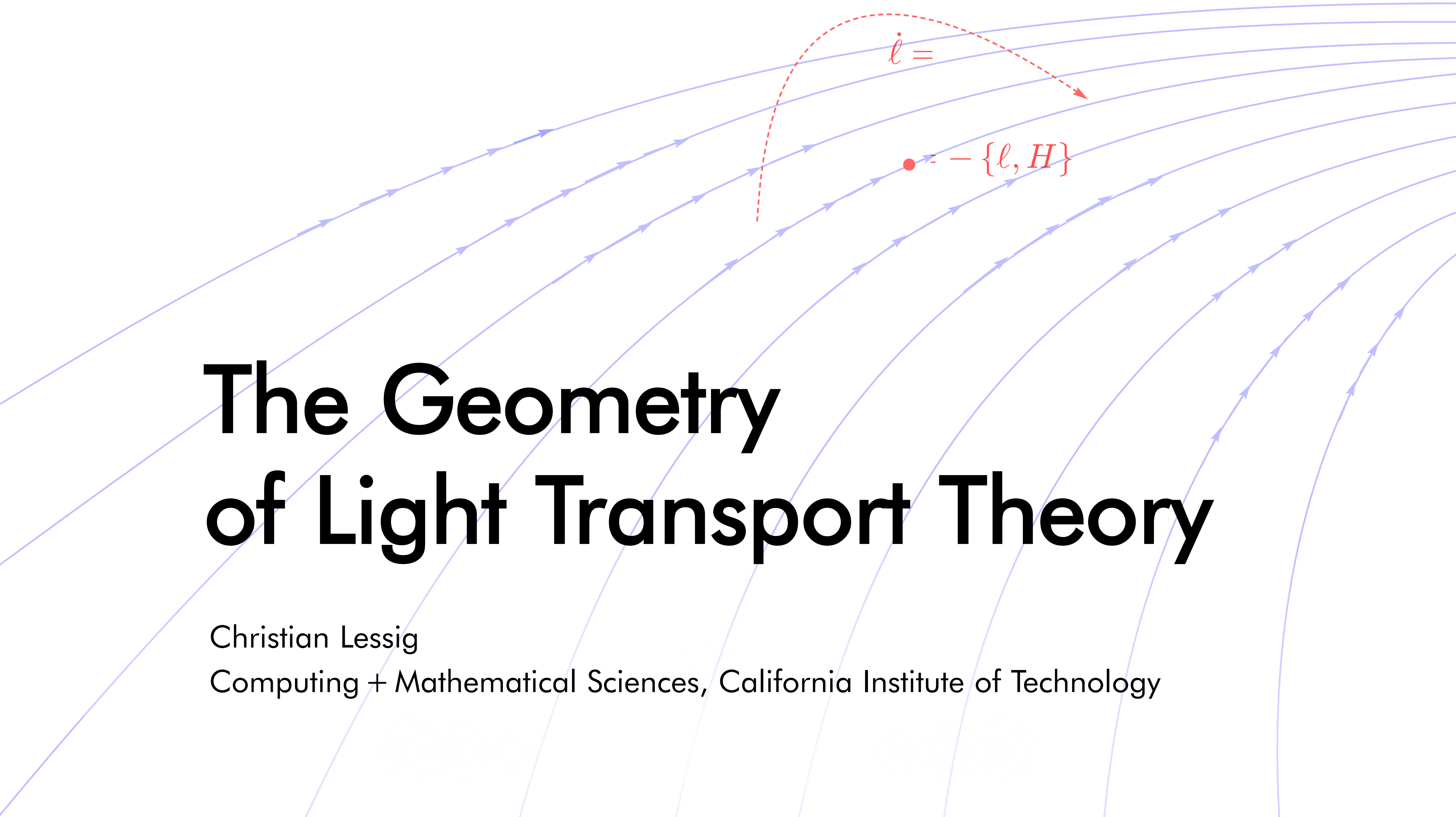
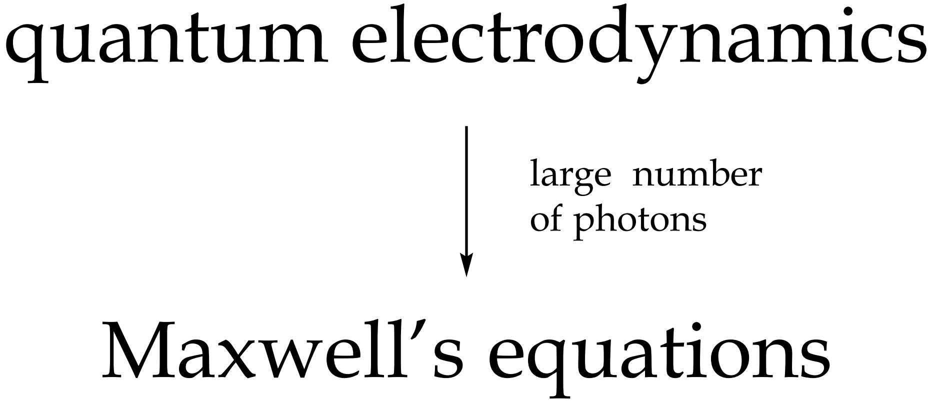


The Geometry of Light Transport Theory

Christian Lessig
Computing + Mathematical Sciences, California Institute of Technology

Light Transport Theory

quantum electrodynamics



quantum electrodynamics

large number
of photons

Maxwell's equations

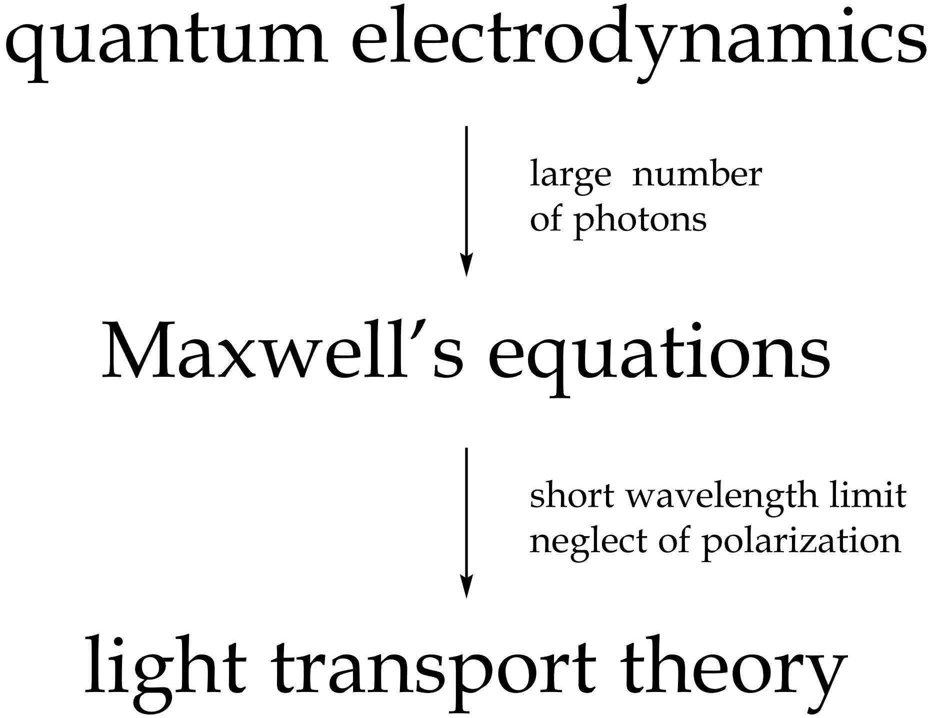
quantum electrodynamics

↓ large number of photons

Maxwell's equations

↓ short wavelength limit
neglect of polarization

geometric optics



quantum electrodynamics

large number
of photons

Maxwell's equations

short wavelength limit
neglect of polarization

light transport theory

quantum electrodynamics

↓ large number of photons

Maxwell's equations

↓ short wavelength limit
neglect of polarization

light transport theory

↓ neglect of intensity

geometric optics

"Theoretical photometry constitutes a case of 'arrested development', and has remained basically unchanged since 1760 while the rest of physics has swept triumphantly ahead. In recent years, however, the increasing needs [. . .] have made the absurdly antiquated concepts of traditional photometric theory more and more untenable."[1]

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[1] Gershun, A. *The Light Field*, Translated by P. Moon, G. Timoshenko, Originally published in Russian (Moscow 1936). Journal of Mathematics and Physics 18 (1939): 51-151, from the translators preface.

Current light transport theory

Transport equation

$$\nabla_x L(x, \omega) = 0$$

Current light transport theory

Transport equation

$$\nabla_x \boxed{L(x, \omega)} = 0$$

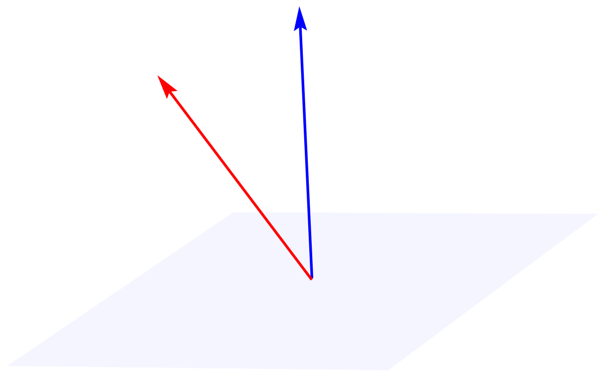
radiance

Current light transport theory

Transport equation

$$\nabla_x \boxed{L(x,\omega)} = 0$$

radiance

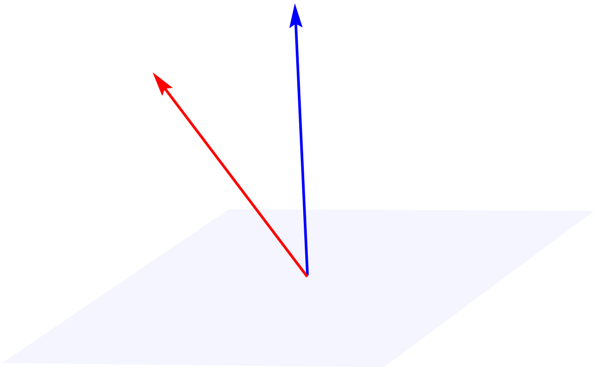


Current light transport theory

Transport equation

$$\nabla_x \boxed{L(x,\omega)} = 0$$

radiance

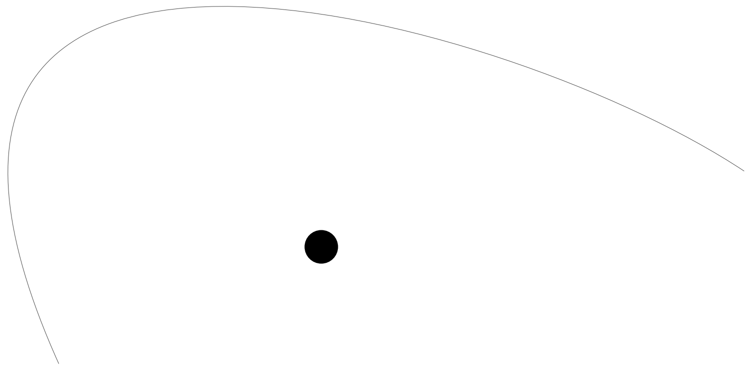


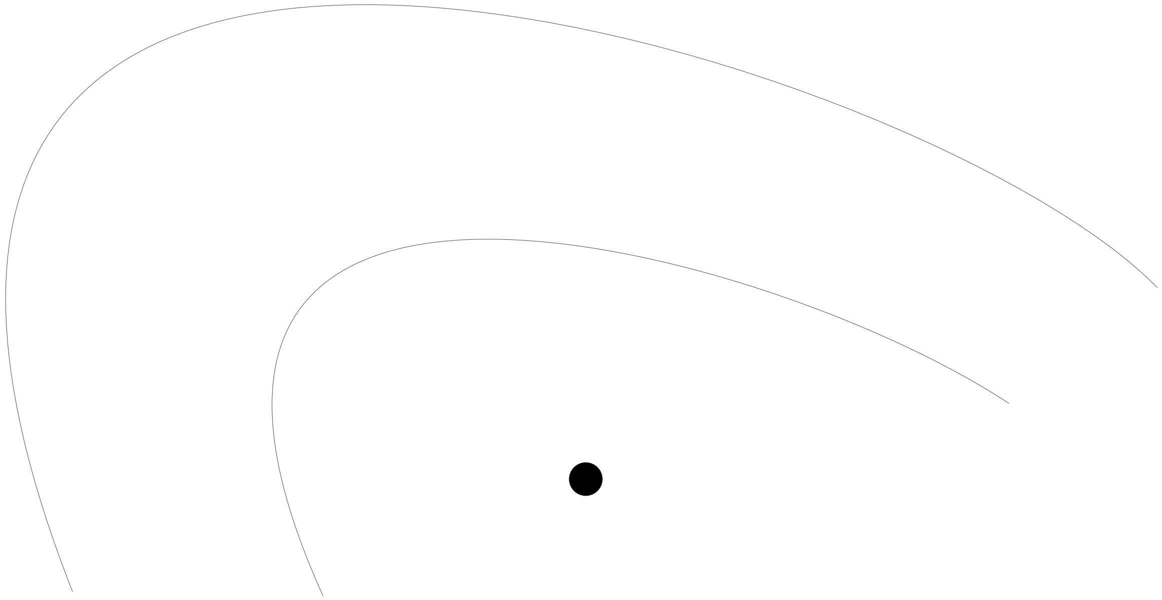
subject to

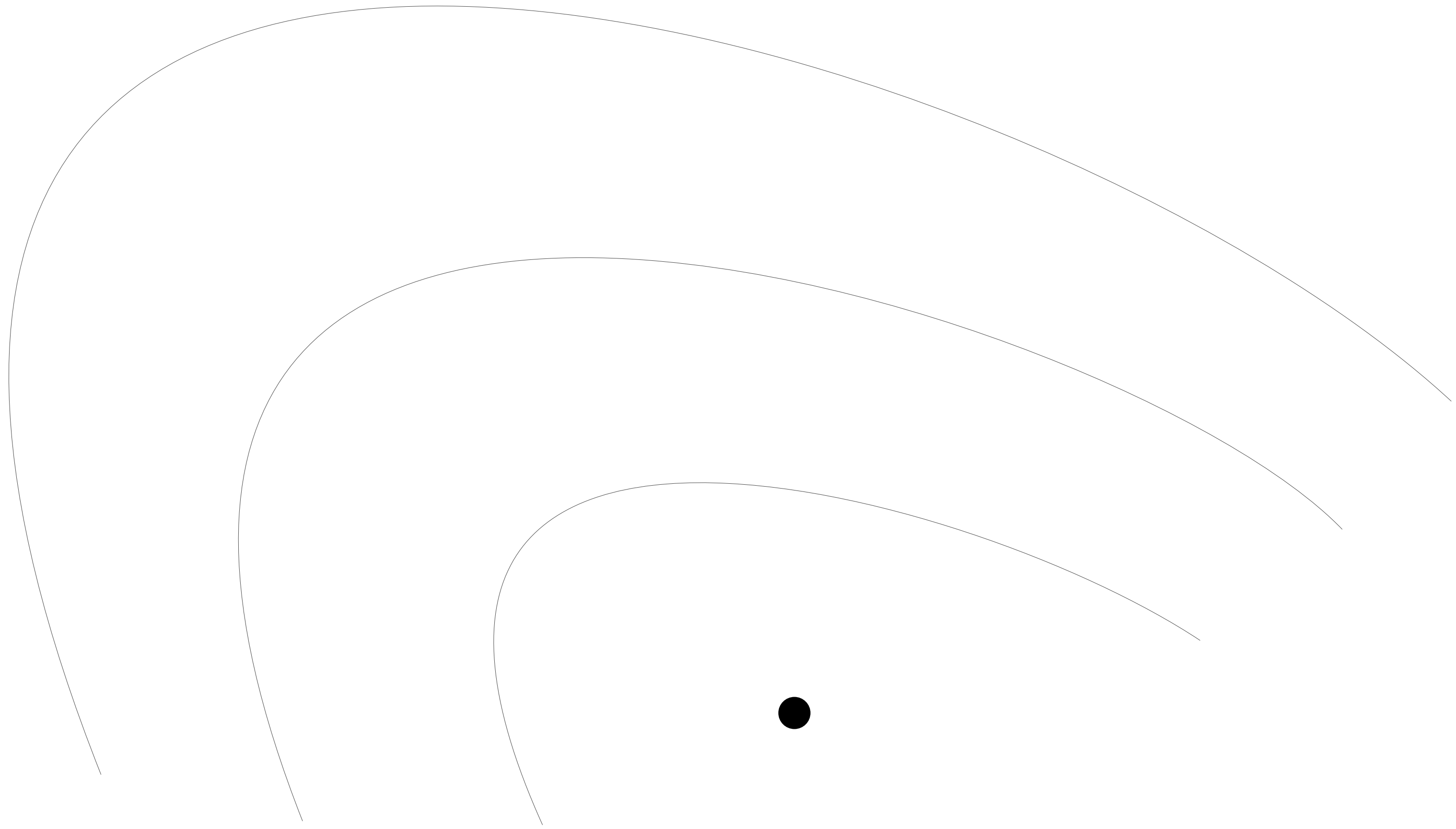
$$L(x,\bar{\omega}) = \int_{H_x^2} L(x,\omega)\, \rho_x(\omega,\bar{\omega})\, d\omega$$

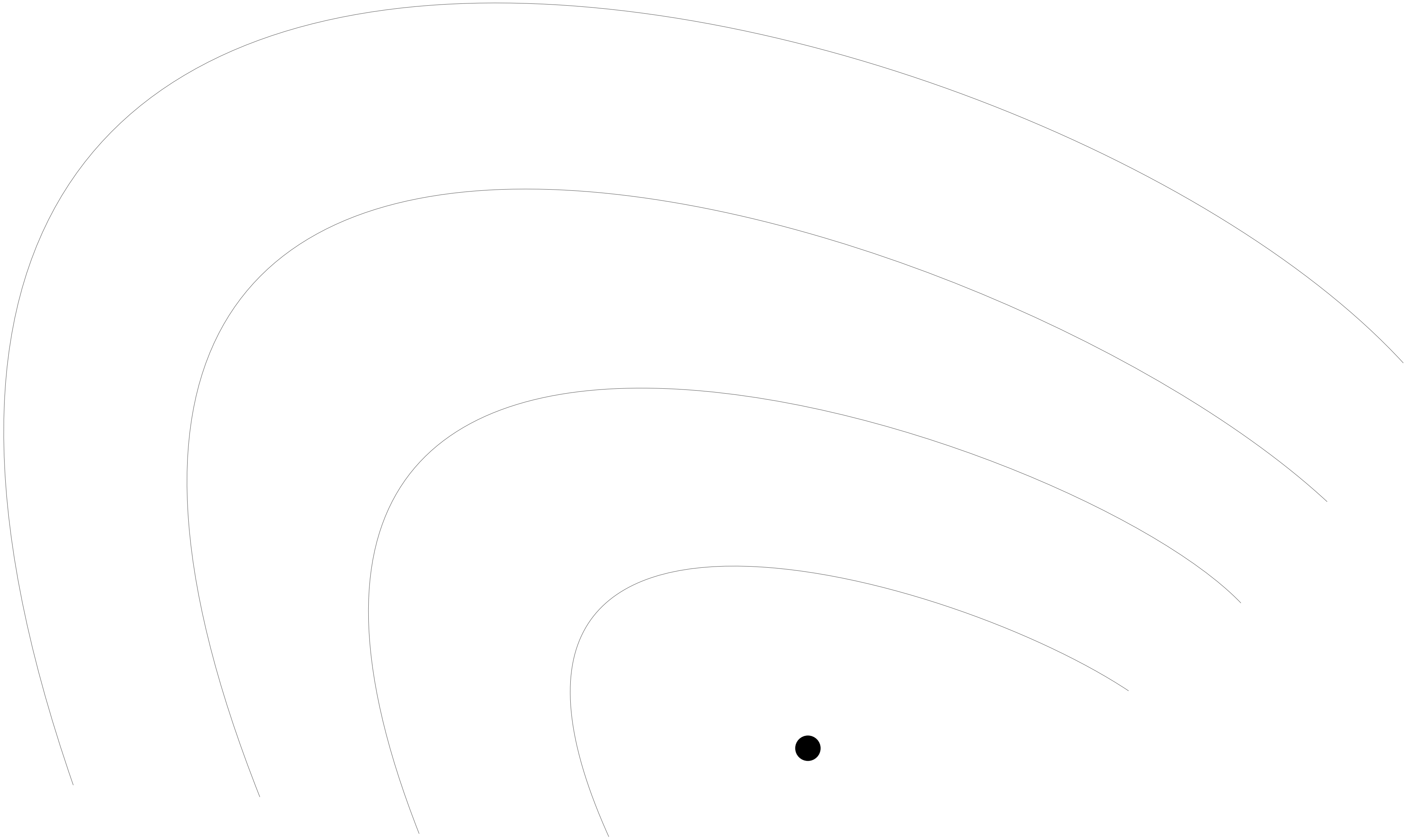
configuration space $Q \subset \mathbb{R}^3$

