1, which comes from viewing G/T as an adjoint orbit.

• Finally identifying the adjoint orbit with a coadjoint orbit we obtain the Kostant Kirilov symplectic structure which, together the fact that G/T is a complex manifold defines a Kahler metric b_2.

If we define b_1 and b_2 in terms of positive self-adjoint operators A_1 and A_2, $A_1 = A_2^2$. In fact b is just $Tr(AB)$, b_1 is $Tr(A^L B^L)$ and b_2 is essentially the square root of b_1.

Gradient flows on the loop group of the circle

Recall that the loop group $\widetilde{\mathrm{L}}(S^1)$ **of the circle** S^1 **consists of smooth maps of** S^1 **to** S^1**. With pointwise multiplication,** $\widetilde{\mathrm{L}}(S^1)$ **is a commutative group. Often, elements of** $\widetilde{\mathrm{L}}(S^1)$ **are written as** $e^{\mathrm{i}f}$**, where** $f \in \widetilde{\mathrm{L}}(\mathbb{R}) := \{g : [-\pi, \pi] \to \mathbb{R} \mid g$ **is** $C^\infty,\ g(\pi) = g(-\pi) + 2n\pi$**, for some** $n \in \mathbb{Z}\}$**;** n **is the** ***winding number*** **of the closed curve** $[-\pi, \pi] \ni t \mapsto e^{\mathrm{i}g(t)} \in S^1$ **about the origin.**

The based loop group of S^1. The inner product on the Hilbert space $L^2(S^1)$ of L^2 real valued functions on S^1 is defined by

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)g(\theta)\mathrm{d}\theta, \qquad f, g \in L^2(S^1).$$

We introduce the closed Hilbert Lie subgroup $\mathrm{L}(S^1) := \{\varphi \in \widetilde{\mathrm{L}}(S^1) \mid \varphi(1) = 1\}$ of $\widetilde{\mathrm{L}}(S^1)$ whose closed commutative Hilbert Lie algebra is $\mathrm{L}(\mathbb{R}) := \{u \in H^s(S^1, \mathbb{R}) \mid u(1) = 0\}$. The exponential map $\exp : \mathrm{L}(\mathbb{R}) \ni u \mapsto e^{\mathrm{i}u} \in \mathrm{L}(S^1)$ is a Lie group isomorphism (with $\mathrm{L}(\mathbb{R})$ thought of as a commutative group relative to addition).

There is a natural 2-cocycle ω on $\mathrm{L}(\mathbb{R})$, namely

$$\omega(u,v) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta\, u'(\theta)v(\theta) = \langle u', v\rangle \ , \tag{0.13}$$

where $u' := du/d\theta$. Therefore, there is a central extension of Lie algebras

$$0 \longrightarrow \mathbb{R} \longrightarrow \widehat{\mathrm{L}(\mathbb{R})} \longrightarrow \mathrm{L}(\mathbb{R}) \longrightarrow 0$$

which, integrates to a central extension of Lie groups

$$1 \longrightarrow S^1 \longrightarrow \widehat{\mathrm{L}(S^1)} \longrightarrow \mathrm{L}(S^1) \longrightarrow 1.$$

The "geometric duals" of $\mathrm{L}(\mathbb{R})$ and $\widehat{\mathrm{L}(\mathbb{R})} = \mathbb{R}\oplus\mathrm{L}(\mathbb{R})$ are themselves, relative to the weak L^2-pairing.

The coadjoint action of $\widehat{\mathrm{L}(S^1)}$ on $\widehat{\mathrm{L}(\mathbb{R})}$ preserves $\{1\} \oplus \mathrm{L}(\mathbb{R})$ so that, as usual, the coadjoint action of $\widehat{\mathrm{L}(S^1)}$ on $\mathrm{L}(\mathbb{R})$ is an affine action which, in this case, because the group is commutative, equals

$$\mathrm{Ad}^*_{e^{\mathrm{i}f}}\,\mu = \frac{f'}{f} = (\log|f|)' \qquad e^{\mathrm{i}f} \in \mathrm{L}(S^1), \qquad \mu \in \mathrm{L}(\mathbb{R}).$$

Thus, the orbit of the constant function 0 is $\widehat{\mathrm{L}(S^1)}/S^1$ (where the denominator is thought of as constant loops), i.e., it equals $\mathrm{L}(S^1)$. Therefore, every element $u \in \mathrm{L}(\mathbb{R})$ of its Lie algebra has vanishing zero order Fourier coefficient , i.e., $\widehat{u}(0) = 0$.

Thus, the based loop group is a coadjoint orbit of its natural central extension has three distinguished weak Riemannian metrics.

Now we introduce the *Hilbert transform* on the circle

$$\mathcal{H}u(\theta) := \frac{1}{2\pi}\fint_{-\pi}^{\pi} u(s)\cot\frac{\theta - s}{2}\mathrm{d}s \;=\; \frac{1}{2\pi}\fint_{-\pi}^{\pi} u(\theta - s)\cot\frac{s}{2}\mathrm{d}s$$
$$:= \lim_{\varepsilon\to 0+}\frac{1}{\pi}\int_{\varepsilon\le|s|\le\pi} u(\theta - s)\cot\frac{s}{2}\mathrm{d}s$$

for any $u \in L^2(S^1)$, **where** $\fint$ **denotes the Cauchy principal value.**

• **If** $u(\theta) = \sum_{n=-\infty}^{\infty}\widehat{u}(n)e^{\mathrm{i}n\theta} \in L^2(S^1)$, **where** $\widehat{u}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\theta)e^{-\mathrm{i}n\theta}\mathrm{d}\theta$, **so** $\overline{\widehat{u}(n)} = \widehat{u}(-n)$ **since** u **is real valued, then**

$$\mathcal{H}u(\theta) = -\mathrm{i}\sum_{n=-\infty}^{\infty}\widehat{u}(n)(\operatorname{sign} n)e^{\mathrm{i}n\theta} \in L^2(S^1) \tag{0.14}$$

which follows from the identity $\widehat{\mathcal{H}f}(n) = -\mathrm{i}\widehat{f}(n)(\operatorname{sign} n)$