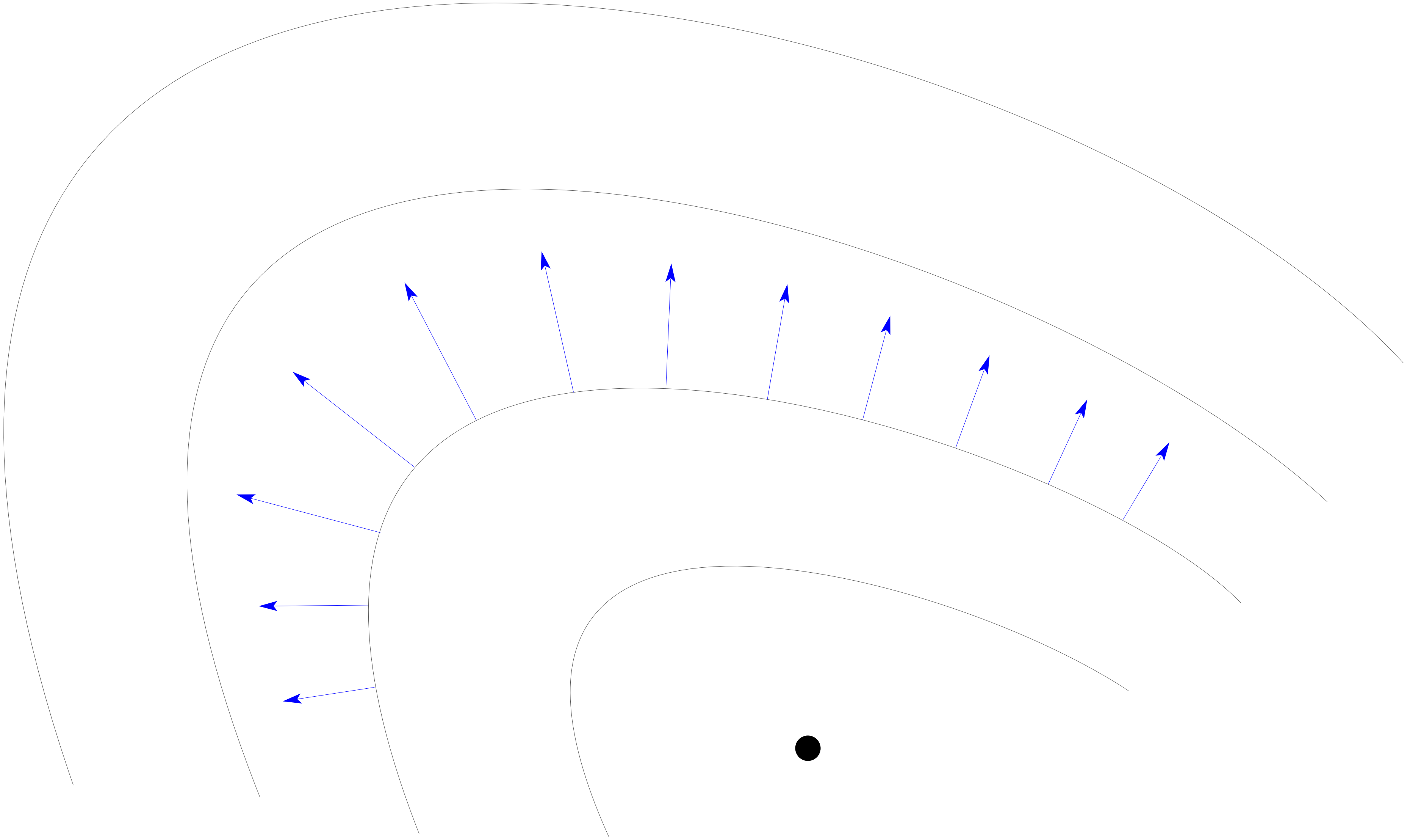
$$\frac{\partial}{\partial t}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\epsilon}\nabla\times \\ \frac{1}{\mu}\nabla\times & 0 \end{pmatrix}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}$$

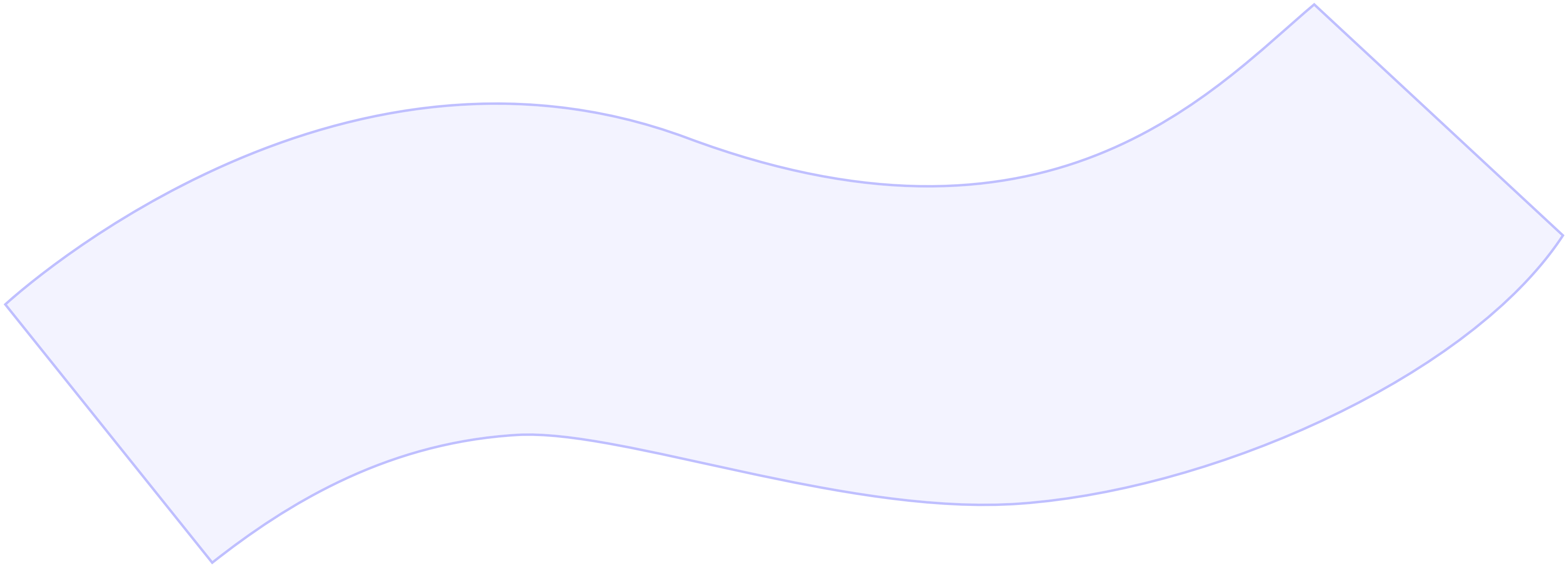


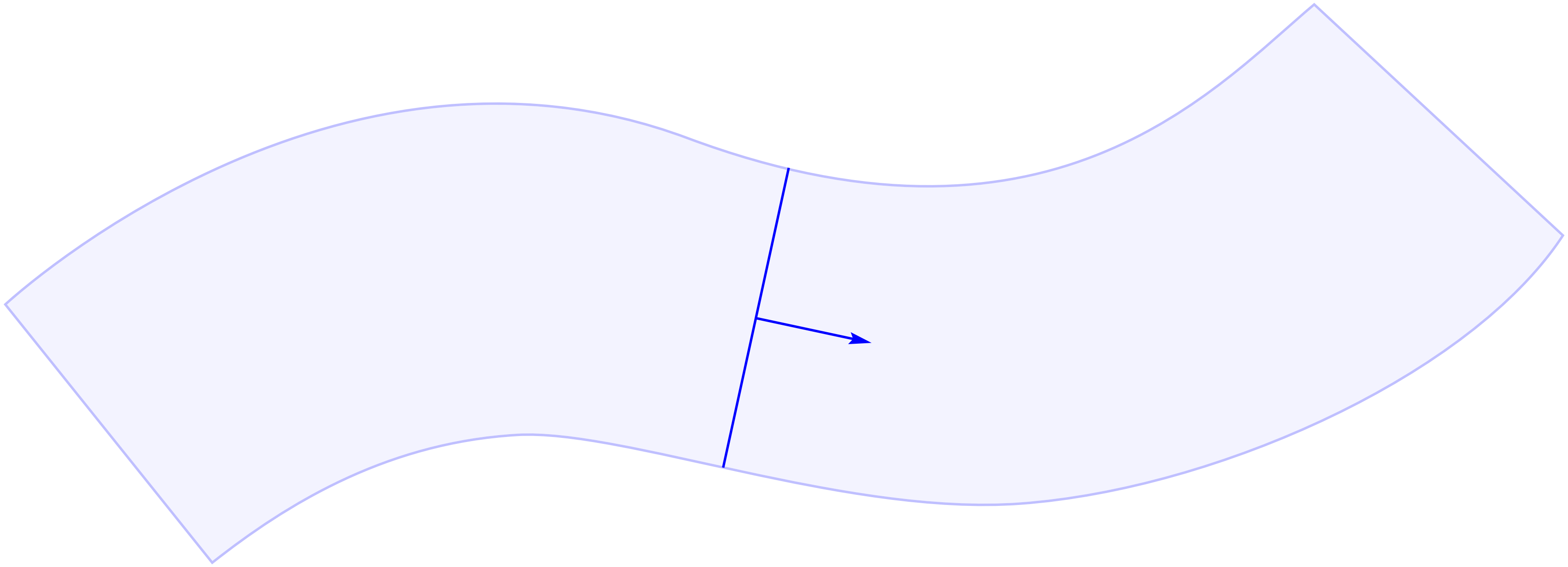


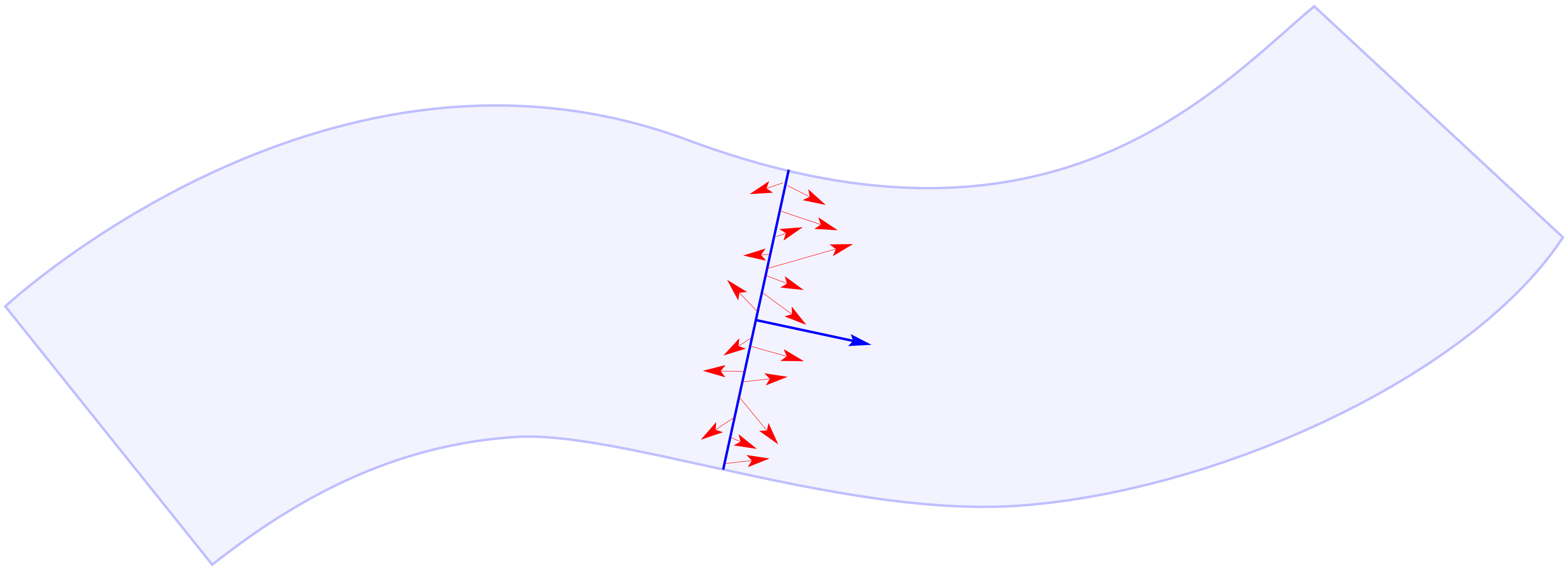


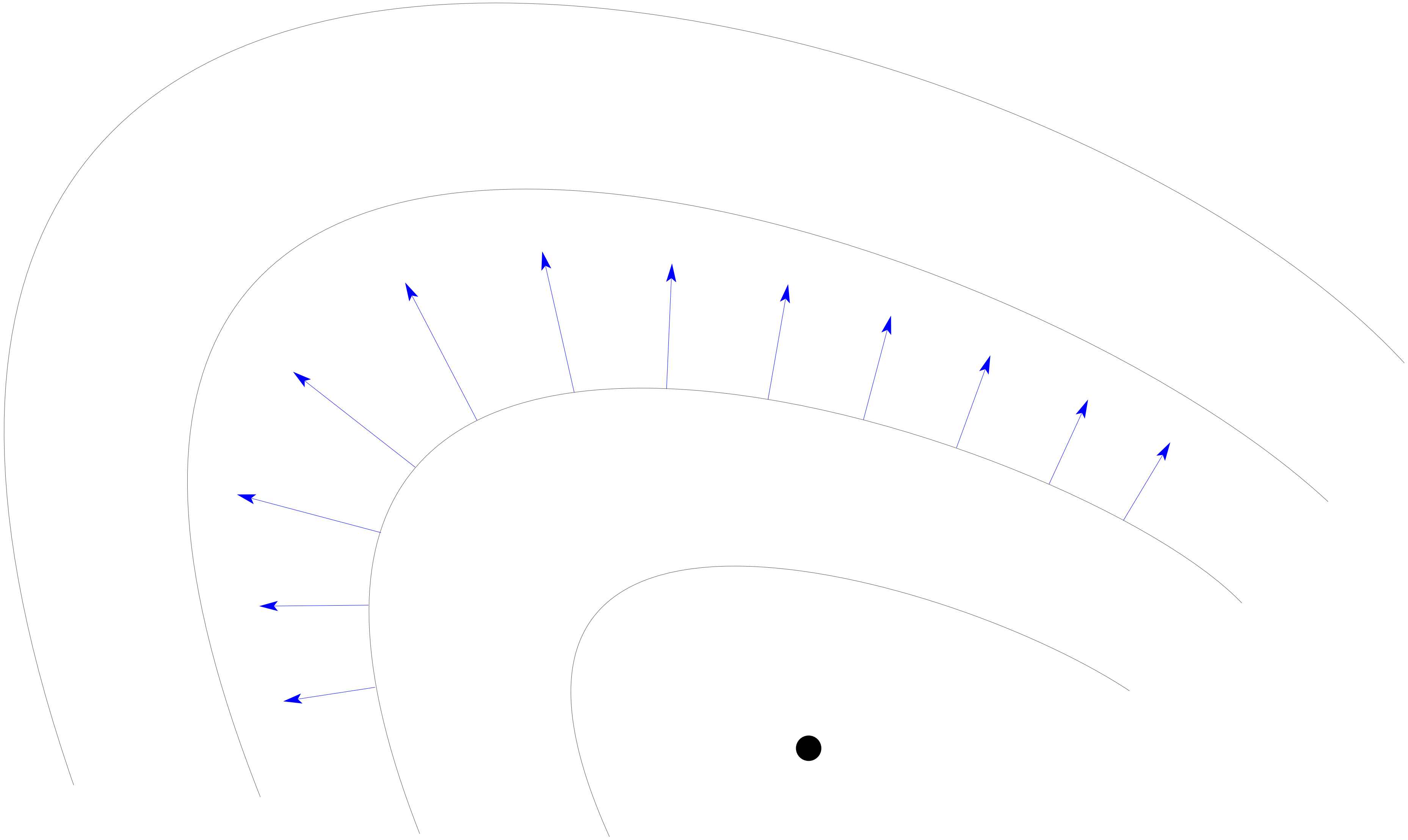


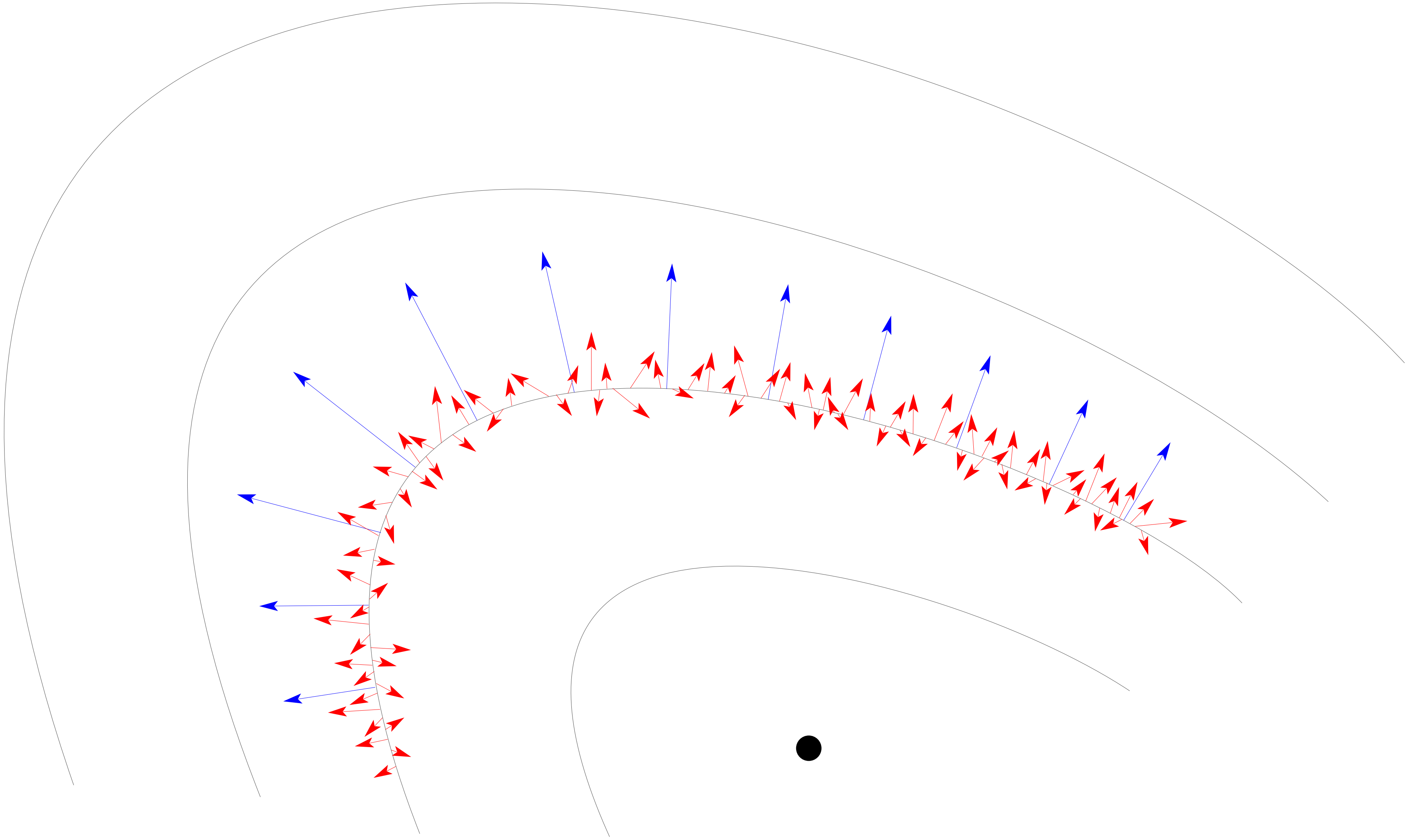


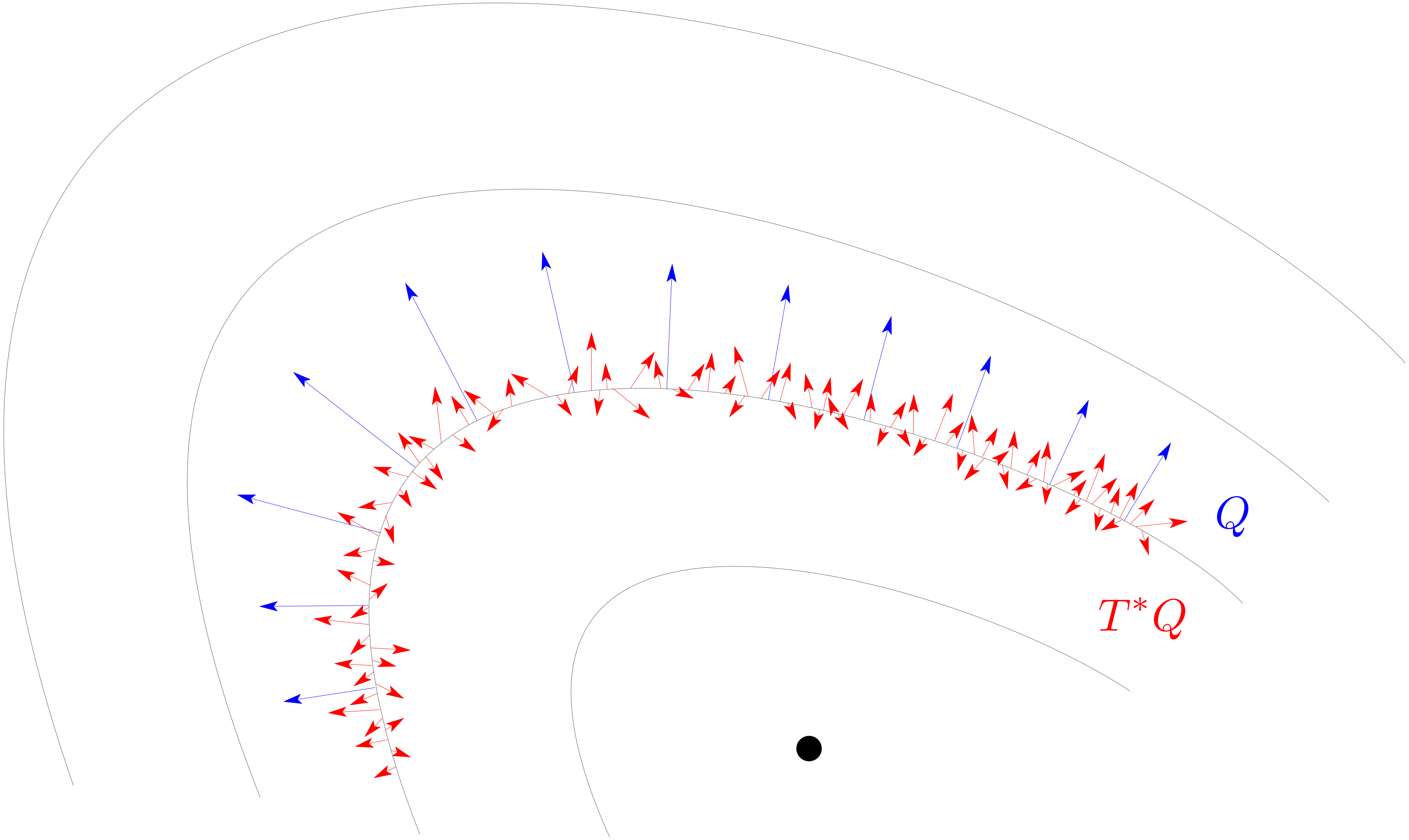











Q
T^*Q

electromagnetic theory

configuration space $Q \subset \mathbb{R}^3$

phase space T^*Q

$$\begin{tikzcd}
\frac{\partial}{\partial t}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{\epsilon}\nabla\times \\ \frac{1}{\mu}\nabla\times & 0 \end{pmatrix}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} \arrow[d, "\substack{\text{microlocal analysis} \\ \text{(Wigner transform)}}"] \\
\dot{W}^{\varepsilon} = -\{\{p^{\varepsilon}, W^{\varepsilon}\}\}
\end{tikzcd}$$

electromagnetic theory

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$\big\downarrow$ microlocal analysis (Wigner transform)

phase space T^*Q

$$-\dot{W}^{\varepsilon} = \frac{1}{\varepsilon}\,[p^{\varepsilon}, W^{\varepsilon}] + \frac{1}{i}\{p^{\varepsilon}, W^{\varepsilon}\} + O(\varepsilon)$$