Can show

$$\mathcal{H}u'(\theta) = (\mathcal{H}u)'(\theta) = \left(-\mathrm{i}\sum_{n=-\infty}^{\infty} \widehat{u}(n)(\operatorname{sign} n)e^{\mathrm{i}n\theta}\right)' = \sum_{n=-\infty}^{\infty} |n|\widehat{u}(n)e^{in\theta}\,. \tag{0.15}$$

On the other hand, if $v \in H^2(S^1)$, then

$$-\frac{d^2}{d\theta^2}v(\theta) = \sum_{n=-\infty}^{\infty} n^2\widehat{v}(n)e^{in\theta} \tag{0.16}$$

and hence if $u \in H^1(S^1)$,

$$\left(-\frac{d^2}{d\theta^2}\right)^{\frac{1}{2}} u(\theta) = \sum_{n=-\infty}^{\infty} |n|\widehat{u}(n)e^{in\theta} = \mathcal{H}u'(\theta) = \left(\left(\mathcal{H}\circ\frac{d}{d\theta}\right)u\right)(\theta) \tag{0.17}$$

by (0.15).

$\mathcal{H}\left(\mathrm{L}(\mathbb{R})\right) \subseteq \mathrm{L}(\mathbb{R})$, $\mathcal{H}$ **is unitary on** $\mathrm{L}(\mathbb{R})$ **(relative to the** H^s**-inner product),** $\mathcal{H} \circ \mathcal{H} = -I$ **on** $\mathrm{L}(\mathbb{R})$**. Concretely, the Hilbert transform on** $\mathrm{L}(\mathbb{R})$ **has the form:**

$$u(\theta) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \widehat{u}(n) e^{\mathrm{i}n\theta} \in \mathrm{L}(\mathbb{R}) \quad \Longrightarrow \quad \mathcal{H}u(\theta) = -\mathrm{i} \sum_{n \in \mathbb{Z} \setminus \{0\}} \widehat{u}(n)(\operatorname{sign} n) e^{\mathrm{i}n\theta}$$

Thus, $\mathcal{H}$ **defines the structure of a complex Hilbert space on** $\mathrm{L}(\mathbb{R})$**, relative to the** H^s **inner product,** $s \geq 1$**. Hence, translating** $\mathcal{H}$ **to any tangent space of** $\mathrm{L}(S^1)$**, we obtain an invariant almost complex structure on the Hilbert Lie group** $\mathrm{L}(S^1)$ **which is, in fact, a complex structure.**

Finally, $\mathrm{L}(S^1)$ is a Kähler manifold. This is immediately seen by noting that

$$g(1)(u,v) := \omega(\mathcal{H}u, v) = \sum_{n=-\infty}^{\infty} |n|\widehat{u}(n)\widehat{v}(n) \tag{0.18}$$

is symmetric and positive definite and so, by translations, defines a weak Riemannian metric on $\mathrm{L}(S^1)$. Note that this metric is *not* the H^s metric for any $s \geq 1$. In fact, the metric g is incomplete, whereas the H^s metric is complete.

Weak Riemannian metrics on $\mathrm{L}(S^1)$. The three metrics for $\mathrm{L}(S^1)$ viewed as a coadjoint orbit of its central extension are as follows.

The *induced metric* is defined by the natural inner product on $\mathrm{L}(\mathbb{R})$, which is the usual L^2-inner product. Hence, the induced metric is obtained by left (equivalently, right) translation of the inner product

$$b(1)(u,v) := \langle u,v\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)v(t)\mathrm{d}t \tag{0.19}$$

for any two functions $u, v \in \mathrm{L}(\mathbb{R})$.

Define the following inner products on $\mathrm{L}(\mathbb{R})$:

$$b_2(1)(u,v) := b(1)(u,\mathcal{H}v') = \langle u,\mathcal{H}v'\rangle\,, \quad \textbf{if} \quad u,v \in H^s(S^1),\ s \geq 1 \tag{0.20}$$

$$b_1(1)(u,v) := b(1)(u',v') = \langle u',v'\rangle\,, \quad \textbf{if} \quad u,v \in H^s(S^1),\ s \geq 1. \tag{0.21}$$

Bilinearity and symmetry of $b_1(1)$ and $b_2(1)$ are obvious. Also

$$b_1(1)(u,u) = \sum_{n=-\infty}^{\infty} n^2 |\widehat{u}(n)|^2 \geq 0.$$

In addition, $b_1(1)(u,u) = 0$ if and only if $\widehat{u}(n) = 0$ for all $n \neq 0$, i.e., $u(\theta) = \widehat{u}(0) = 0$. This shows that $b_1(1)$ is indeed an inner product on $\mathrm{L}(\mathbb{R})$ which coincides with the H^1 inner product. Hence, if $\mathrm{L}(\mathbb{R})$ is endowed with the H^s topology for $s \geq 1$, this inner product is strong if $s = 1$ and weak if $s > 1$. Left translating this inner product to any tangent space of $\mathrm{L}(S^1)$ (endowed with the H^s topology for $s \geq 1$), yields a Riemannian metric on $\mathrm{L}(S^1)$ which is strong for $s = 1$ and weak for $s > 1$.

This Riemannian metric is the the *normal metric* on $\mathrm{L}(S^1)$.

The inner product $b_2(1)$ is identical to $g(1)$ by (0.20). Thus, translating this inner product to the tangent space at every point of the Hilbert Lie group $\mathrm{L}(S^1)$, yields the *standard Kähler metric* $b_2 = g$ on $\mathrm{L}(S^1)$, endowed with the H^s topology for $s \geq 1$. Note that if $u \in \mathrm{L}(S^1)$, then

$$b_2(1)(u, u) = \sum_{n=-\infty}^{\infty} |n||\widehat{u}(n)|^2$$

which shows that the Kähler metric b_2 coincides with the $H^{1/2}$ metric and is, therefore, a weak metric on $\mathrm{L}(S^1)$.

The gradient vector fields in the three metrics of $\mathrm{L}(S^1)$. We compute now the gradients of a specific function the three metrics.