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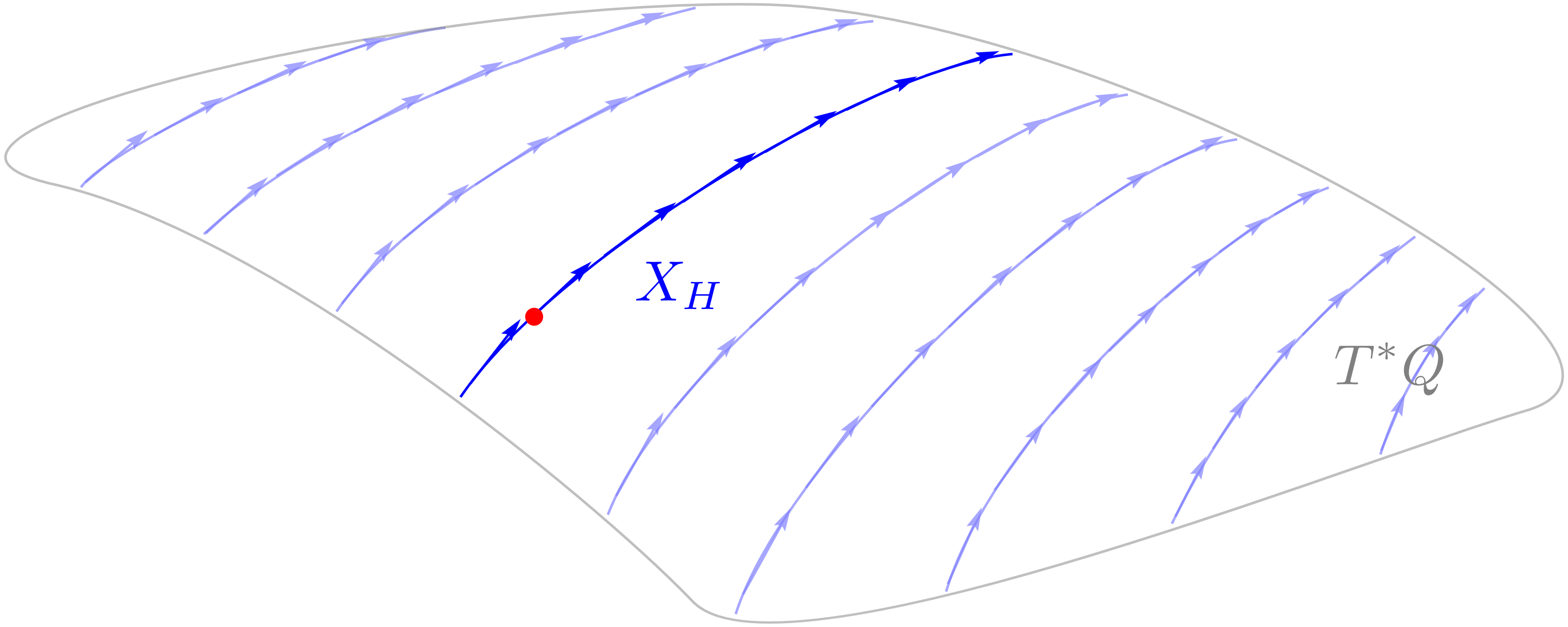
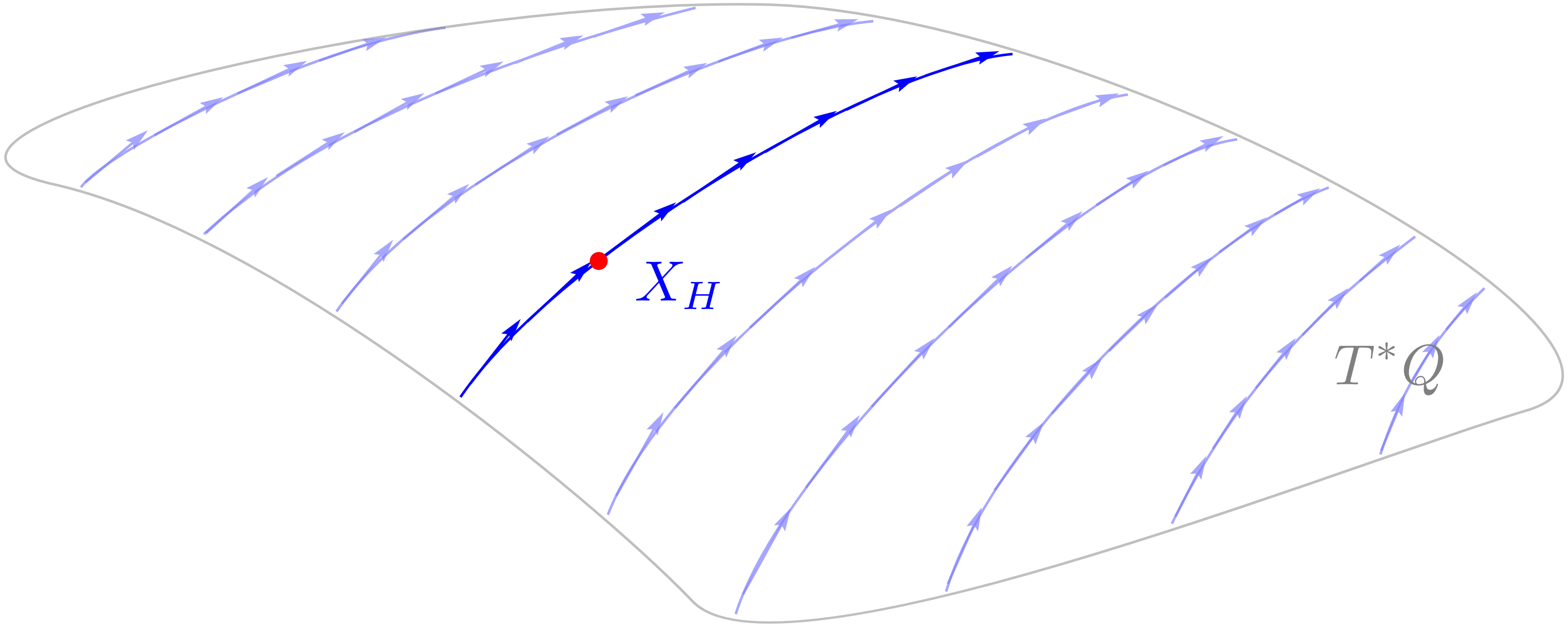
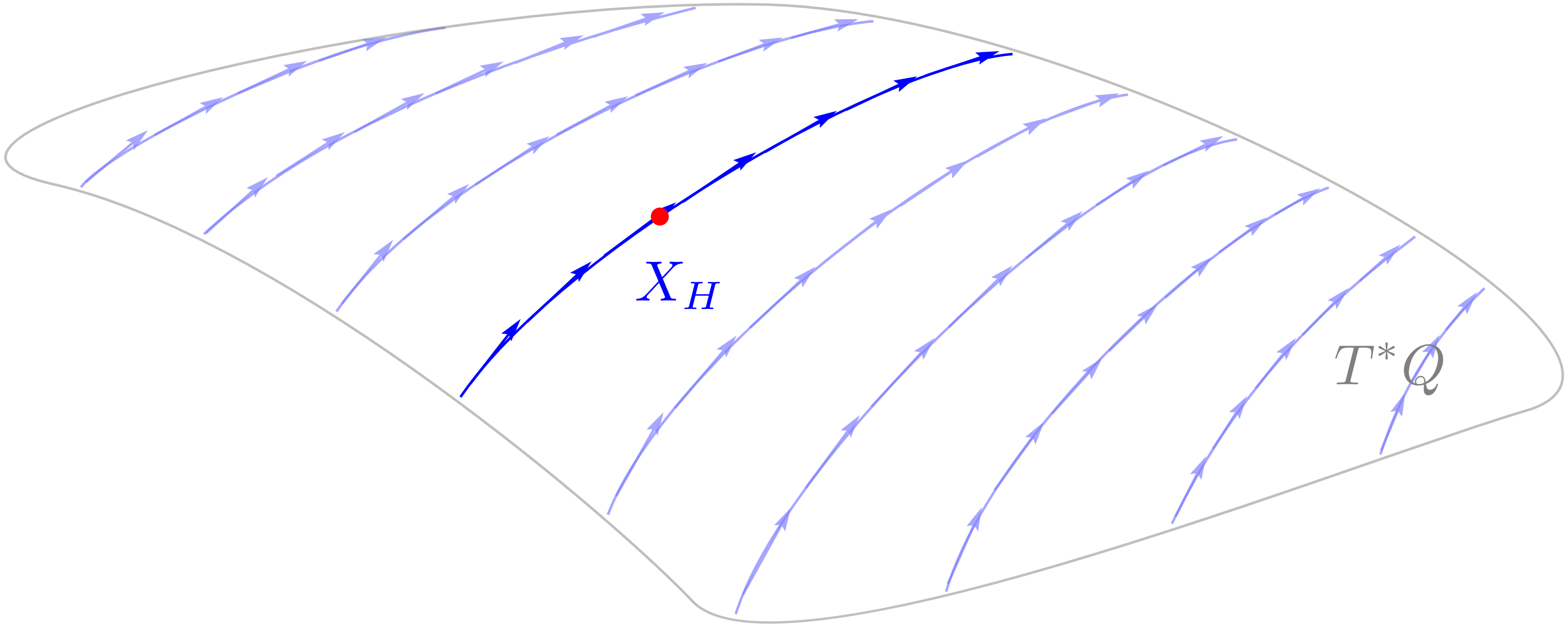
$\downarrow$ unpolarized radiation

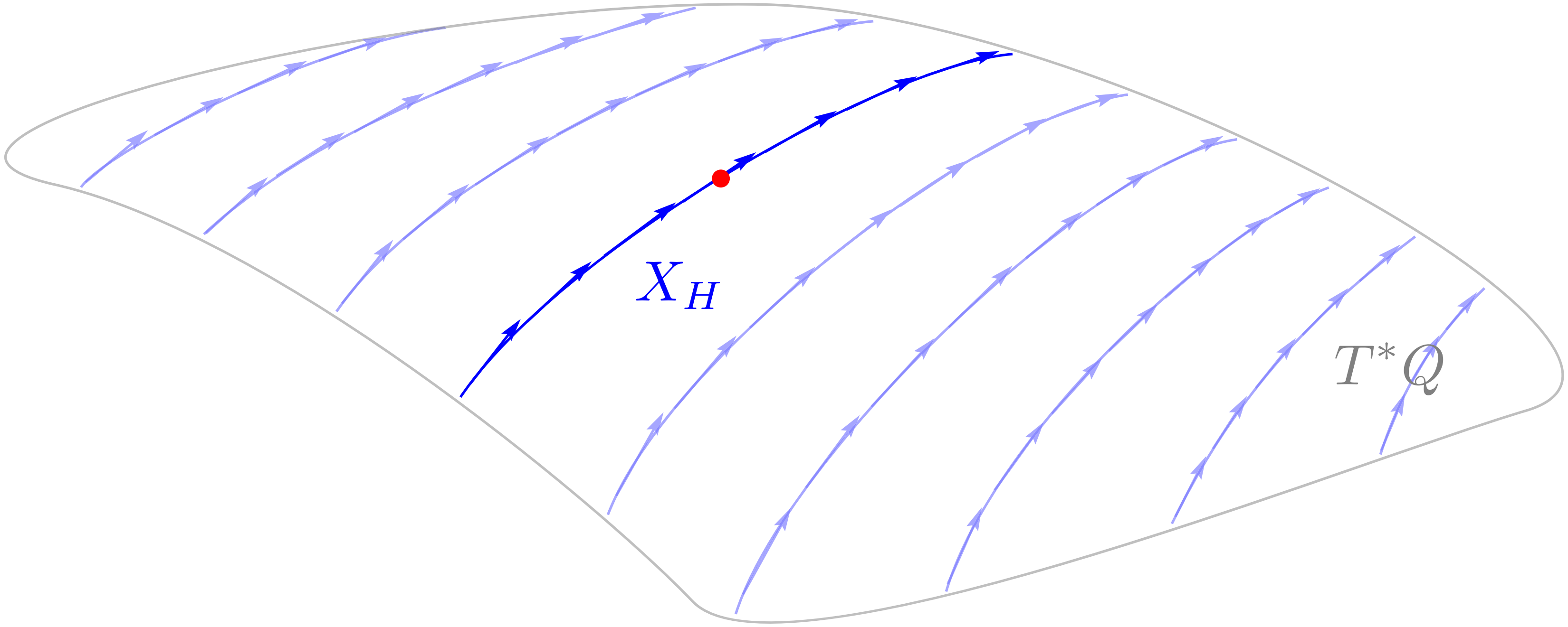
light transport equation

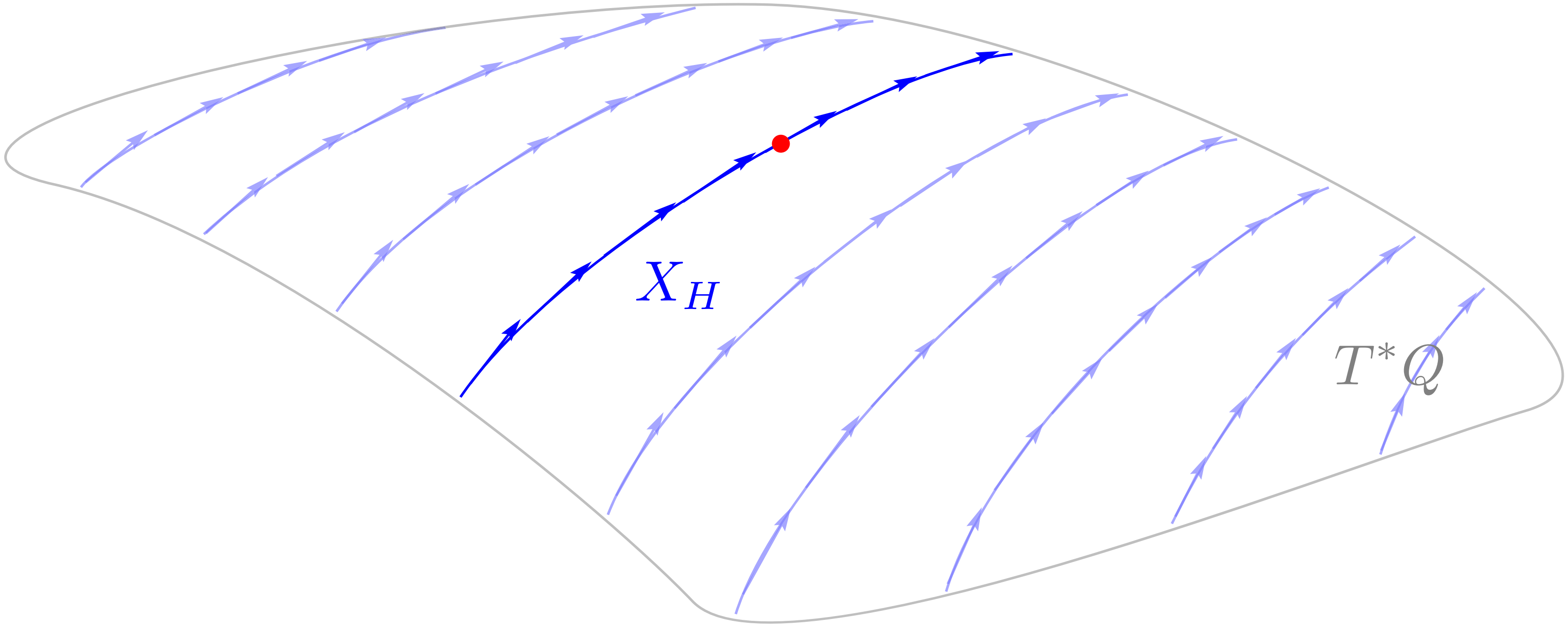
$$\dot{\ell} = -\{\ell, H\}$$

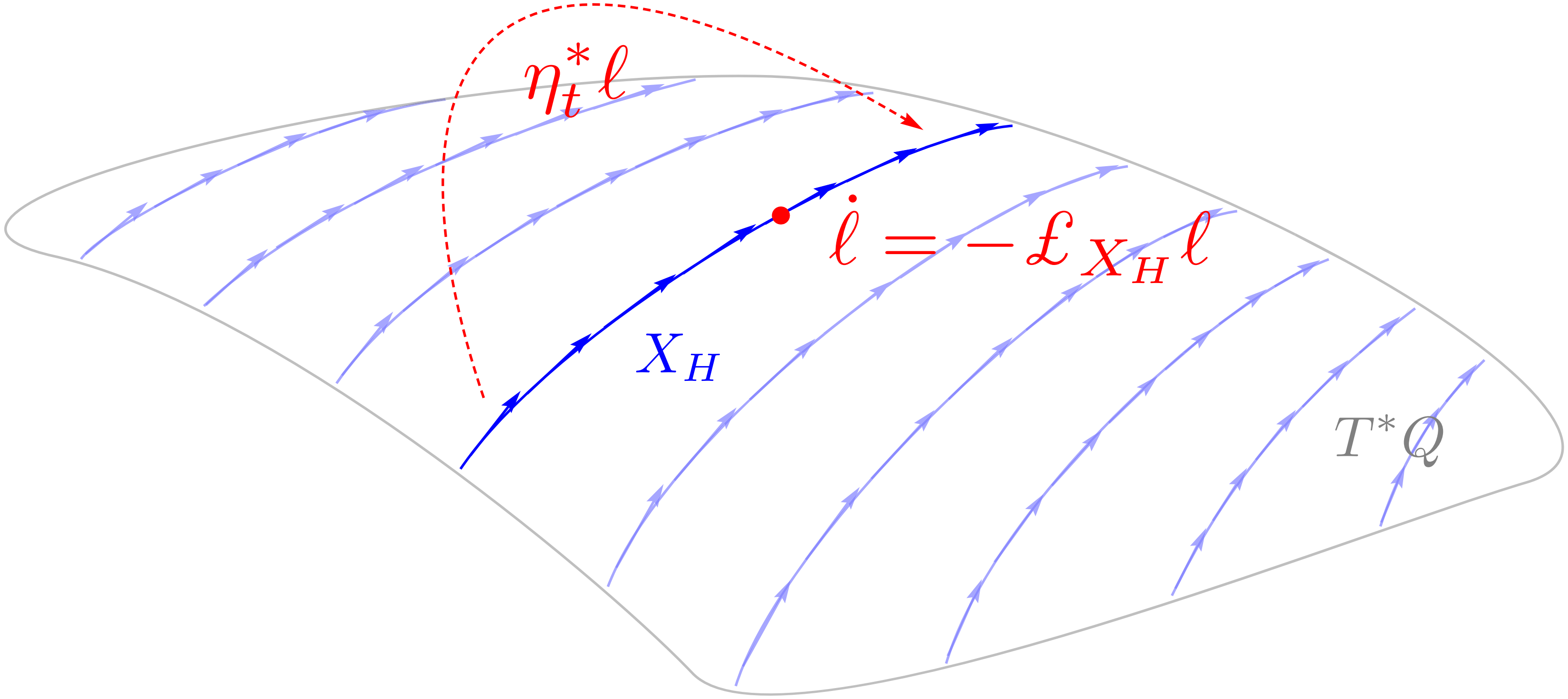

X_H
T^*Q


X_H
T^*Q


X_H
T^*Q


X_H
T^*Q


X_H
T^*Q


$\eta_t^* \ell$
$\dot{\ell} = -\pounds_{X_H} \ell$
X_H
T^*Q

electromagnetic theory

configuration space $Q \subset \mathbb{R}^3$

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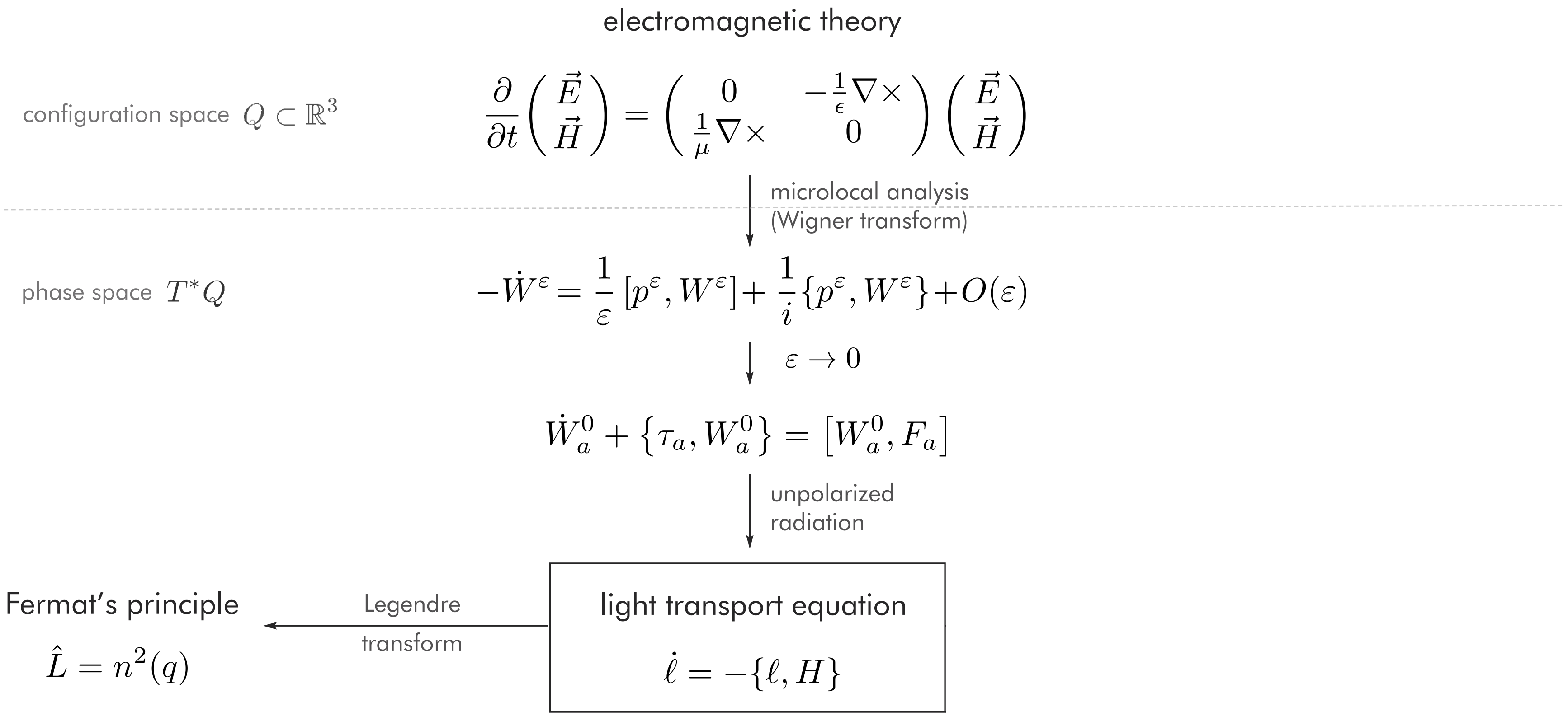
$\downarrow$ unpolarized radiation

light transport equation

$$\dot{\ell} = -\{\ell, H\}$$



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unpolarized
radiation

light transport equation

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Legendre
transform

Fermat's principle

$$\hat{L} = n^2(q)$$

Fermat's principle $L = n(q)$

$$\begin{tikzcd} \text{geodesic energy} & \hat{L} = n^2(q) \\ \text{Fermat's principle} & L = n(q) \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd}
\text{geodesic energy} & \hat{L} = n^2(q) \arrow[r, leftrightarrow, "\text{Legendre}", "\text{transform}"'] & \hat{H} = g^{ij}\hat{p}_i\hat{p}_j & \text{geodesic Hamiltonian} \\
\text{Fermat's principle} & L = n(q) \arrow[u] & &
\end{tikzcd}$$

$$\begin{tikzcd}
\text{geodesic energy} & \hat{L} = n^2(q) \arrow[r, leftrightarrow, "\text{Legendre}", "\text{transform}"'] & \hat{H} = g^{ij}\hat{p}_i\hat{p}_j \arrow[d] & \text{geodesic Hamiltonian} \\
\text{Fermat's principle} & L = n(q) \arrow[u] & \sqrt{\hat{H}} = \frac{c}{n(q)}\|\hat{p}\| &
\end{tikzcd}$$

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\text{Fermat's principle} & L = n(q) \arrow[u] & \sqrt{\hat{H}} = \frac{c}{n(q)}\|\hat{p}\| \arrow[d] & \\
 & & H = \omega\check{H} = \frac{c}{n(q)}\|p\| & \text{light transport}
\end{tikzcd}$$

electromagnetic theory

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↓ microlocal analysis (Wigner transform)

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↓ unpolarized radiation

light transport equation

$$\dot{\ell} = -\{\ell, H\}$$

← Legendre transform

Fermat's principle

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electromagnetic theory

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$$\Big\downarrow$$

microlocal analysis
(Wigner transform)

phase space T^*Q

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$$\Big\downarrow \quad \varepsilon \to 0$$

$$\dot{W}_a^0 + \left\{\tau_a, W_a^0\right\} = \left[W_a^0, F_a\right]$$

conservation of
frequency

$$\Big\downarrow$$

unpolarized
radiation

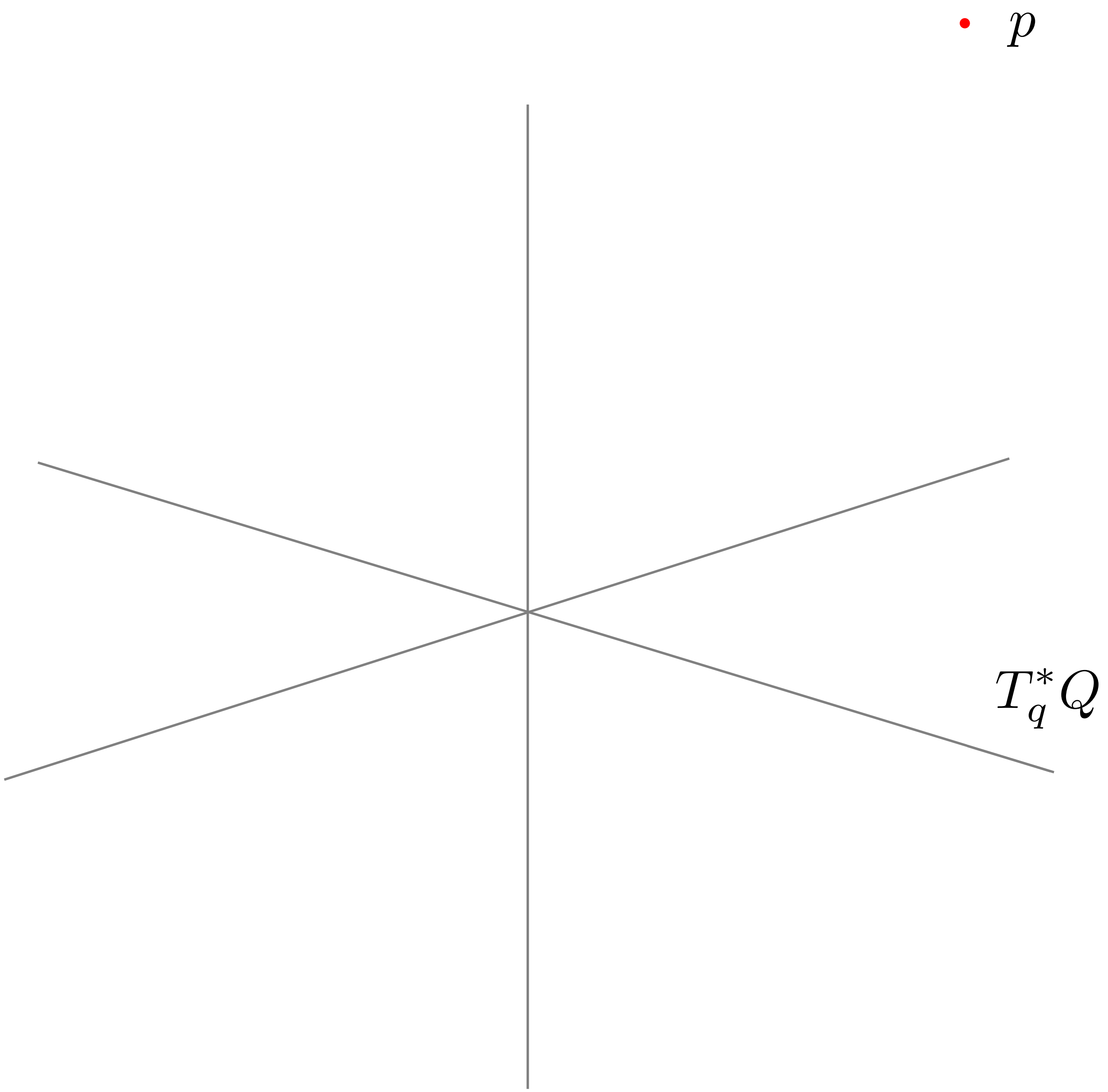
light transport equation

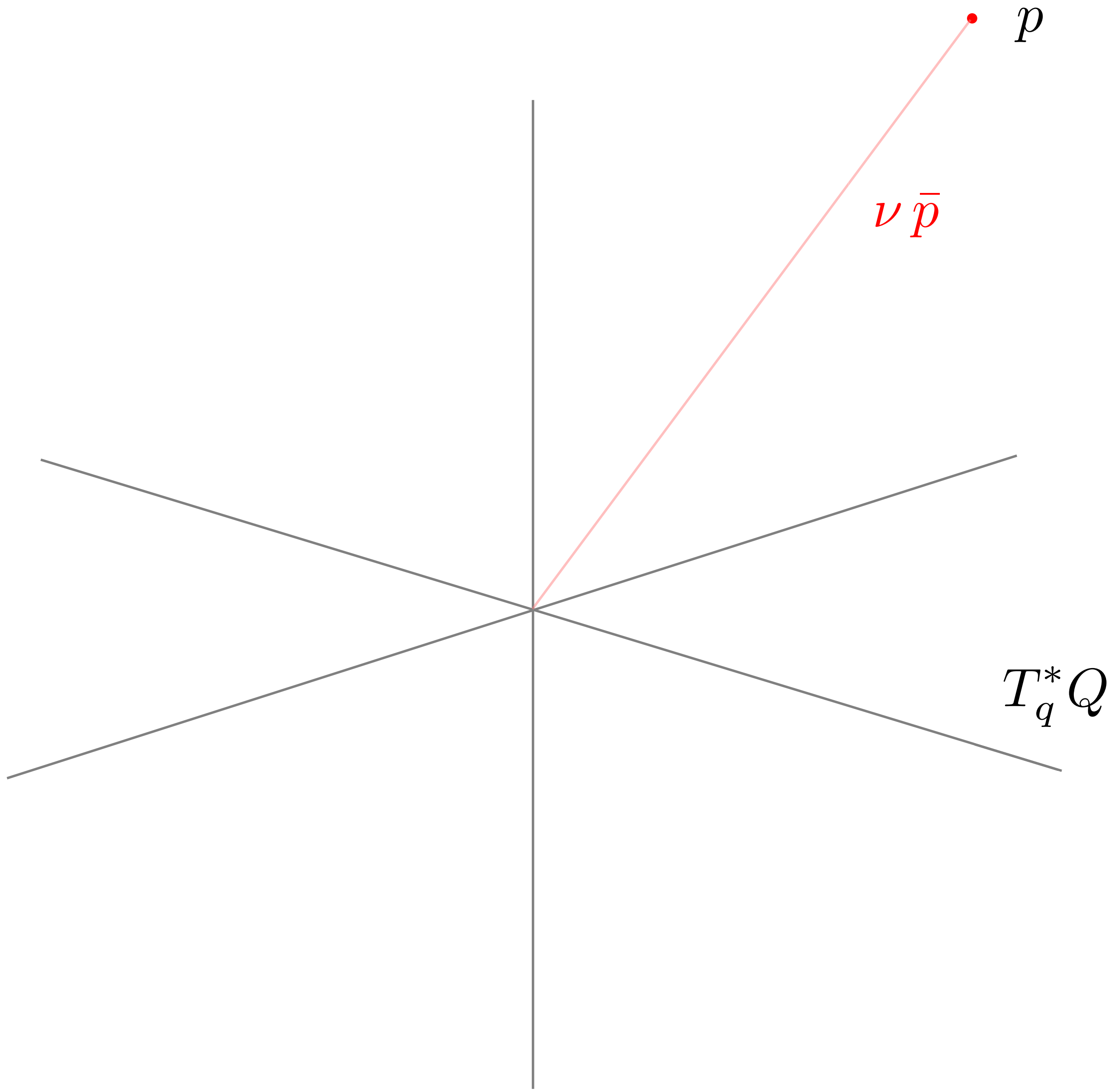
$$\dot{\ell} = -\{\ell, H\}$$

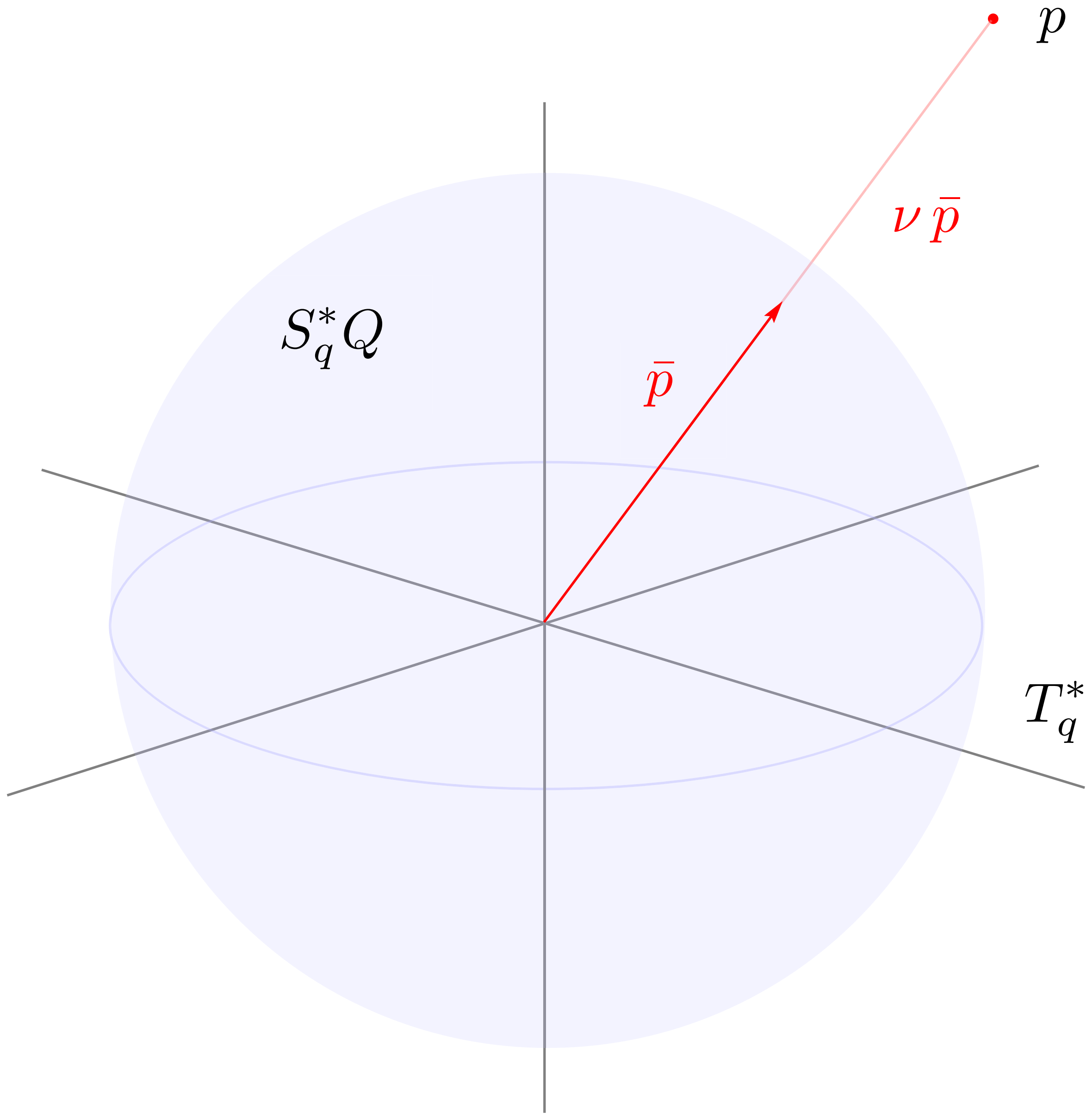
$$\xleftarrow[\text{transform}]{\text{Legendre}}$$

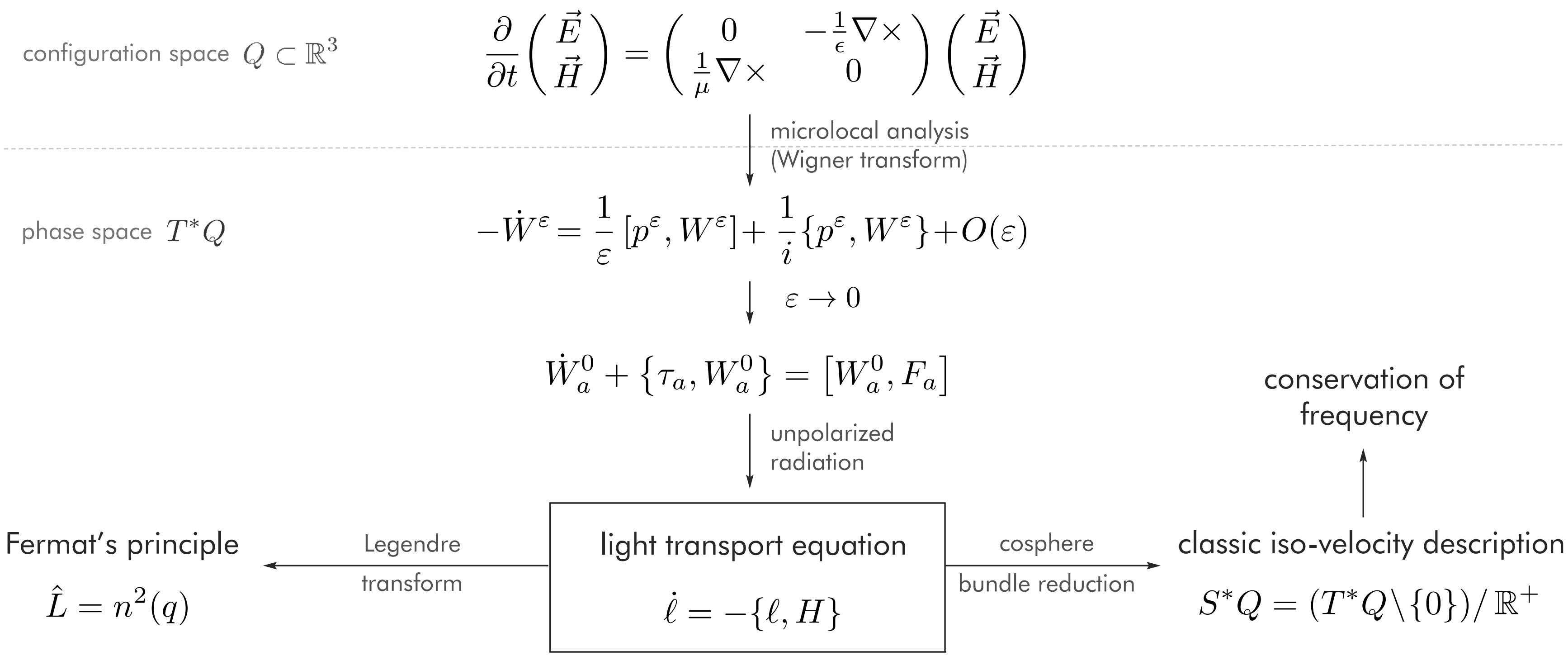
Fermat's principle

$$\hat{L} = n^2(q)$$


p
T_q^*Q


p
$\nu\,\bar{p}$
T_q^*Q


p
$\nu\,\bar{p}$
$S^*_q Q$
$\bar{p}$
$T^*_q Q$



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unpolarized
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conservation of
frequency