Theorem 0.2. *The gradients of the smooth function $H : \mathrm{L}(S^1) \to \mathbb{R}$ given by*

$$H\left(e^{\mathrm{i}f}\right) = \frac{1}{4\pi}\int_{-\pi}^{\pi} f'(\theta)^2 \mathrm{d}\theta$$

are

(i) $\nabla^1 H\left(e^{\mathrm{i}f}\right) = f e^{\mathrm{i}f}$ *for the normal metric* b_1*;*

(ii) $\nabla H\left(e^{\mathrm{i}f}\right) = -f'' e^{\mathrm{i}f}$ *with respect to the induced metric* b *for* $f \in H^s(S^1)$ *with* $s \geq 2$*;*

(iii) $\nabla^2 H\left(e^{\mathrm{i}f}\right) = (\mathcal{H} f')e^{\mathrm{i}f}$ *with respect to the weak Kähler metric* b_2*.*

Since

$$\omega\left(e^{\mathrm{i}f}\right)\left(\mathcal{H}\nabla^2 H\left(e^{\mathrm{i}f}\right), ue^{\mathrm{i}f}\right) \overset{(0.27)}{=} b_2\left(e^{\mathrm{i}f}\right)\left(\nabla^2 H\left(e^{\mathrm{i}f}\right), ue^{\mathrm{i}f}\right) = \mathbf{d}H\left(e^{\mathrm{i}f}\right)\left(ue^{\mathrm{i}f}\right)$$

it follows that the Hamiltonian vector field on $\left(\mathrm{L}(S^1), \omega\right)$ for the function H is $X_H = \mathcal{H}\nabla^2 H$. Since $\mathcal{H}$ commutes with the tangent lift to group translations, Theorem 0.2(iii) implies that

$$X_H\left(e^{\mathrm{i}f}\right) = \left(\mathcal{H}\nabla^2 H\right)\left(e^{\mathrm{i}f}\right) = \mathcal{H}\left(\nabla^2 H\left(e^{\mathrm{i}f}\right)\right) = \mathcal{H}\left(\left(\mathcal{H}f'\right)e^{\mathrm{i}f}\right) = -f'e^{\mathrm{i}f}.$$

This proves the first part of the following statement.

Corollary 0.3. *The Hamiltonian vector field of H relative to the translation invariant symplectic form ω on $\mathrm{L}(S^1)$ whose value at the identity element is given by (0.13) has the expression $X_H\left(\mathrm{e}^{\mathrm{i}f}\right) = -f'\mathrm{e}^{\mathrm{i}f}$. Its flow is the rotation*

$$\left(F_t\left(\mathrm{e}^{\mathrm{i}f}\right)\right)(\theta) = e^{-\mathrm{i}(f(t+\theta)-f(t))}.$$

Theorem 0.4. *Let $H : \mathrm{L}(S^1) \to \mathbb{R}$ be a smooth function (with $\mathrm{L}(S^1)$ endowed, as usual, with the H^s topology for $s \geq 1$) and assume that the functional derivative $\delta H/\delta u \in \mathrm{L}(S^1)$ exists. Then the gradient vector fields are*

(i) $\nabla H(u) = \frac{\delta H}{\delta u}$ *with respect the weak inner product $b(1)$ defining the induced metric;*

(ii) $\left(\nabla^1 H(u)\right)(\theta) = -\int_0^\theta \left(\int_0^\varphi \frac{\delta H}{\delta u}(\psi)\mathrm{d}\psi\right)\mathrm{d}\varphi$ *with respect to the (weak) inner product $b_1(1)$ defining the normal metric, provided both $\int_0^\theta \frac{\delta H}{\delta u}(\varphi)\mathrm{d}\varphi$ and $\int_0^\theta \left(\int_0^\varphi \frac{\delta H}{\delta u}(\psi)\mathrm{d}\psi\right)\mathrm{d}\varphi$ are periodic;*

(iii) $\left(\nabla^2 H(u)\right)(\theta) = -\mathcal{H}\int_0^\theta \frac{\delta H}{\delta u}(\varphi)\mathrm{d}\varphi$ *wrt the weak inner product $b_2(1)$ defining the Kähler metric, provided $\int_0^\theta \frac{\delta H}{\delta u}(\varphi)\mathrm{d}\varphi$ is periodic.*

Vector fields on $\mathrm{L}(S^1)$ and $\mathrm{L}(\mathbb{R})$:

Note the exponential map $\exp : \mathrm{L}(\mathbb{R}) \ni u \mapsto e^{\mathrm{i}u} \in \mathrm{L}(S^1)$ **is a Lie group isomorphism**

Here, we identified the Lie algebra of S^1 with $\mathbb{R}$, even though, naturally, it is the imaginary axis, the tangent space at $1 \in S^1$ to S^1.

Proposition 0.5. *Let $X \in \mathfrak{X}(\mathrm{L}(\mathbb{R}))$ be an arbitrary vector field . Then its push-forward to $\mathrm{L}(S^1))$ has the expression*

$$(\exp_* X)\left(e^{\mathrm{i}u}\right) = X(u)e^{\mathrm{i}u}$$

for any $u \in \mathrm{L}(\mathbb{R})$.

Applying Proposition 0.5 to Theorem 0.2, we get the following result:

Corollary 0.6. *The three gradient vector fields for the smooth function* $H_1 : \mathrm{L}(\mathbb{R}) \to \mathbb{R}$ *given by*

$$H_1(u) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \, (u')^2$$

are

(i) $\nabla^1 H_1(u) = u$ *for the weak inner product* $b_1(1)$ *defining the normal metric;*

(ii) $\nabla H_1(u) = -u''$ *for the weak inner product* $b(1)$ *defining the induced metric, where for* $u \in H^s(\mathbb{R})$ *with* $s \geq 2$*;*

(iii) $\nabla^2 H_1(u) = (\mathcal{H} u')$ *for the weak inner product* $b_2(1)$ *defining the Kähler metric.*

Since the exponential map is a Lie group isomorphism and the three metrics coincide with the respective inner products at the identity, their left invariance guarantees that the three inner products on $\mathrm{L}(\mathbb{R})$ correspond to the three invariant metrics on $\mathrm{L}(S^1)$.