Fermat's principle

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Legendre
transform

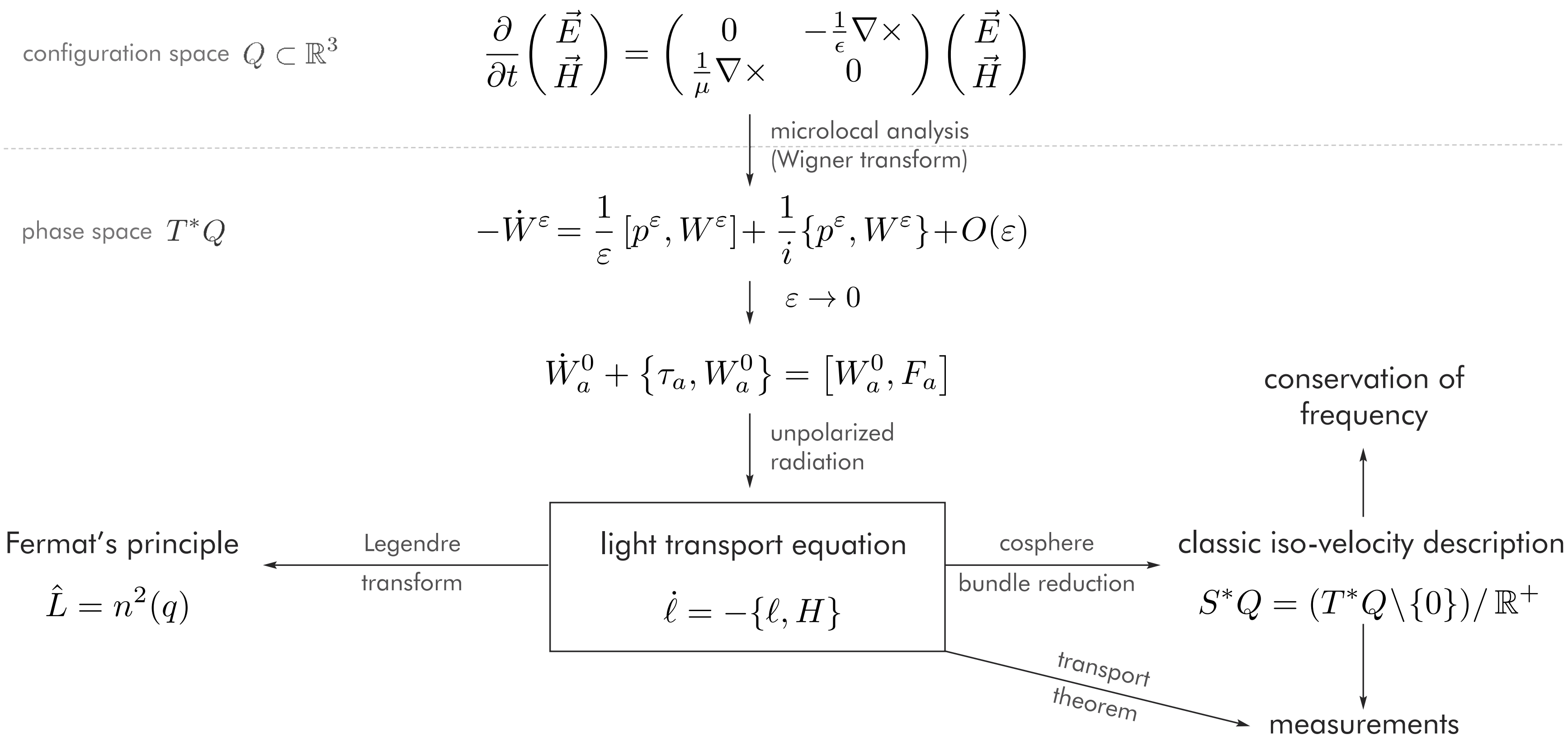
light transport equation

$\dot{\ell} = -\{\ell, H\}$

cosphere
bundle reduction

classic iso-velocity description

$S^*Q = (T^*Q \backslash \{0\}) / \mathbb{R}^+$



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microlocal analysis (Wigner transform)

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Legendre transform

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cosphere bundle reduction

classic iso-velocity description

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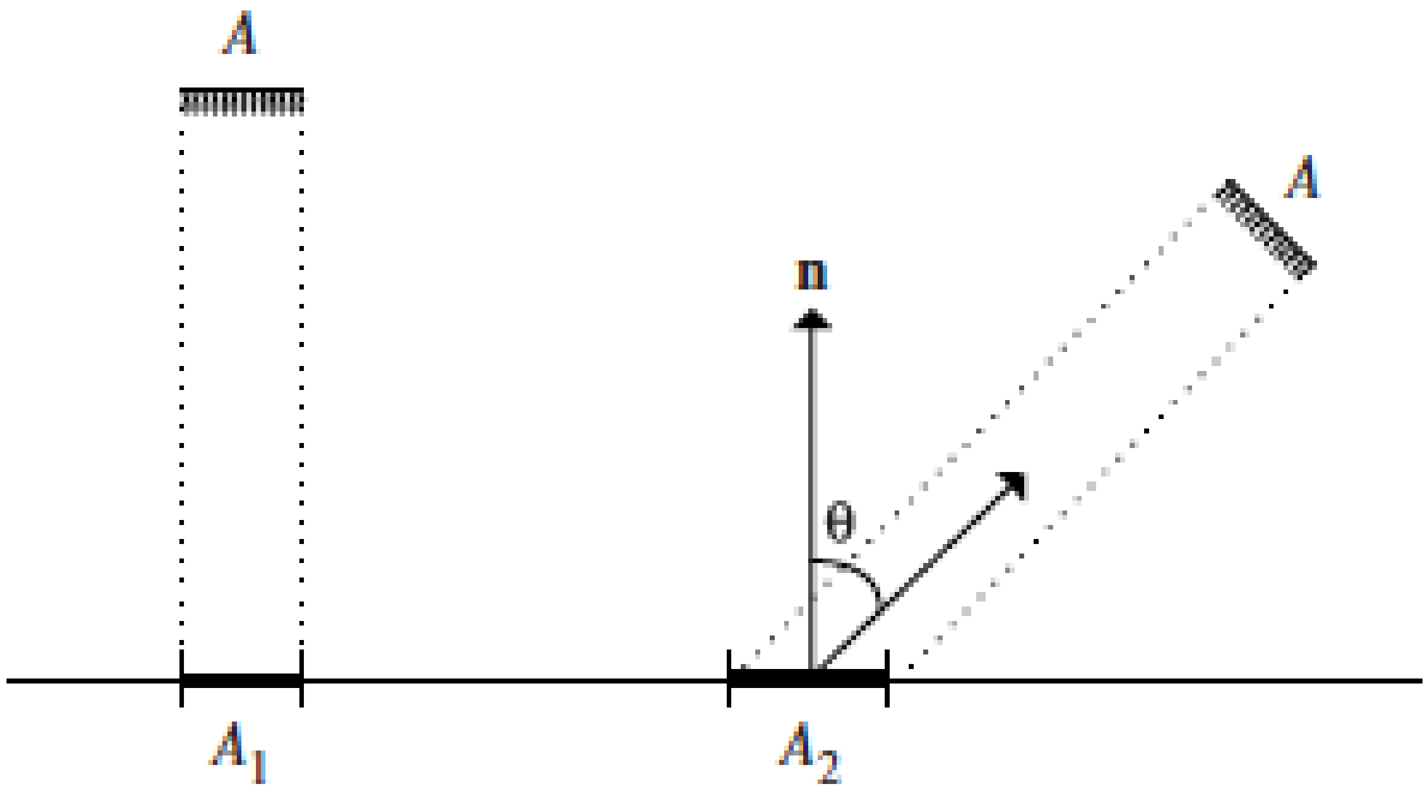
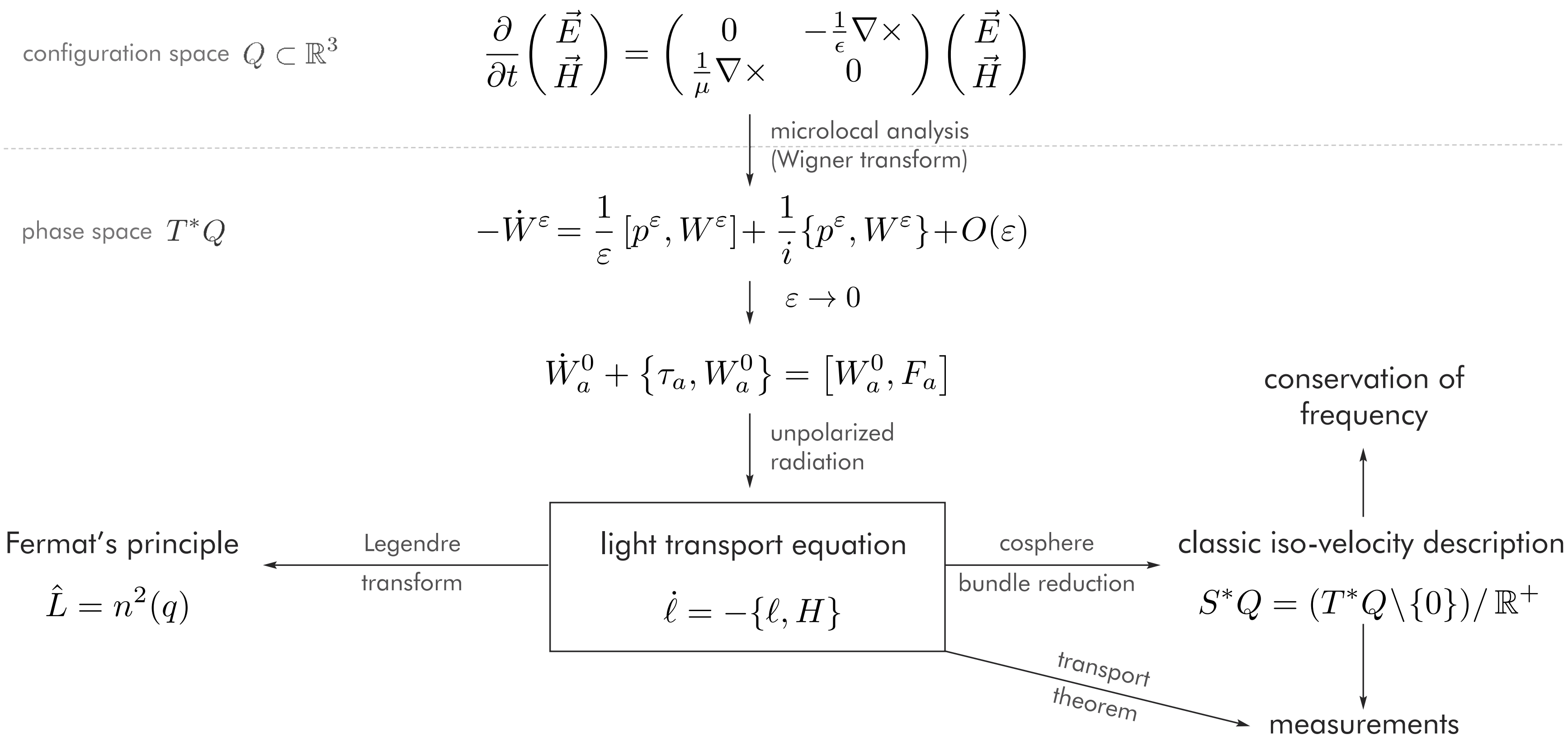
conservation of frequency

transport theorem

measurements


A
A
n
θ
A_1
A_2

electromagnetic theory



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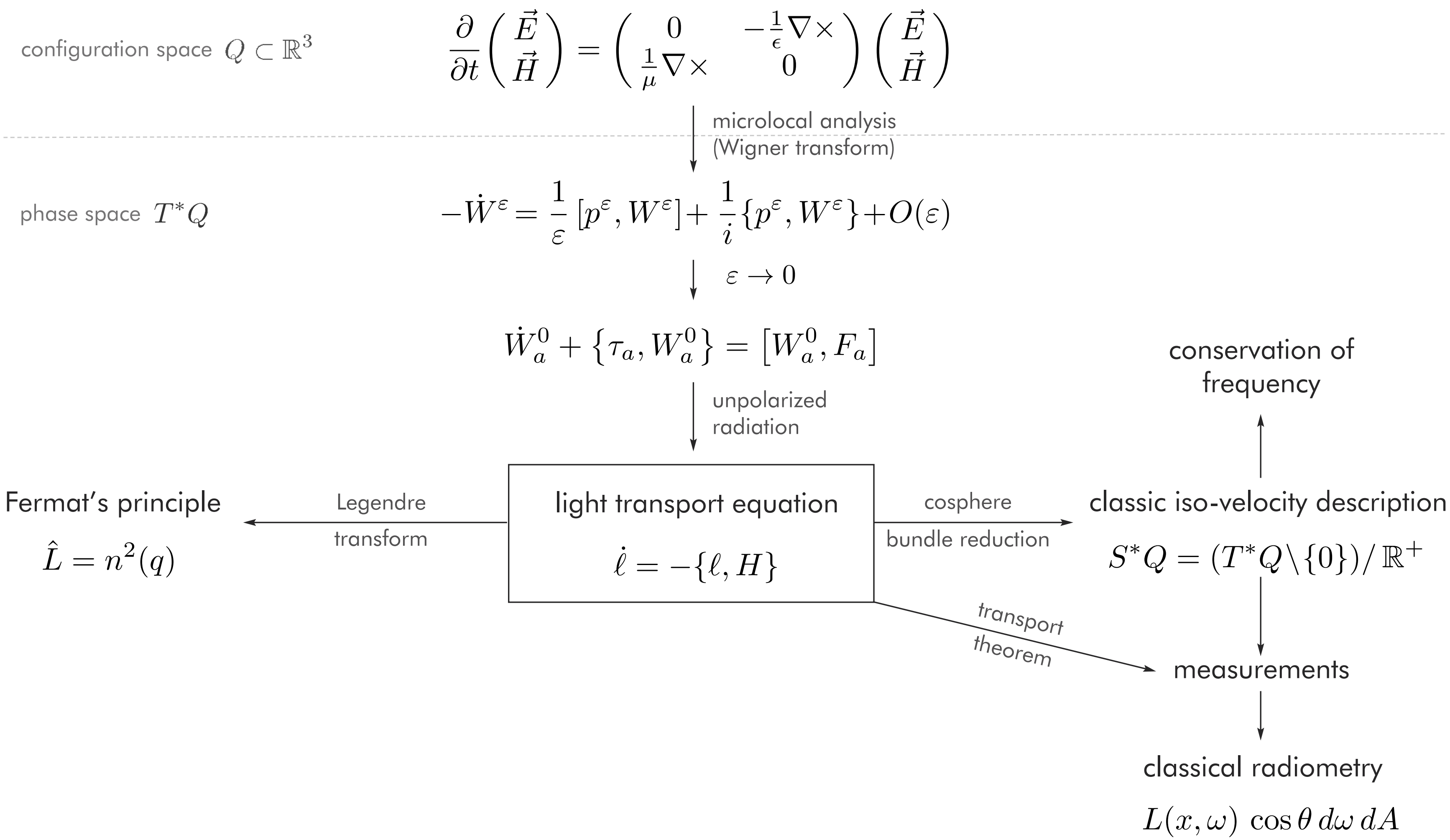

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cosphere bundle reduction

classic iso-velocity description

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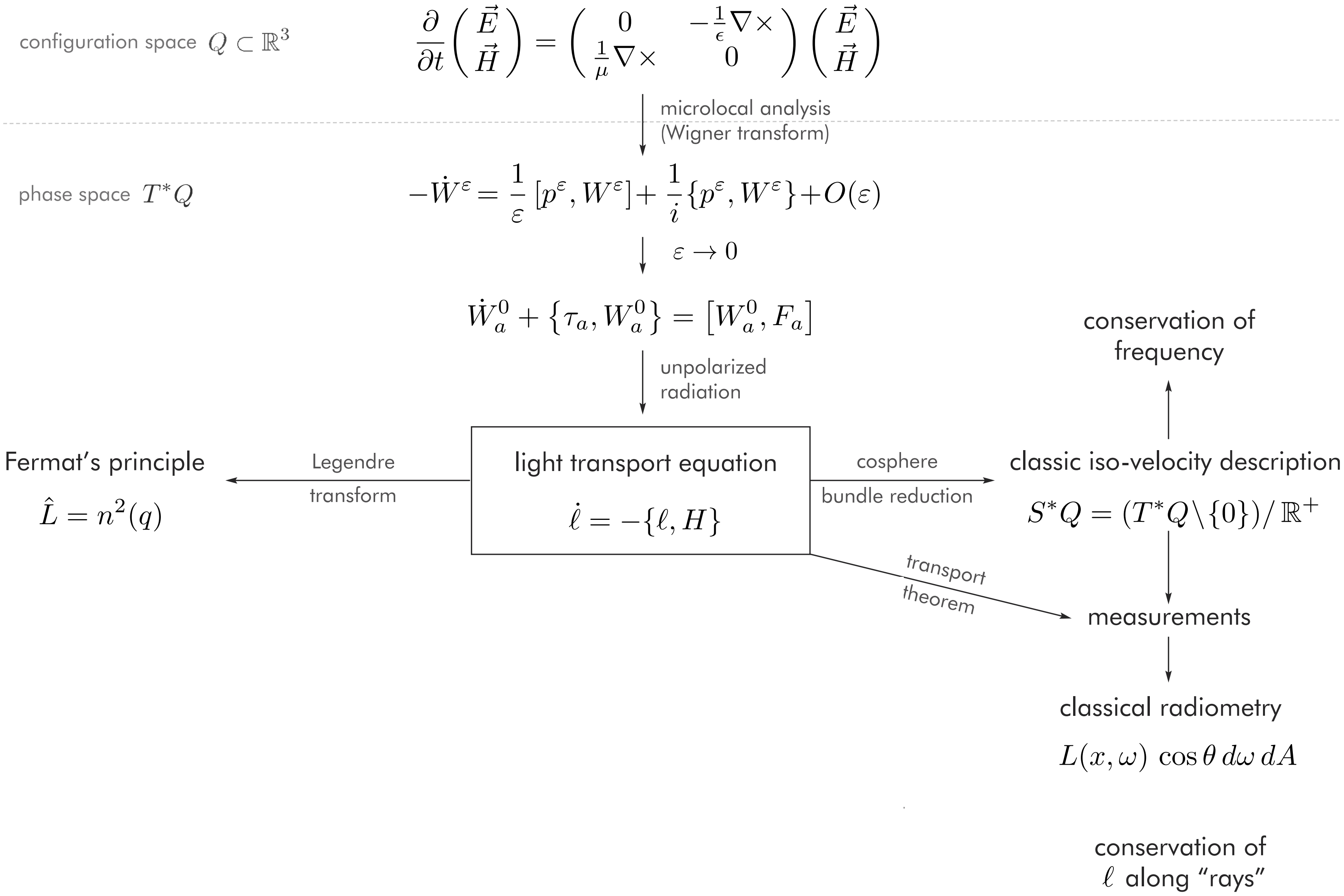
conservation of frequency

transport theorem

measurements

classical radiometry

$$L(x,\omega)\ \cos\theta\, d\omega\, dA$$



configuration space $Q \subset \mathbb{R}^3$

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conservation of frequency

transport theorem

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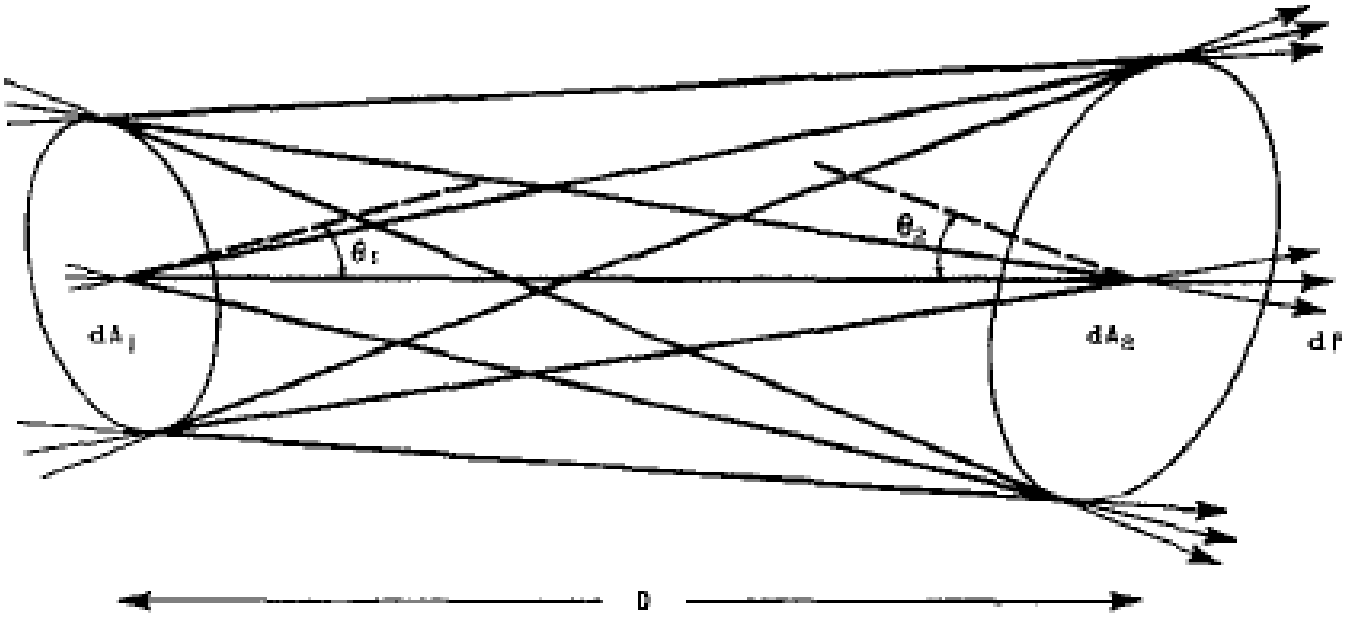
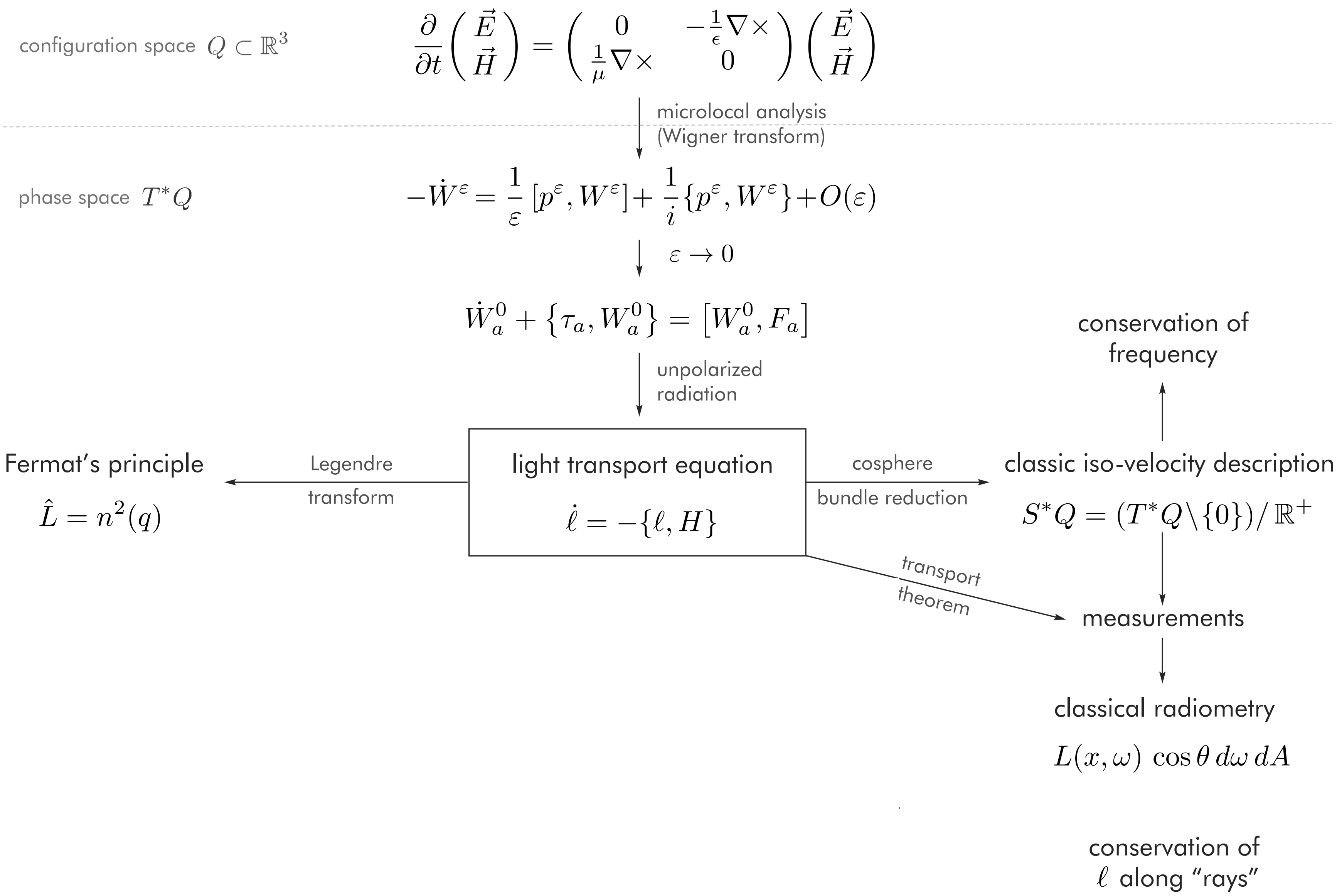
$$L(x,\omega)\,\cos\theta\,d\omega\,dA$$

conservation of ℓ along "rays"


θ_1
θ_2
dA_1
dA_2
dr
D



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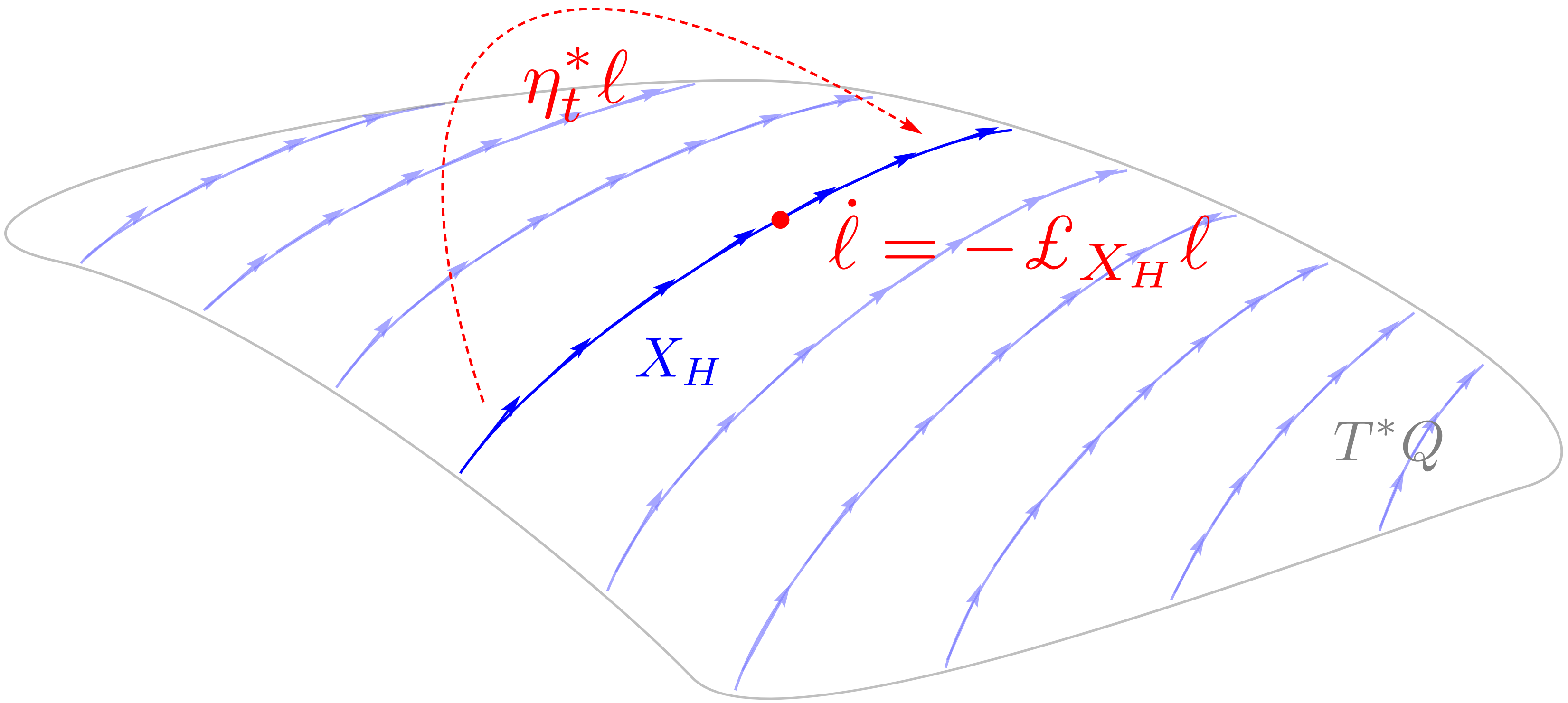
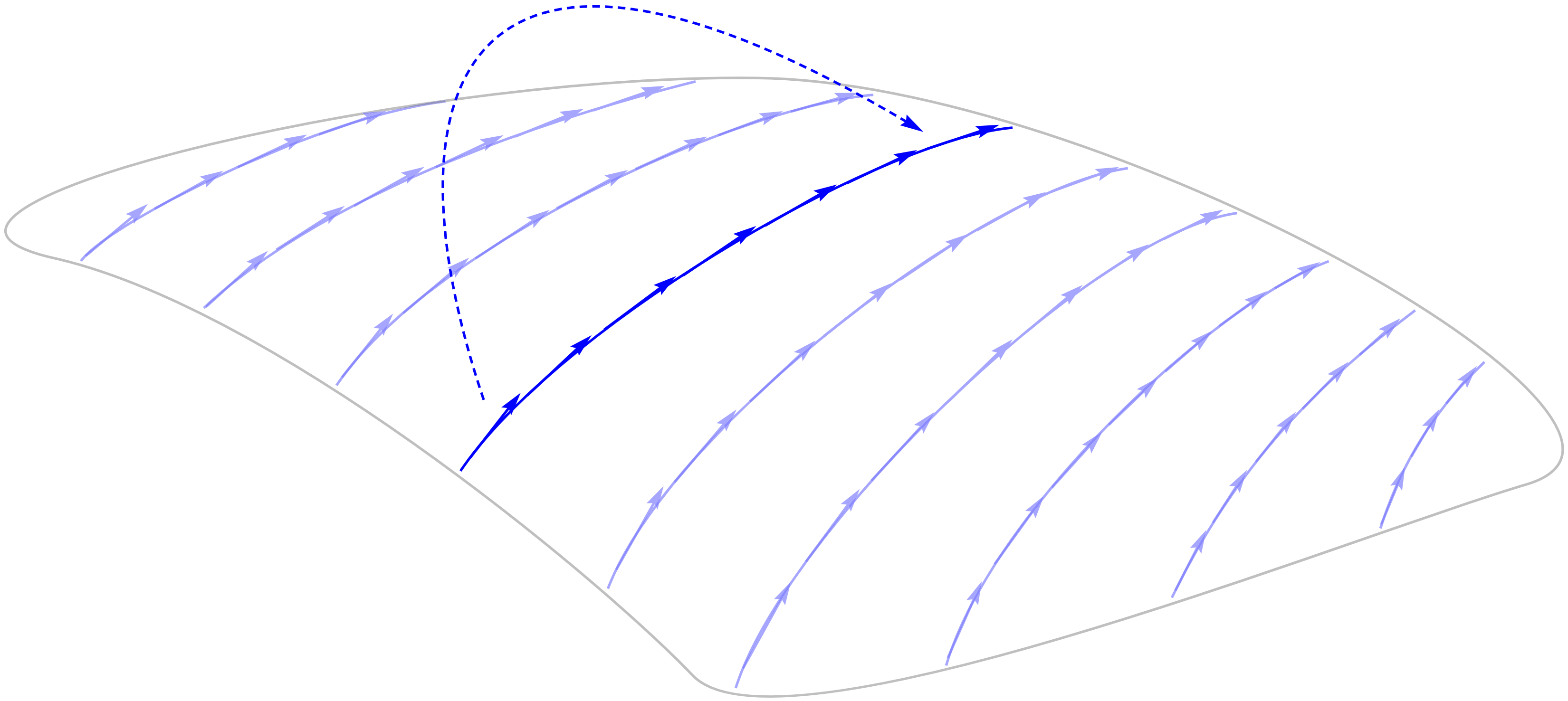
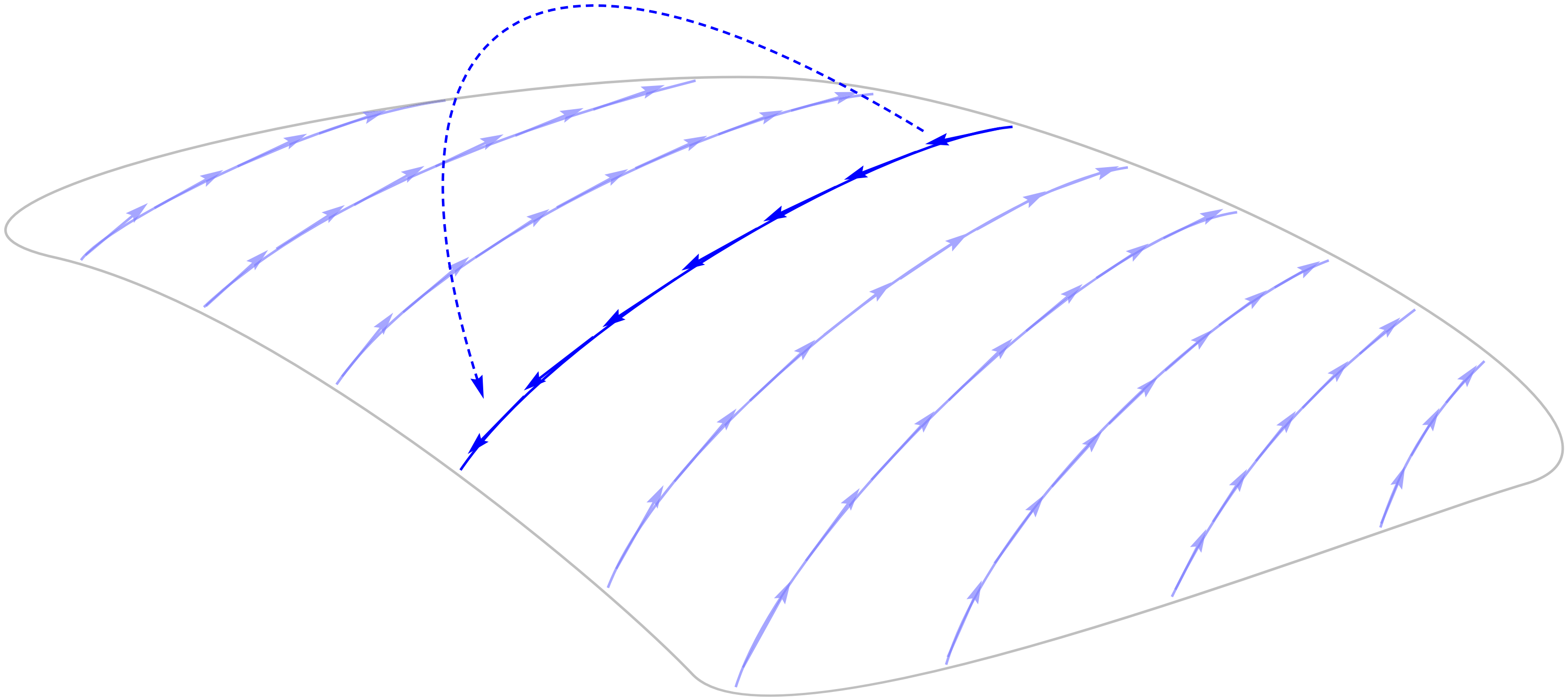
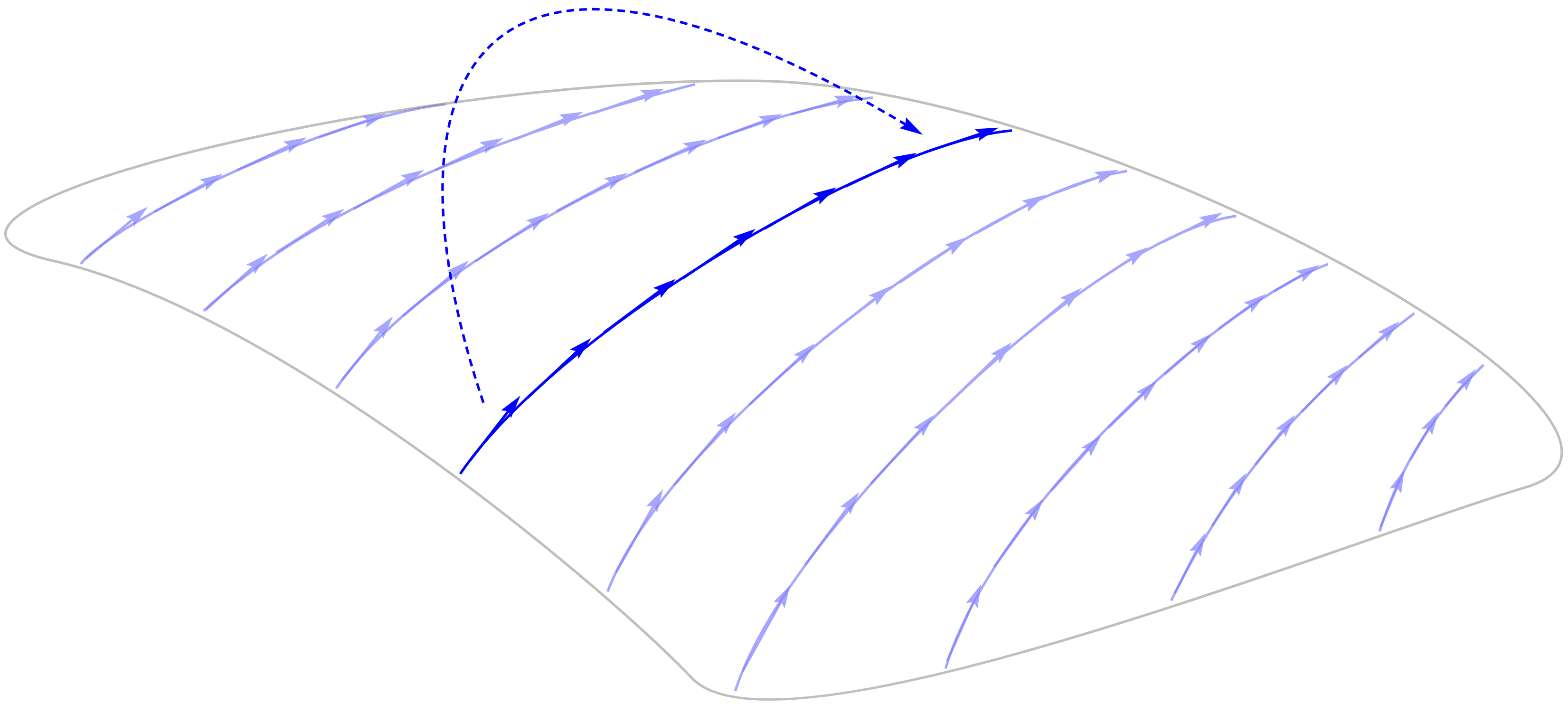
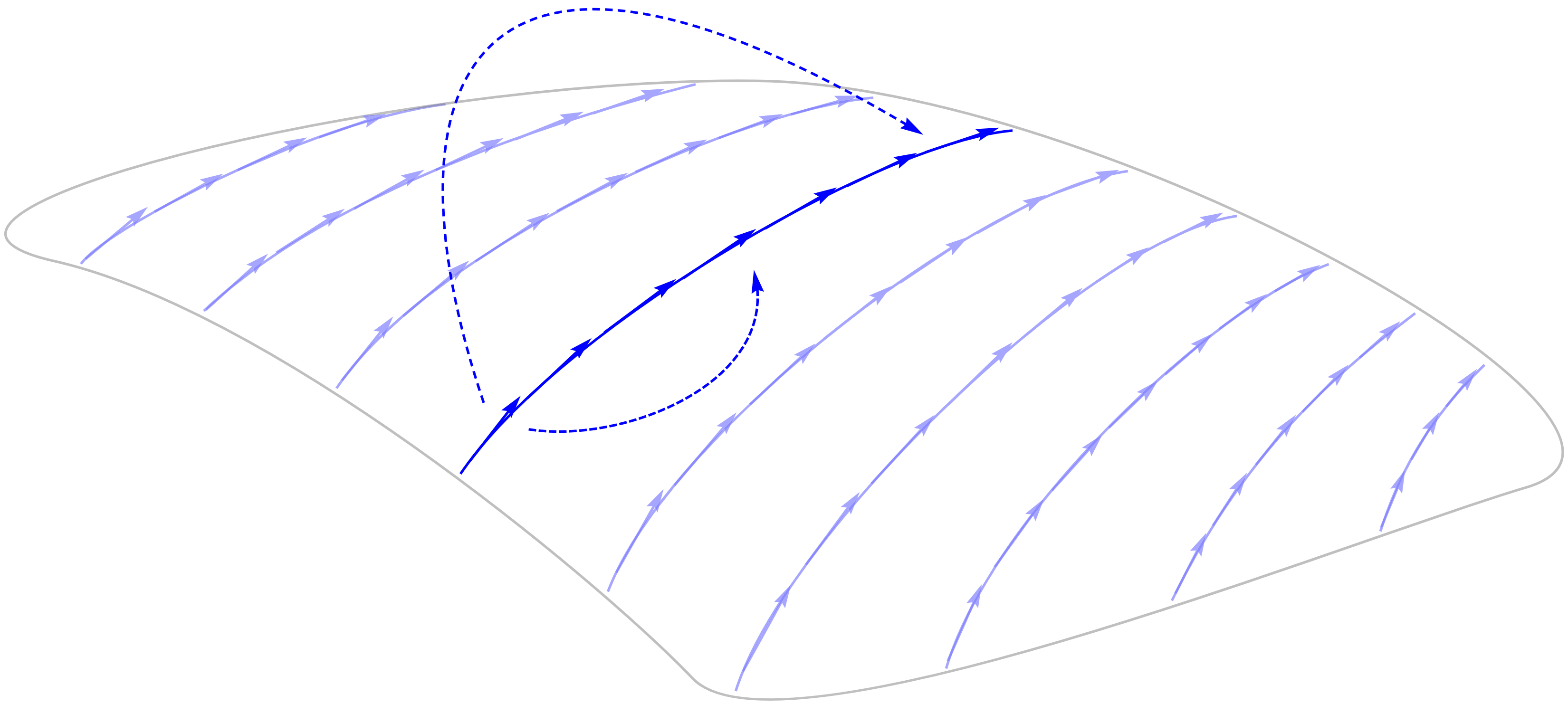


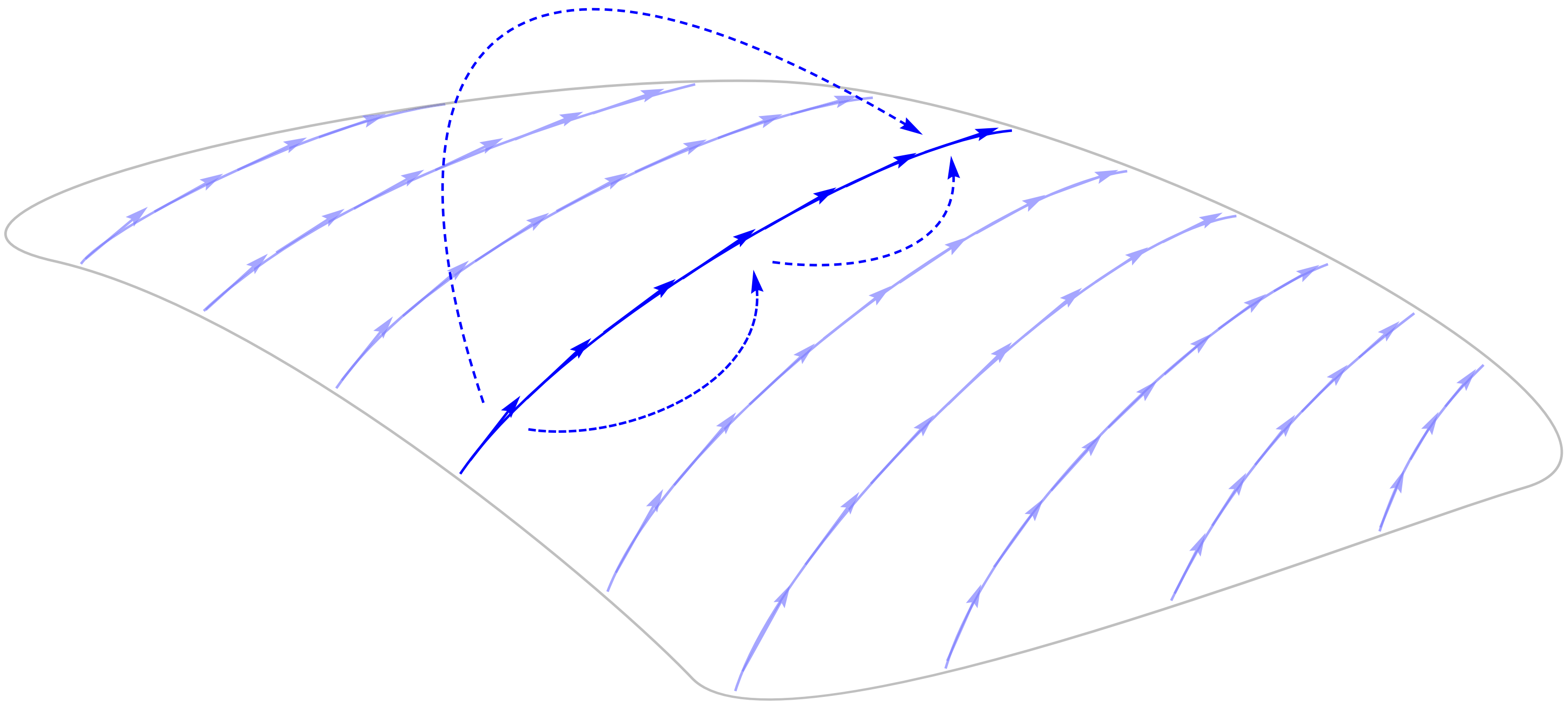
$\eta_t^* \ell$
$\dot{\ell} = -\pounds_{X_H} \ell$
X_H
T^*Q