Applying Proposition 0.5 to Corollary 0.3, we conclude:

Corollary 0.7. *The Hamiltonian vector field of H_1 relative to the symplectic form ω given by (0.13) has the expression $X_H(u) = -u'$. Its flow is $(F_t(u))(\theta) = u(\theta - t)$.*

The verification of the statement about the flow is immediate:

$$\frac{d}{dt}(F_t(u))(\theta) = \frac{d}{dt}u(\theta - t) = -u'(\theta - t) = (X_H(F_t(u)))(\theta).$$

More stringent hypotheses on the functional give a more general result:

Theorem 0.8. *Let* $H : \mathrm{L}(S^1) \to \mathbb{R}$ *be a smooth function (with* $\mathrm{L}(S^1)$ *endowed, as usual, with the* H^s *topology for* $s \geq 1$*) and assume that the functional derivative* $\delta H/\delta u \in \mathrm{L}(S^1)$ *exists. Then the gradient vector fields are*

(i) $\nabla H(u) = \frac{\delta H}{\delta u}$ *with respect the weak inner product* $b(1)$ *defining the induced metric;*

(ii) $\left(\nabla^1 H(u)\right)(\theta) = -\int_0^\theta \mathrm{d}\varphi \left(\int_0^\varphi \mathrm{d}\psi \, \frac{\delta H}{\delta u}(\psi)\right)$ *with respect to the (weak) inner product* $b_1(1)$ *defining the normal metric, provided both* $\int_0^\theta \mathrm{d}\varphi \, \frac{\delta H}{\delta u}(\varphi)$ *and* $\int_0^\theta \mathrm{d}\varphi \left(\int_0^\varphi \mathrm{d}\psi \, \frac{\delta H}{\delta u}(\psi)\right)$ *are periodic;*

(iii) $\left(\nabla^2 H(u)\right)(\theta) = -\mathcal{H} \int_0^\theta \mathrm{d}\varphi \, \frac{\delta H}{\delta u}(\varphi)$ *wrt weak inner product* $b_2(1)$ *defining the Kähler metric, provided* $\int_0^\theta \mathrm{d}\varphi \, \frac{\delta H}{\delta u}(\varphi)$ *is periodic.*

Corollary 0.9. *Under the same hypothesis as in Theorem 0.8*(iii)*, the Hamiltonian vector field of the smooth function* $H : \mathrm{L}(S^1) \to \mathbb{R}$ *relative to the symplectic form* ω *on* $\mathrm{L}(\mathbb{R})$ *given by (0.13) has the expression* $X_H(u) = \int_0^\theta \mathrm{d}\varphi \, \frac{\delta H}{\delta u}(\varphi)$

The theorem can be applied to the functional H_1 in Corollary 0.6, but one needs additional smoothness. Indeed, the first thing to check is if this functional has a functional derivative. In fact, it does not, unless we assume that $u \in H^s(S^1)$ for $s \geq 2$, in which case we have

$$\mathbf{D}H_1(u) \cdot v = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}s\, u'(s) v'(s) == \langle -u'', v \rangle\,,$$

i.e., $\delta H / \delta u = -u''$. With this additional hypothesis, the gradient flow with respect to the weak inner product $b(1)$ defining the induced metric is given by $u_t = -u''$.

Similarly $\nabla^1 H(u) = u$.

The same situation occurs in the computation of the third gradient. In the hypotheses of the theorem, we have

$$\left(\nabla^2 H(u)\right)(\theta) = -\mathcal{H}\int_0^\theta \mathrm{d}\varphi\,\frac{\delta H}{\delta u}(\varphi) = \mathcal{H}(u' - u'(0)) = \mathcal{H}u'$$

because the Hilbert transform of a constant is zero. Thus, the gradient flow is given in this case by

$$u_t = \mathcal{H}u' \overset{(0.17)}{=} \left(-\frac{d^2}{d\theta^2}\right)^{\frac{1}{2}} u.$$

Symplectic structures on loop groups: The periodic Korteweg-de Vries (KdV) equation

$$u_t - 6uu_\theta + u_{\theta\theta\theta} = 0, \tag{0.22}$$

where $u(t, \theta)$ is a real valued function of $t \in \mathbb{R}$ and $\theta \in [-\pi, \pi]$, periodic in θ, and $u_\theta := \partial u / \partial \theta$. The KdV equation is, of course, a most famous integrable Hamiltonian system. It is Hamiltonian on the Poisson manifold of all periodic functions relative to the Gardner bracket

$$\{F, G\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta \, \frac{\delta F}{\delta u} \frac{d}{d\theta} \frac{\delta G}{\delta u} \,, \tag{0.23}$$

where

$$F(u) = \int_{S^1} \mathrm{d}\theta \, f(u, u_\theta, u_{\theta\theta}, \ldots)$$

and similarly for G.

The functional derivative $\delta F/\delta u$ is the usual one relative to the $L^2(S^1)$ inner product, i.e.,

$$\frac{\delta F}{\delta u} = \frac{\partial f}{\partial u} - \frac{d}{d\theta}\left(\frac{\partial f}{\partial u_\theta}\right) + \frac{d^2}{d\theta^2}\left(\frac{\partial f}{\partial u_{\theta\theta}}\right) - \cdots .$$

The Hamiltonian vector field of $H(u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta\, h(u, u_\theta, u_{\theta\theta}, \ldots)$ has the expression

$$X_H(u) = \frac{d}{d\theta}\left(\frac{\delta H}{\delta u}\right).$$

For the KdV equation one takes

$$H(u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta \, \left(2u^3 + \frac{1}{2}u_\theta^2\right). \tag{0.24}$$

The Casimir functions of the Gardner bracket are all smooth functionals C for which $\delta C/\delta u = c$ is a constant function, i.e.,

$$C(u) = \langle c, u \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta\, c u(\theta) = c\widehat{u}(0).$$

Thus $C^{-1}(0)$ is a candidate weak symplectic leaf in the phase space of all periodic functions. The situation in infinite dimensions is not as clear as in finite dimensions, where this would be a conclusion, because there is no general stratification theorem and one cannot expect, in general, more than a weak symplectic form. However, in our case, this actually holds.