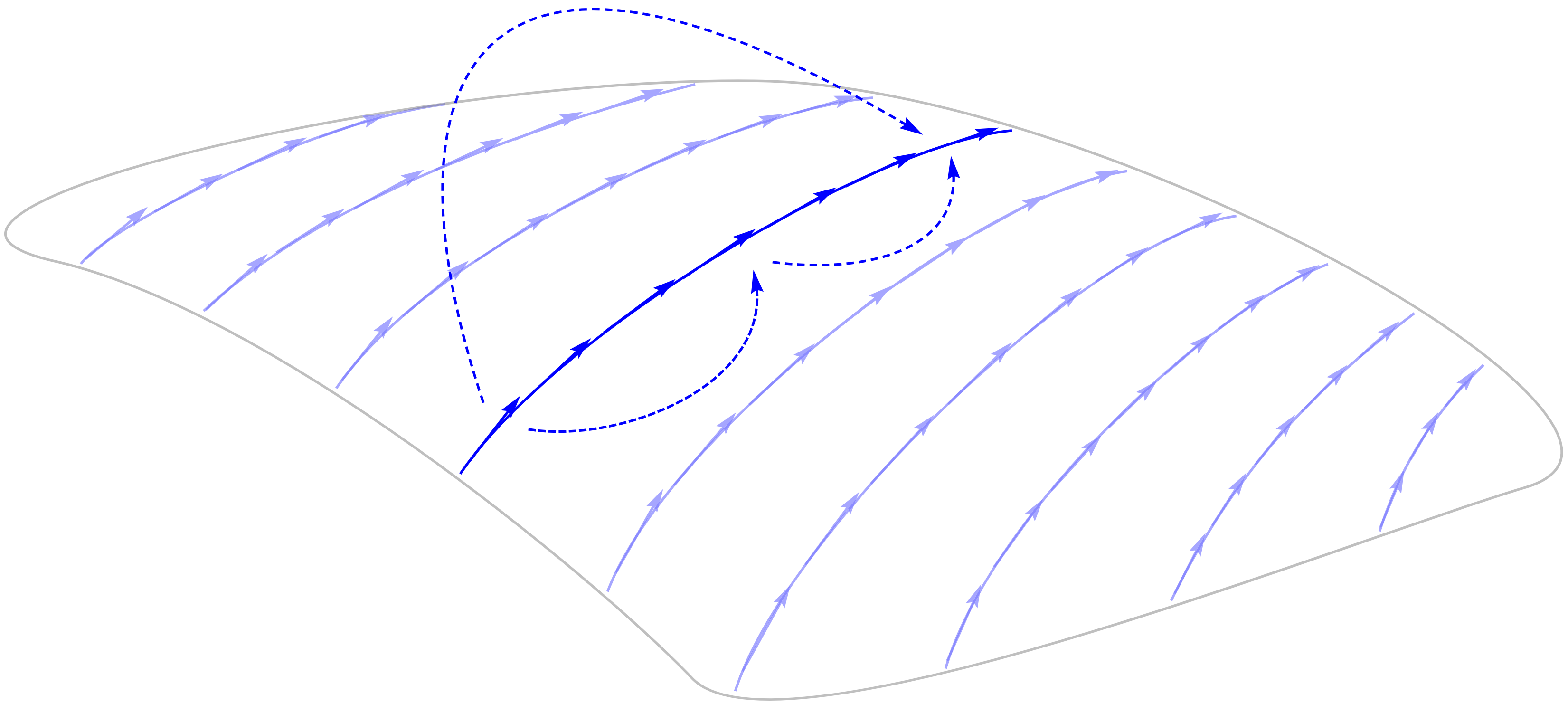


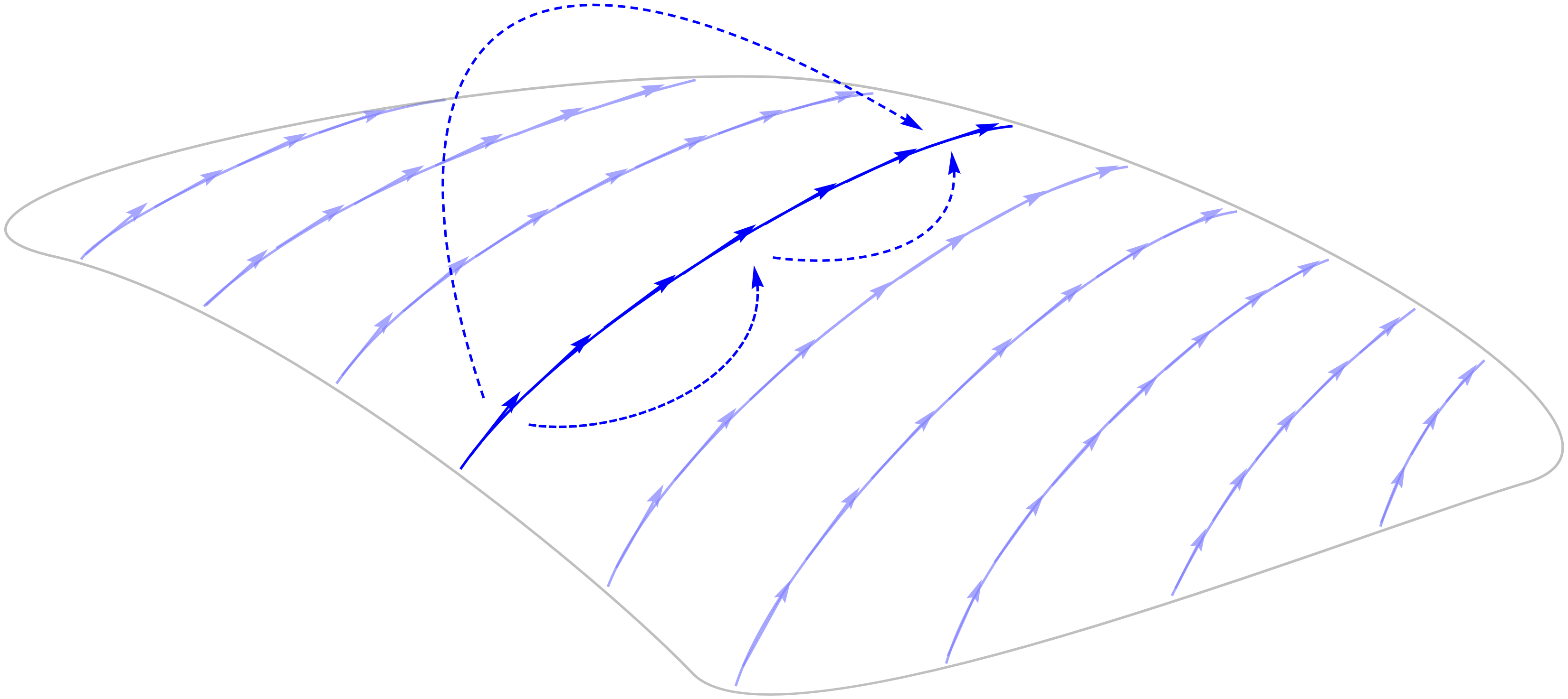




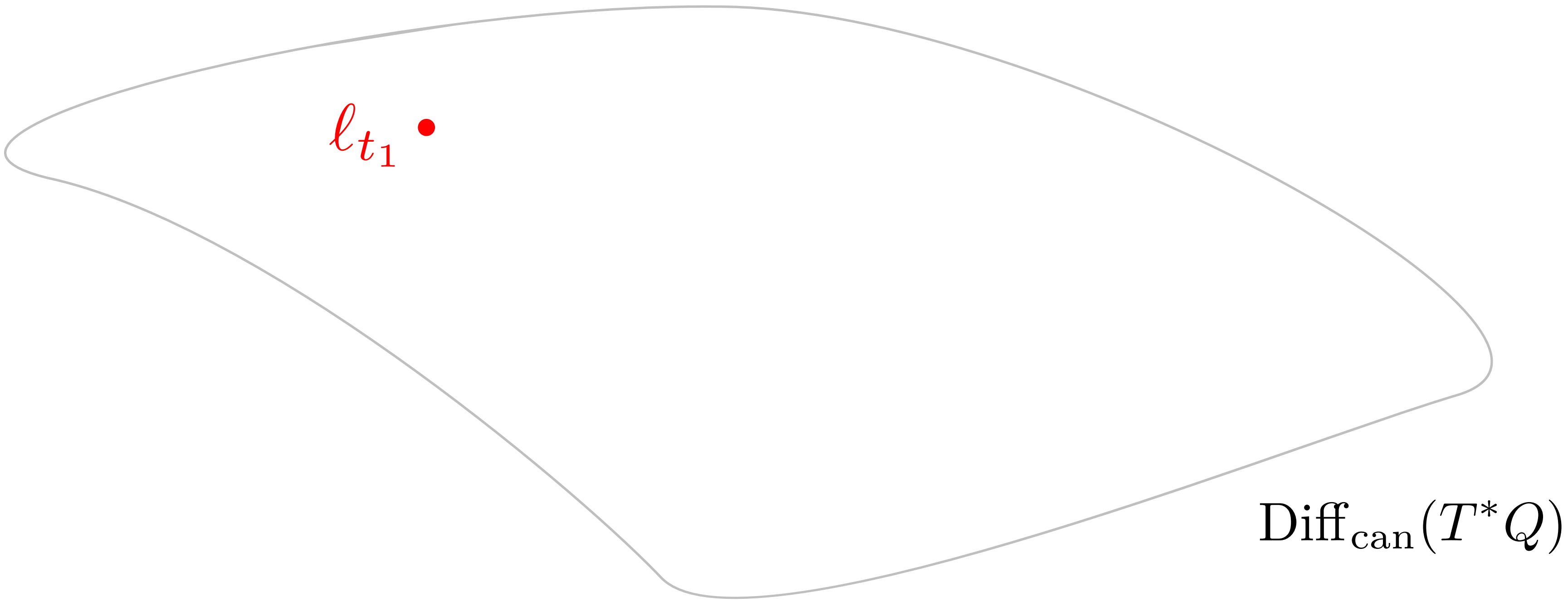


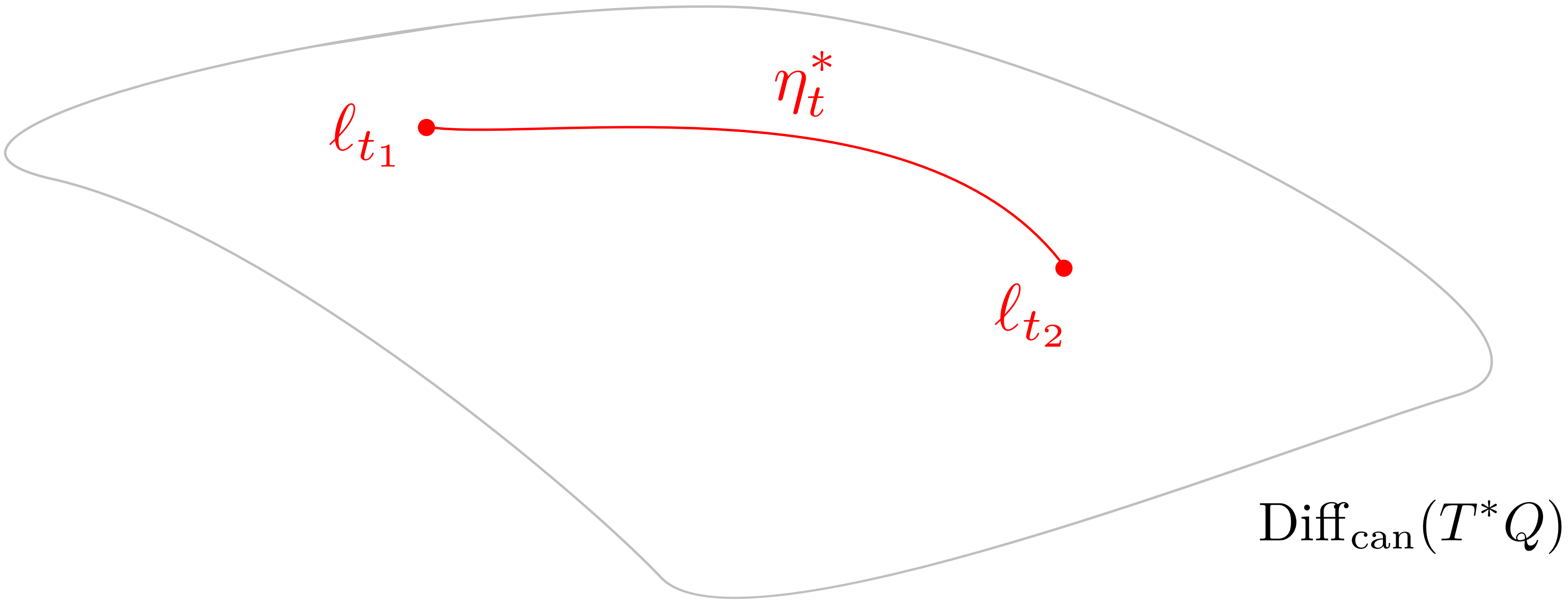






$Q = G = \mathrm{Diff}_{\mathrm{can}}(T^*Q)$


ℓ_{t_1}
$\mathrm{Diff}_{\mathrm{can}}(T^*Q)$


η_t^*
ℓ_{t_1}
ℓ_{t_2}
$\mathrm{Diff}_{\mathrm{can}}(T^*Q)$

TG

Lagrangian

T^*G

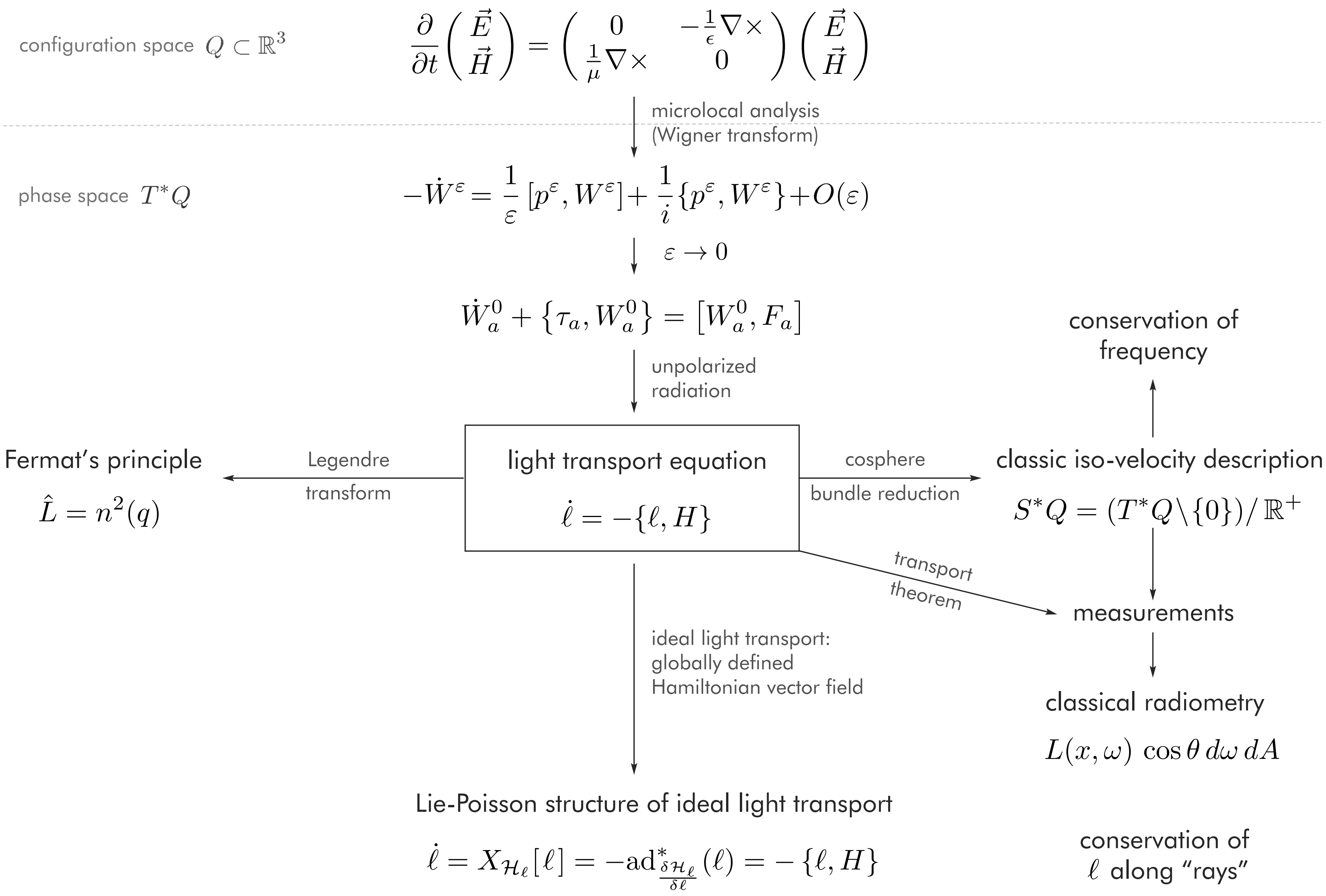
$$\begin{tikzcd}
TG \arrow[rr, "(T_g R_{g^{-1}})\, V_t = v_t"] & & \mathfrak{g}_+ \\
\text{Lagrangian} & \begin{array}{c}\text{right}\\ \text{translation}\end{array} & \text{Eulerian} \\
T^*G \arrow[rr] & & \mathfrak{g}_+^*
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{g}_- & & TG \arrow[ll, "\mathcal{V}_t = (T_g L_{g^{-1}})\, V_t"'] \arrow[rr, "(T_g R_{g^{-1}})\, V_t = v_t"] & & \mathfrak{g}_+ \\
\text{Convective} & \text{left translation} & \text{Lagrangian} & \text{right translation} & \text{Eulerian} \\
\mathfrak{g}^*_- & & T^*G \arrow[ll] \arrow[rr] & & \mathfrak{g}^*_+
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{g}_- & TG \arrow[l, "\mathcal{V}_t = (T_g L_{g^{-1}})\, V_t"'] \arrow[l, "\begin{array}{c}\text{left}\\ \text{translation}\end{array}"] \arrow[r, "(T_g R_{g^{-1}})\, V_t = v_t"] \arrow[r, "\begin{array}{c}\text{right}\\ \text{translation}\end{array}"'] & \mathfrak{g}_+ \\
\text{Convective} & \text{Lagrangian} & \text{Eulerian} \\
\mathfrak{g}^*_- & T^*G \arrow[l] \arrow[r] & \mathfrak{g}^*_+
\end{tikzcd}$$

right invariant
Hamiltonian

$$\begin{tikzcd}
\mathfrak{g}_- & & TG \arrow[ll, "\mathcal{V}_t = (T_g L_{g^{-1}})\, V_t"'] \arrow[rr, "(T_g R_{g^{-1}})\, V_t = v_t"] & & \mathfrak{g}_+ \\
\text{Convective} & \text{left translation} & \text{Lagrangian} & \text{right translation} & \text{Eulerian} \\
\mathfrak{g}_-^* & & T^*G \arrow[ll] \arrow[rr] & & \mathfrak{g}_+^* \\
 & & & & \text{Dynamics} \\
 & & & \mathbf{J}_L = T_e^* R \arrow[uur] & \\
 & & \text{right invariant Hamiltonian} \arrow[ur, dash] & &
\end{tikzcd}$$