This immediately shows that there is a tight relationship with the symplectic form ω of the complex Hilbert space $\mathrm{L}(R^1)$ namely

$$\sigma\left(\frac{d^2}{d\theta^2}u, v\right) = \omega(u, v)$$

for all $u, v \in \mathrm{L}(S^1)$ of class H^s, $s \geq 2$. Defining

$$\left(\frac{d}{d\theta}\right)^{-1} u := \int_0^\theta \mathrm{d}\varphi \, u(\varphi),$$

the KdV symplectic form σ has the suggestive expression

$$\sigma(u_1, u_2) = \left\langle \left(\frac{d}{d\theta}\right)^{-1} u_1, u_2 \right\rangle ,$$

which is well defined on $H^{-\frac{1}{2}}(S^1, \mathbb{R})$.

On the other hand, the Poisson bracket given by the Kähler symplectic form on $\mathrm{L}(S^1)$ is

$$\{F, G\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta \, \frac{\delta F}{\delta u} \left(\frac{d}{d\theta}\right)^{-1} \frac{\delta G}{\delta u}\,, \tag{0.25}$$

which is similarly well defined on $H^{-\frac{1}{2}}$, and the Hamiltonian vector field defined by this bracket is given by Corollary 0.9, i.e.,

$$u_t = X_H(u) = \left(\frac{d}{d\theta}\right)^{-1} \frac{\delta H}{\delta u}\,. \tag{0.26}$$

Now, the gradient vector field for the corresponding Kähler metric, as computed in Theorem 0.8(iii), is written as

$$u_t = -\mathcal{H}\left(\frac{d}{d\theta}\right)^{-1} \frac{\delta H}{\delta u}\,. \tag{0.27}$$

Metriplectic Systems.

A *metriplectic system* consists of a smooth manifold P, two smooth vector bundle maps $\pi, \kappa : T^*P \to TP$ covering the identity, and two functions $H, S \in C^\infty(P)$, the *Hamiltonian* or *total energy* and the *entropy* of the system, such that

(i) $\{F, G\} := \langle \mathbf{d}F, \pi(\mathbf{d}G)\rangle$ **is a Poisson bracket; in particular** $\pi^* = -\pi$**;**

(ii) $(F, G) := \langle \mathbf{d}F, \kappa(\mathbf{d}G)\rangle$ **is a positive semidefinite symmetric bracket, i.e.,** $(\,,\,)$ **is** $\mathbb{R}$**-bilinear and symmetric, so** $\kappa^* = \kappa$**, and** $(F, F) \geq 0$ **for every** $F \in C^\infty(P)$**;**

(iii) $\{S, F\} = 0$ **and** $(H, F) = 0$ **for all** $F \in C^\infty(P) \iff \pi(\mathbf{d}S) = \kappa(\mathbf{d}H) = 0$**.**

The *metriplectic dynamics* of the system is given in terms of the two brackets by

$$\frac{d}{dt}F = \{F, H+S\} + (F, H+S) = \{F, H\} + (F, S), \quad \textbf{for all} \quad F \in C^\infty(P), \tag{0.28}$$

or, equivalently, as an ordinary differential equation, by

$$\frac{d}{dt}c(t) = \pi(c(t))\mathbf{d}H(c(t)) + \kappa(c(t))\mathbf{d}S(c(t)). \tag{0.29}$$

The Hamiltonian vector field $X_H := \pi(\mathbf{d}H) \in \mathfrak{X}(P)$ represents the *conservative* or *Hamiltonian part*, whereas $Y_S := \kappa(\mathbf{d}S) \in \mathfrak{X}(P)$ the *dissipative part* of the full metriplectic dynamics (0.28) or (0.29).

The definition of metriplectic systems has three immediate important consequences. Let $c(t)$ be an integral curve of the system (0.29).

(1) ***Energy conservation*:**

$$\frac{d}{dt}H(c(t)) = \{H, H\}(c(t)) + (H, S)(c(t)) = 0. \tag{0.30}$$

(2) ***Entropy production*:**

$$\frac{d}{dt}S(c(t)) = \{S, H\}(c(t)) + (S, S)(c(t)) \geq 0. \tag{0.31}$$

(3) ***Maximum entropy principle yields equilibria*: Suppose that there are n functions $C_1, \ldots, C_n \in C^\infty(P)$ such that $\{F, C_i\} = (F, C_i) = 0$ for all $F \in C^\infty(P)$, i.e., these functions are simultaneously conserved by the conservative and dissipative part of the metriplectic dynamics.**

Let $p_0 \in P$ be a maximum of the entropy S subject to the constraints $H^{-1}(h) \cap C_1^{-1}(c_1) \cap \ldots C_n^{-1}(c_n)$, for given regular values $h, c_1, \ldots, c_n \in \mathbb{R}$ of $H, C_1, \ldots, C_n$, respectively. By the Lagrange Multiplier Theorem, there exist $\alpha, \beta_1, \ldots, \beta_n \in \mathbb{R}$ such that

$$\mathbf{d}S(p_0) = \alpha \mathbf{d}H(p_0) + \beta_1 \mathbf{d}C_1(p_0) + \cdots + \mathbf{d}C_n(p_0).$$

But then, assuming that $\alpha \neq 0$, for every $F \in C^\infty(P)$, we have

$$\begin{aligned} &\{F, H\}(p_0) + (F, S)(p_0) \\ &= \langle \mathbf{d}F(p_0), \pi(p_0)\left(\mathbf{d}H(p_0)\right)\rangle + \langle \mathbf{d}F(p_0), \kappa(p_0)\left(\mathbf{d}S(p_0)\right)\rangle = 0 \end{aligned}$$

which means that p_0 is an equilibrium of the metriplectic dynamics (0.28) or (0.29). This is akin to the free energy extremization of thermodynamics, as noted by Morrison and Mielke.

Suppose that $K \in C^\infty(P)$ is a conserved quantity for the Hamiltonian part of the metriplectic dynamics, i.e., $\{K, H\} = 0$. Then, if $c(t)$ is an integral curve of the metriplectic dynamics, we have

$$\begin{aligned}\frac{d}{dt}K(c(t)) &= \mathbf{d}K(c(t))\,(\dot{c}(t)) \\ &= \langle \mathbf{d}F(c(t)), \pi(c(t))\,(\mathbf{d}H(c(t)))\rangle + \langle \mathbf{d}F(c(t)), \kappa(c(t))\,(\mathbf{d}S(c(t)))\rangle \\ &= \{K, H\}(c(t)) + (K, S)(c(t)) = (K, S)(c(t)).\end{aligned}$$

As pointed out by Morrison, this immediately implies that a function that is simultaneously conserved for the full metriplectic dynamics and its Hamiltonian part, is necessarily conserved for the dissipative part. Physically, it is advantageous for general metriplectic systems to conserve dynamical constraints, i.e., conserved quantitates of its Hamiltonian part.

Metriplectic systems based on Lie algebra triple brackets:

Let $\mathfrak{g}$ be an arbitrary finite dimensional Lie algebra. Recall that the Killing form is defined by $\kappa(\xi, \eta) := \mathrm{Trace}(\mathrm{ad}_\xi \circ \mathrm{ad}_\eta)$. If $\{e_i\}$, $i = 1, \ldots \dim \mathfrak{g}$, is an arbitrary basis of $\mathfrak{g}$ and $c^p{}_{ij}$ are the structure constants of $\mathfrak{g}$, i.e., $[e_i, e_j] = c^p{}_{ij} e_p$, then

$$\kappa(\xi, \eta) = \xi^i c^p{}_{iq} \eta^j c^q{}_{jp}$$

and hence the components of κ in the basis $\{e_i\}$, $i = 1, \ldots \dim \mathfrak{g}$, are given by

$$\kappa_{ij} = \kappa(e_i, e_j) = c^p{}_{iq} c^q{}_{jp}.$$

The Killing form is bilinear symmetric and invariant; it is non-degenerate if and only if $\mathfrak{g}$ is semisimple. Moreover, $-\kappa$ is a positive definite inner product if and only if the Lie algebra $\mathfrak{g}$ is compact (i.e., it is the Lie algebra of a compact Lie group).

In general, let κ be a bilinear symmetric non-degenerate invariant form and define the completely antiymmetric covariant 3-tensor

$$c(\xi,\eta,\zeta) := \kappa(\xi,[\eta,\zeta]) = -c(\xi,\zeta,\eta) = -c(\eta,\xi,\zeta) = -c(\zeta,\eta,\xi).$$

In the coordinates given by the basis $\{e_i\}$, $i = 1,\ldots \dim \mathfrak{g}$, the components of c are

$$c_{ijk} := \kappa_{im}c^{m}{}_{jk} = -c_{ikj} = -c_{jik} = -c_{kji}.$$

This construction immediately leads to the triple bracket introduced by Bialnycki-Birula and Morrison, 1991 $\{\cdot\,,\cdot,\cdot\}: C^\infty(\mathfrak{g}) \times C^\infty(\mathfrak{g}) \times C^\infty(\mathfrak{g}) \to C^\infty(\mathfrak{g})$ **defined by**

$$\{f,g,h\}(\xi) := c(\nabla f(\xi), \nabla g(\xi), \nabla h(\xi)) := \kappa\left(\nabla f(\xi), [\nabla g(\xi), \nabla h(\xi)]\right), \tag{0.32}$$

where the gradient is taken relative to the non-degenerate bilinear form κ**, i.e., for any** $\xi \in \mathfrak{g}$ **we have**

$$\kappa(\nabla f(\xi), \cdot) := \mathbf{d}f(\xi)$$

or, in coordinates

$$\nabla^i f(\xi) = \kappa^{ij} \frac{\partial f}{\partial \xi^i}$$

where $[\kappa^{ij}] = [\kappa_{kl}]^{-1}$**, i.e.,** $\kappa^{ij}\kappa_{jk} = \delta^i_k$**. This triple bracket is trilinear over** $\mathbb{R}$**, completely antisymmetric, and satisfies the Leibniz rule in any of its variables.**

This construction extends the bracket due to Nambu to a Lie algebra setting. Nambu considered ordinary vectors in $\mathbb{R}^3$ and defined

$$\{f, g, h\}_{\text{Nambu}}(\mathbf{\Pi}) = \nabla f(\mathbf{\Pi}) \cdot (\nabla g(\mathbf{\Pi}) \times \nabla h(\mathbf{\Pi}))\,, \tag{0.33}$$

where '$\cdot$' and '$\times$' are the ordinary dot and cross products. Thus, the Nambu bracket is a special case of the triple bracket (0.32) in the case of $\mathfrak{g} = \mathfrak{so}(3)$, whose the structure constants are the completely antisymmetric Levi-Civita symbol ϵ_{ijk}. Such 'modified rigid body brackets' were also described in Bloch and Marsden [1990], Holm and Marsden [1991], and and Marsden and Ratiu [1999].

If $\mathfrak{g}$ is an arbitrary quadratic Lie algebra with bilinear symmetric non-degenerate invariant form κ, the quadratic function

$$C_2(\xi) := \frac{1}{2}\kappa(\xi, \xi) \tag{0.34}$$

is a Casimir function for the Lie-Poisson bracket on $\mathfrak{g}$, identified with $\mathfrak{g}^*$ via κ, i.e.,

$$\{f, g\}_\pm(\xi) = \pm\kappa\left(\xi, [\nabla f(\xi), \nabla g(\xi)]\right), \tag{0.35}$$

as an easy verification shows since $\nabla C_2(\xi) = \xi$. In view of (0.35), the following identity is obvious

$$\{f, g\}_+ = \{C_2, f, g\}.$$

For example, if $\mathfrak{g} = \mathfrak{so}(3)$, the (-)Lie-Poisson bracket

$$\{f, g\}_{-}^{\mathfrak{so}(3)}(\boldsymbol{\Pi}) = -\{C_2, f, g\}_{\mathrm{Nambu}}(\boldsymbol{\Pi}) = -\boldsymbol{\Pi} \cdot (\nabla f(\boldsymbol{\Pi}) \times \nabla g(\boldsymbol{\Pi})) \quad (0.36)$$

is the rigid body bracket, i.e., if $h(\boldsymbol{\Pi}) = \frac{1}{2}\boldsymbol{\Pi} \cdot \boldsymbol{\Omega}$, where $\boldsymbol{\Pi}_i = I_i \boldsymbol{\Omega}_i$, $I_i > 0$, $i = 1, 2, 3$, and I_i are the principal moments of inertia of the body, then Hamilton's equations $\frac{d}{dt}F(\boldsymbol{\Pi}) = \{f, h\}_{-}^{\mathfrak{so}(3)}(\boldsymbol{\Pi})$ are equivalent to Euler's equations $\dot{\boldsymbol{\Pi}} = \boldsymbol{\Pi} \times \boldsymbol{\Omega}$.

Note that given any two functions, $f, g \in C^\infty(\mathfrak{g})$, because the triple bracket satisfies the Leibniz identity in every factor, the map $C^\infty(\mathfrak{g}) \ni h \mapsto \{h, f, g\} \in C^\infty(\mathfrak{g})$ is a derivation and hence defines a vector field on $\mathfrak{g}$, denoted by $X_{f,g} : \mathfrak{g} \to \mathfrak{g}$, i.e.,

$$\langle \mathbf{d}h(\xi), X_{f,g}(\xi)\rangle = \kappa\left(\nabla h(\xi), X_{f,g}(\xi)\right) = \{h, f, g\}(\xi) \quad \textbf{for all} \quad h \in C^\infty(\mathfrak{g}). \tag{0.37}$$

Note that $X_{f,f} = 0$. Thus, for triple brackets, two functions define a vector field, analogous to the Hamiltonian vector field defined by a single function associated to a standard Poisson bracket.

We have the following result.

Proposition 0.10. *The vector field $X_{f,g}$ on $\mathfrak{g}$ corresponding to the pair of functions f, g is given by*

$$X_{f,g}(\xi) = [\nabla f(\xi), \nabla g(\xi)]\,. \tag{0.38}$$

Triple brackets of the form (0.32) can be used to construct metriplectic systems on a quadratic Lie algebra $\mathfrak{g}$ in the following manner. Let κ be the bilinear symmetric non-degenerate form on $\mathfrak{g}$ defining the quadratic structure and fix some $h \in C^\infty(\mathfrak{g})$. Define the symmetric bracket

$$(f, g)_h^\kappa(\xi) := -\kappa\left(X_{h,f}(\xi), X_{h,g}(\xi)\right). \tag{0.39}$$

Assume that $-\kappa$ is a positive definite inner product. Then $(f, f) \geq 0$. Thus we have the manifold $\mathfrak{g}$ endowed with the Lie-Poisson bracket (0.35), the symmetric bracket (0.39), the Hamiltonian h, and for the entropy S we take any Casimir function of the Lie-Poisson bracket.

Then the conditions (i)–(iii) of metriplectic are all satisfied, because $(h,g)^{\kappa}_{h} = -\kappa(X_{h,h}, X_{h,g}) = -\kappa(0, X_{h,g}) = 0$ **for any** $g \in C^{\infty}(\mathfrak{g})$**. The equations of motion (0.28) are in this case given by**

$$\frac{d}{dt}f(\xi) = \kappa\left(\nabla f(\xi), \frac{d}{dt}\xi\right) = \{f,h\}_{\pm}(\xi) + (f,S)(\xi) = \pm\kappa\left(\xi, [\nabla f(\xi), \nabla h(\xi)]\right) - \kappa$$
$$= \mp\kappa\left(\nabla f(\xi), [\xi, \nabla h(\xi)]\right) - \kappa\left([\nabla h(\xi), \nabla f(\xi)], [\nabla h(\xi), \nabla S(\xi)]\right)$$

for any $f \in C^{\infty}(\mathfrak{g})$**.**

This gives the equations of motion

$$\dot{\xi} = \pm[\xi, \nabla h(\xi)] + [\nabla h(\xi), [\nabla h(\xi), \nabla S(\xi)]] \,. \tag{0.40}$$

Note that the flow corresponding to S is a generalized double bracket flow. Observe also that this flow reduces to a double bracket flow and is tangent to an orbit of the group if $\nabla h(\xi) = \xi$. Indeed if $h = \frac{1}{2}\kappa(\xi, \xi)$ the symmetric bracket (0.39) reduces to the symmetric bracket induced from the normal metric.

Special Case of $\mathfrak{so}(3)$: If the quadratic Lie algebra is $\mathfrak{so}(3)$, we identify it with $\mathbb{R}^3$ with the cross product as Lie bracket via the Lie algebra isomorphism $\hat{}: \mathbb{R}^3 \to \mathfrak{so}(3)$ given by $\hat{\mathbf{u}}\mathbf{v} := \mathbf{u} \times \mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Since $\mathrm{Ad}_A \hat{\mathbf{u}} = \widehat{A\mathbf{u}}$, for any $A \in SO(3)$ and $\mathbf{u} \in \mathbb{R}^3$, we conclude that the usual inner product on $\mathbb{R}^3$ is an invariant inner product. In terms of elements of $\mathfrak{so}(3)$ we have $\mathbf{u} \cdot \mathbf{v} = -\frac{1}{2} \operatorname{Trace}(\hat{\mathbf{u}}\hat{\mathbf{v}})$. We shall show below that the metriplectic structure on $\mathbb{R}^3$ is precisely the one given in Morrison [1986].

Recall that the Nambu bracket is given for $\mathfrak{so}(3)$ by (0.33) and hence the symmetric bracket (0.39) has the form

$$\begin{aligned}\kappa(\{\Pi, h, f\}, \{\Pi, h, g\}) &= \epsilon^{imn}\frac{\partial h}{\partial \Pi^m}\frac{\partial f}{\partial \Pi^n}\,\delta_{ij}\,\epsilon^{jst}\frac{\partial h}{\partial \Pi^s}\frac{\partial g}{\partial \Pi^t} \\ &= \epsilon^{imn}\,\epsilon_i^{\,st}\,\frac{\partial h}{\partial \Pi^m}\frac{\partial f}{\partial \Pi^n}\frac{\partial h}{\partial \Pi^s}\frac{\partial g}{\partial \Pi^t} \\ &= \|\nabla h\|^2 \nabla g \cdot \nabla f - (\nabla f \cdot \nabla h)(\nabla g \cdot \nabla h) \quad (0.41)\end{aligned}$$

where in the third equality we have used the identity $\epsilon^{imn}\epsilon_i^{\,st} = \delta^{ms}\delta^{nt} - \delta^{mt}\delta^{ns}$.

With the choice $S(\mathbf{\Pi}) = \|\mathbf{\Pi}\|^2/2$ and the usual rigid body Hamiltonian, the equations of motion (0.40) are those for the relaxing rigid body of Morrison [1986].

Comments.

- **In three dimensions any Poisson bracket can be written as**

$$\{f,g\} = J^{ij}\frac{\partial f}{\partial \Pi^i}\frac{\partial g}{\partial \Pi^j} = \epsilon^{ij}{}_k V^k(\mathbf{\Pi})\frac{\partial f}{\partial \Pi^i}\frac{\partial g}{\partial \Pi^j} \tag{0.42}$$

where $i,j,k = 1,2,3$, and $V \in \mathbb{R}^3$. Using the well known fact that brackets of the form of (0.42) satisfy the Jacobi identity if

$$V \cdot \nabla \times V = 0\,, \tag{0.43}$$

we conclude that

$$\{F,G\}_f = \{f,F,G\}_{\text{Nambu}} \tag{0.44}$$

satisfies the Jacobi identity for *any* smooth function f; i.e., unlike the general case.

- **Thinking in terms of $\mathfrak{so}(3)^*$, the setting arising from reduction, this construction leads to a natural geometric interpretation of a metriplectic system on the manifold $P = \mathbb{R}^3$. With the Poisson bracket on $\mathbb{R}^3$ of (0.44), the bundle map $\pi : T^*\mathbb{R}^3 \to T\mathbb{R}^3$ has the expression**

$$\pi_f(x, \Pi) = \big(x, \nabla f(\Pi) \times (\cdot)^\top\big)$$

since d$H(\Pi)^\top = \nabla H(\Pi)$ (d$H(\Pi)$ is a row vector and $\nabla H(\Pi)$ is its transpose, a column vector). Now the triple bracket associated to the equation (0.40) can be used to generate a symmetric bracket :

$$\begin{aligned} (F, G)_{BKMR}(\Pi) &= (F, G)^\kappa_C = \kappa(\{\Pi, C, F\}, \{\Pi, C, G\}) \\ &= (\Pi \times \nabla F(\Pi)) \cdot (\Pi \times \nabla G(\Pi)) \,. \end{aligned} \tag{0.45}$$

where now $C = ||\Pi||^2/2$. Hence the bundle map $\kappa : T^*\mathbb{R}^3 \to T\mathbb{R}^3$

has the expression

$$\kappa(x,\Pi) = -\Pi \times \left(\Pi \times (\cdot)^{\top}\right).$$

Thus, with the freedom to choose any quantity $S = f$ as an entropy, with the assurance that (0.43) will be satisfied because $\nabla \times V = \nabla \times \nabla f = 0$, we can take $H = C$ and have $\{F, S\}_f = 0$ and $(F, H) = 0$ for all $F \in C^\infty(\mathbb{R}^3)$. The equations of motion for this metriplectic system are

$$\dot{\Pi} = -\Pi \times \nabla f(\Pi) - \Pi \times (\Pi \times \nabla f(\Pi)). \tag{0.46}$$

The symmetric bracket is the inner product of the two Hamiltonian vector fields on each concentric sphere. As discussed in BKMR, this symmetric bracket can be defined on any compact Lie algebra by taking the normal metric on each coadjoint orbit.

- **The following set of equations were given in Fish (2005):**

$$\dot{\Pi} = \nabla S \times \nabla H - \nabla H \times (\nabla H \times \nabla S). \tag{0.47}$$

This metriplectic system is equivalent to.

$$\dot{\Pi} = \{\Pi, S, H\} + \kappa\left(\{\Pi, H, \Pi\}, \{\Pi, H, S\}\right)\ , \tag{0.48}$$

$$(F, G)_g(\Pi) = \kappa\left(\{\Pi, g, F\}, \{\Pi, g, G\}\right) = \ (\nabla g(\Pi) \times \nabla F(\Pi)) \cdot (\nabla g(\Pi) \times \nabla G(\Pi)). \tag{0.49}$$

Thus, the bundle map $\kappa : T^*\mathbb{R}^3 \to T\mathbb{R}^3$ **has the expression**

$$\kappa_g(x, \Pi) = -\nabla g(\Pi) \times \left(\nabla(\Pi) \times (\cdot)^\top\right).$$

Examples: Two special cases of the equation (0.47) are of interest.

(i) If we take $H = \frac{1}{2}||\Pi||^2$ **and** $S = c \cdot \Pi$**,** c **a constant vector, we obtain**

$$\dot{\Pi} = c \times \Pi - \Pi \times (\Pi \times c). \tag{0.50}$$

(ii) If we take $S = \frac{1}{2}\|\Pi\|^2$ **and** $H = c \cdot \Pi$**,** c **a constant, we obtain**

$$\dot{\Pi} = \Pi \times c - c \times (\mathfrak{c} \times \Pi)\,. \tag{0.51}$$

The equations of motion (0.50) is an instance of double bracket damping, where the damping is due to the normal metric, whereas (0.51) gives linear damping of the sort arising in quantum systems. See also Gay-Balmaz/Holm fluids.

Metriplectic extends to full Toda with dissipation:

Triple bracket and associate symmetric brackets extends to fields and PDE's (see paper.)

Related work: dispersionless Toda with Hermann Flaschka and Tudor.