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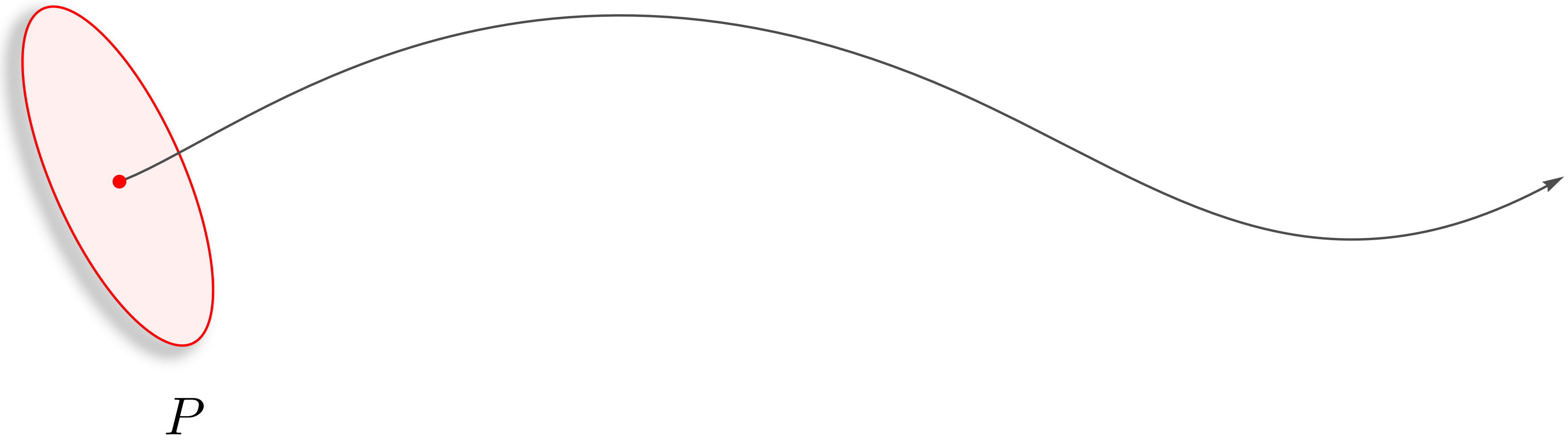
\mathfrak{g}^*_- & & T^*G \arrow[ll] \arrow[rr] & & \mathfrak{g}^*_+ \\
& \mathbf{J}_R = T_e^* L \arrow[ul, "\text{Noether's theorem}"] & & \mathbf{J}_L = T_e^* R \arrow[ur, "\text{Dynamics}"'] & \\
& & \text{right invariant Hamiltonian} \arrow[ul, dash] \arrow[ur, dash] & &
\end{tikzcd}$$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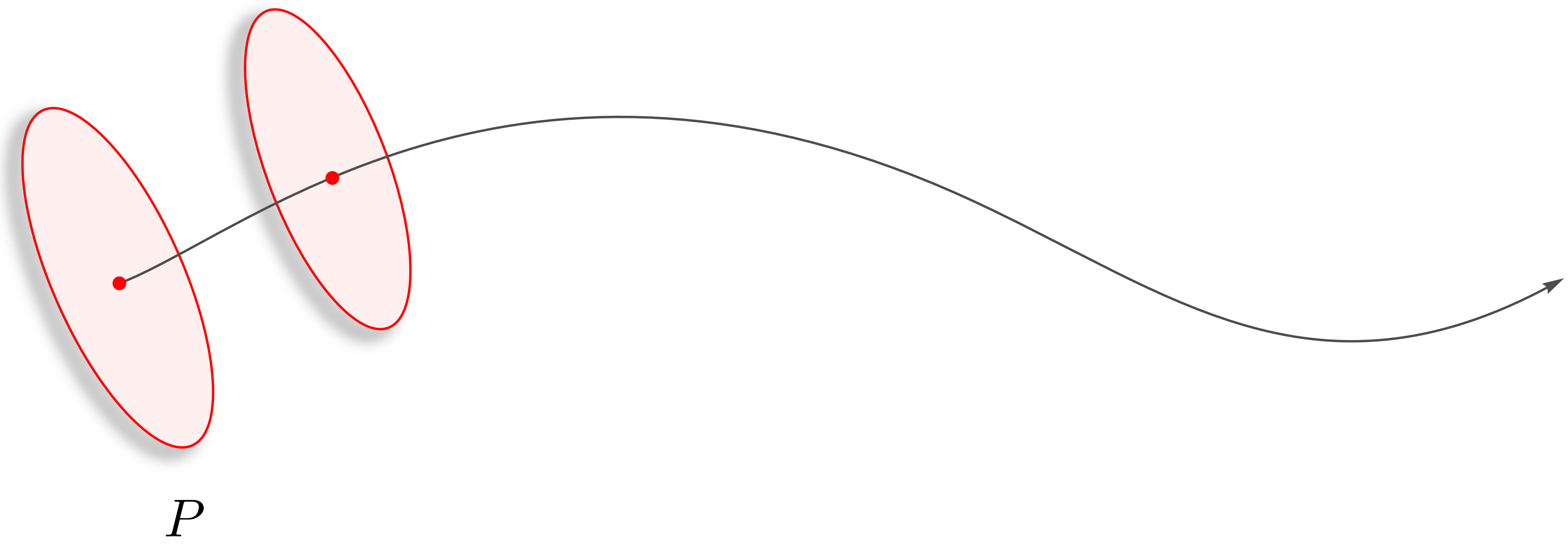
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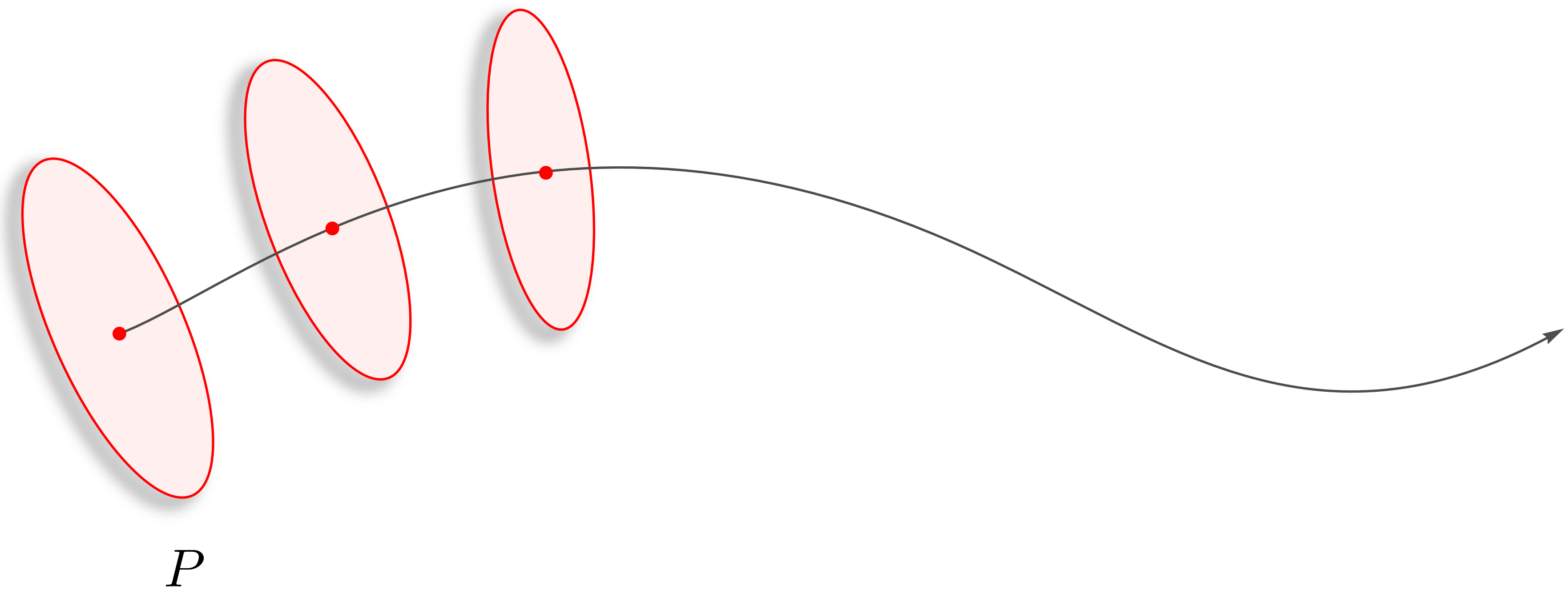
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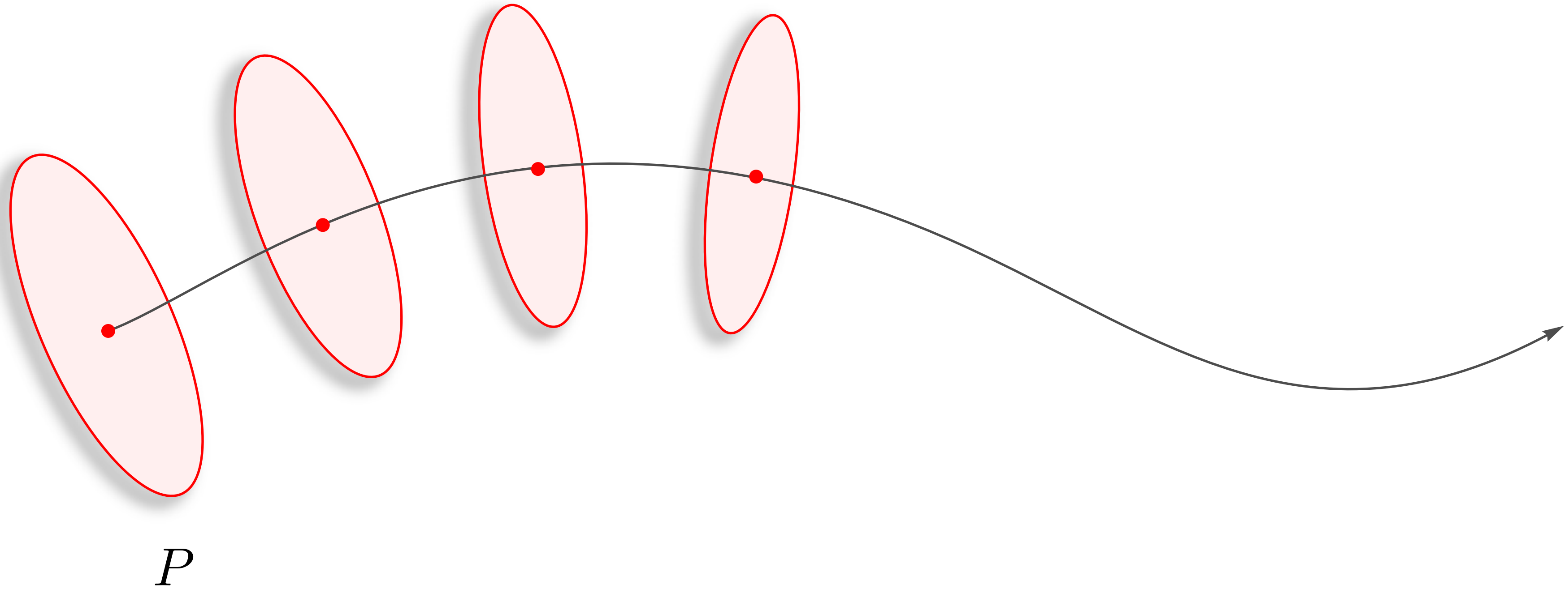
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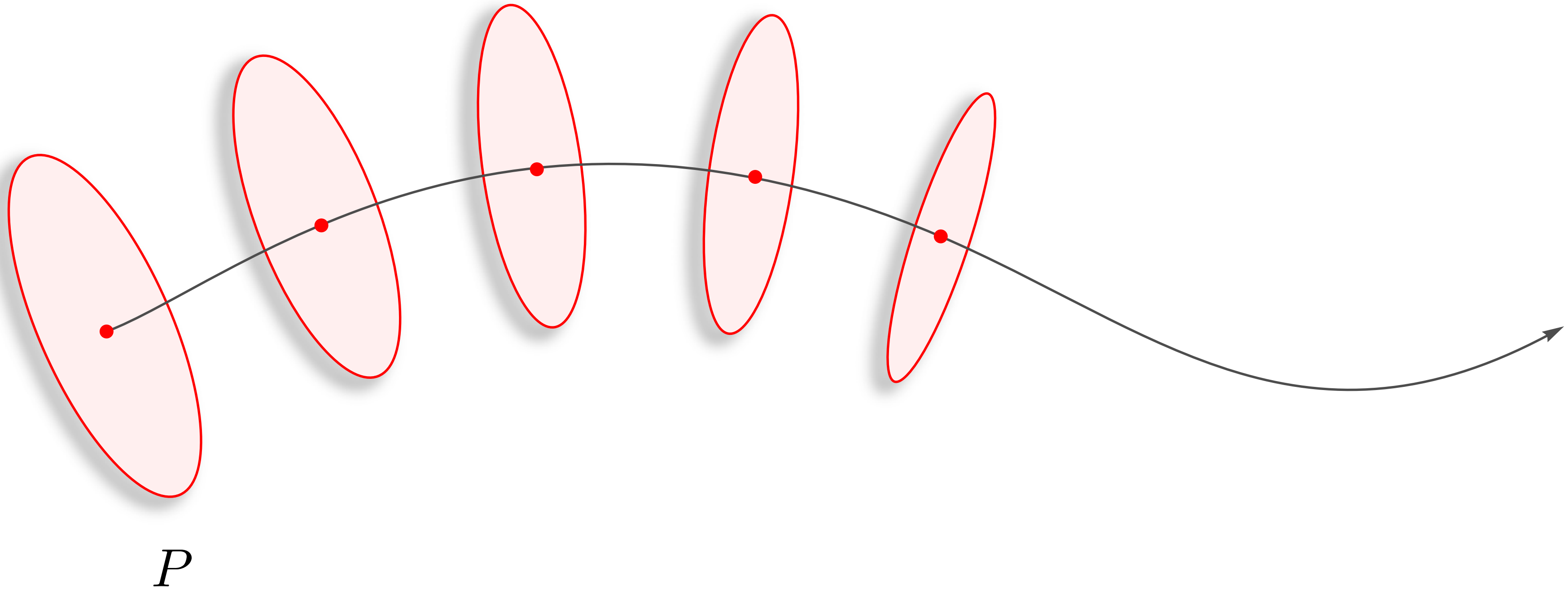
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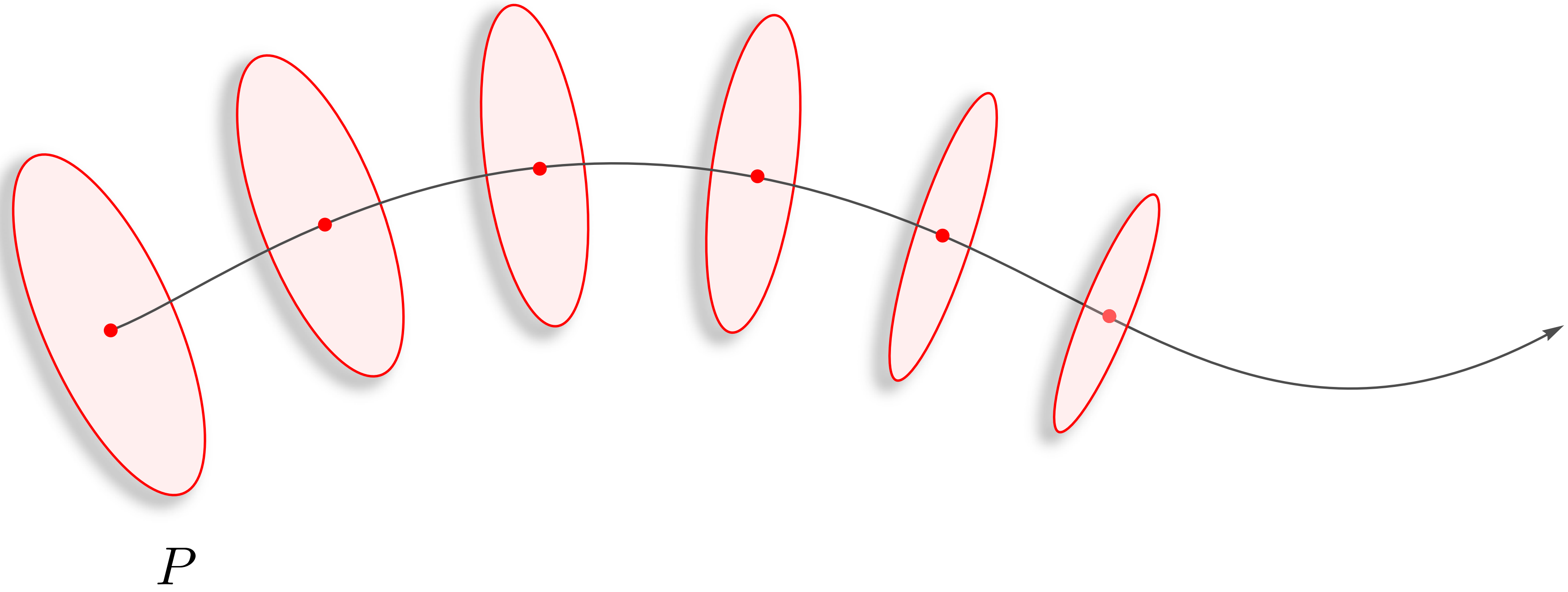
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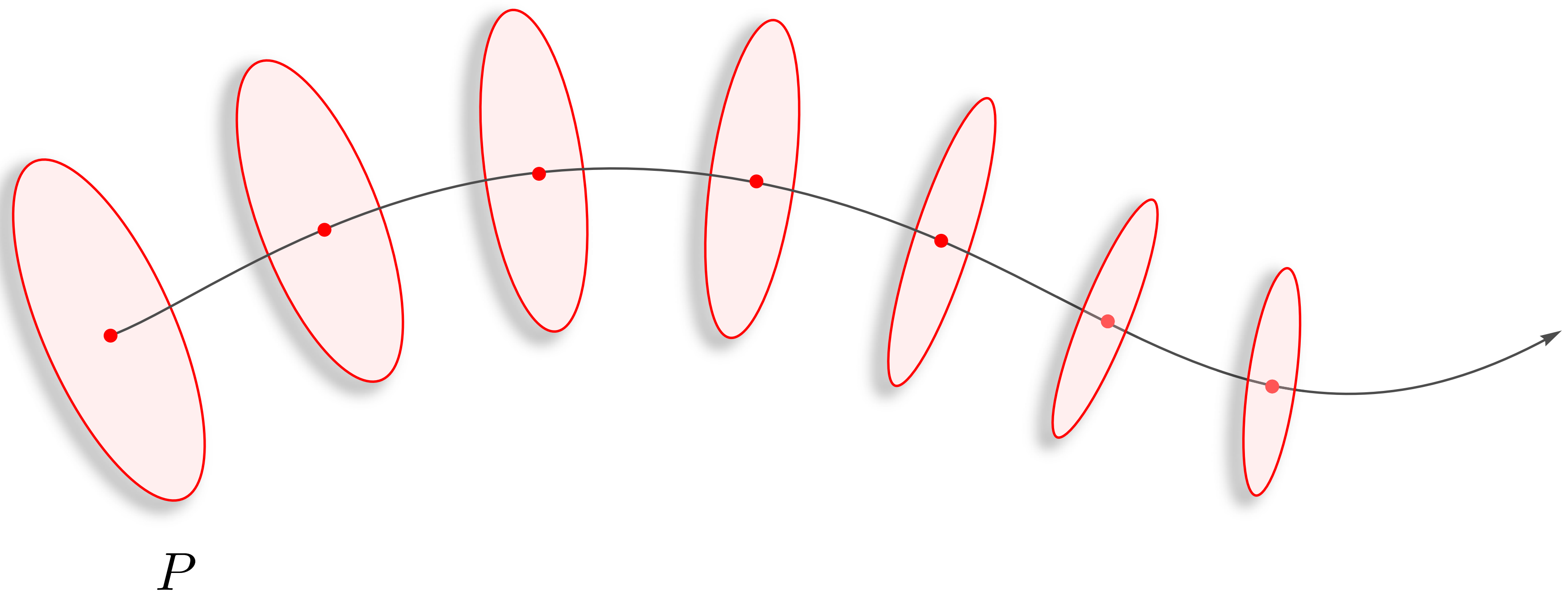
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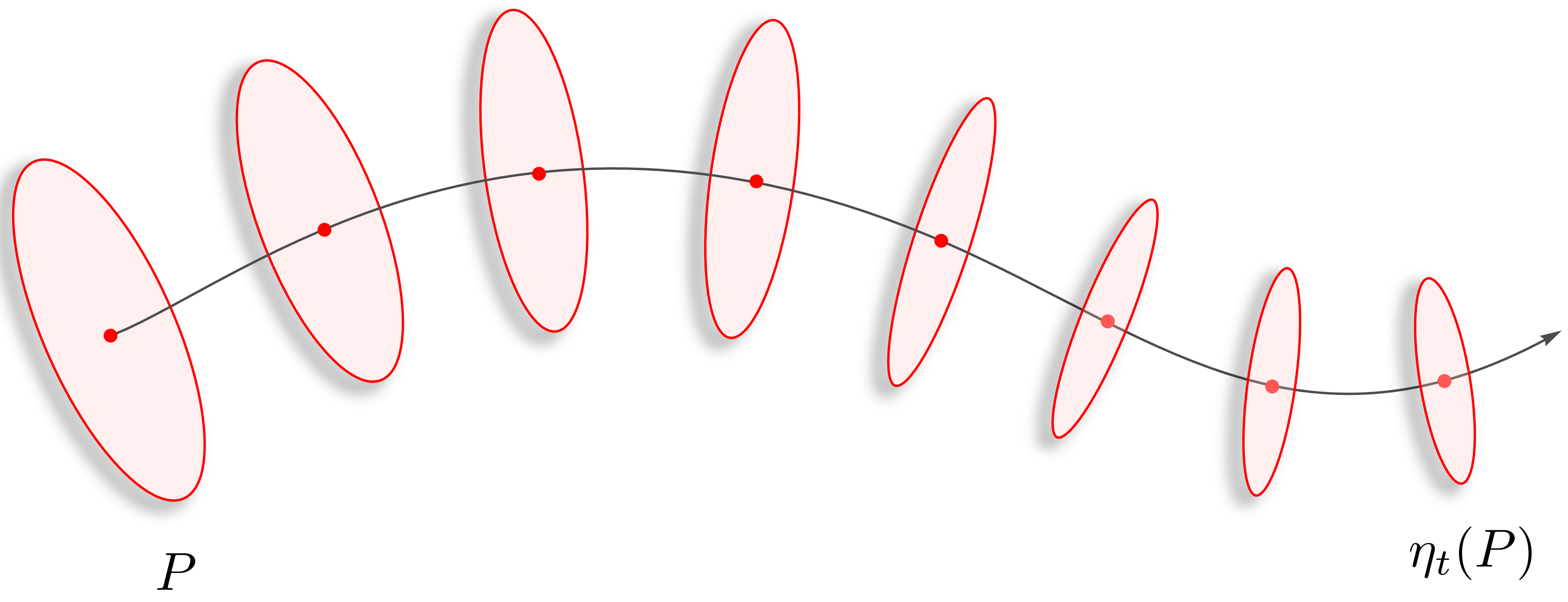
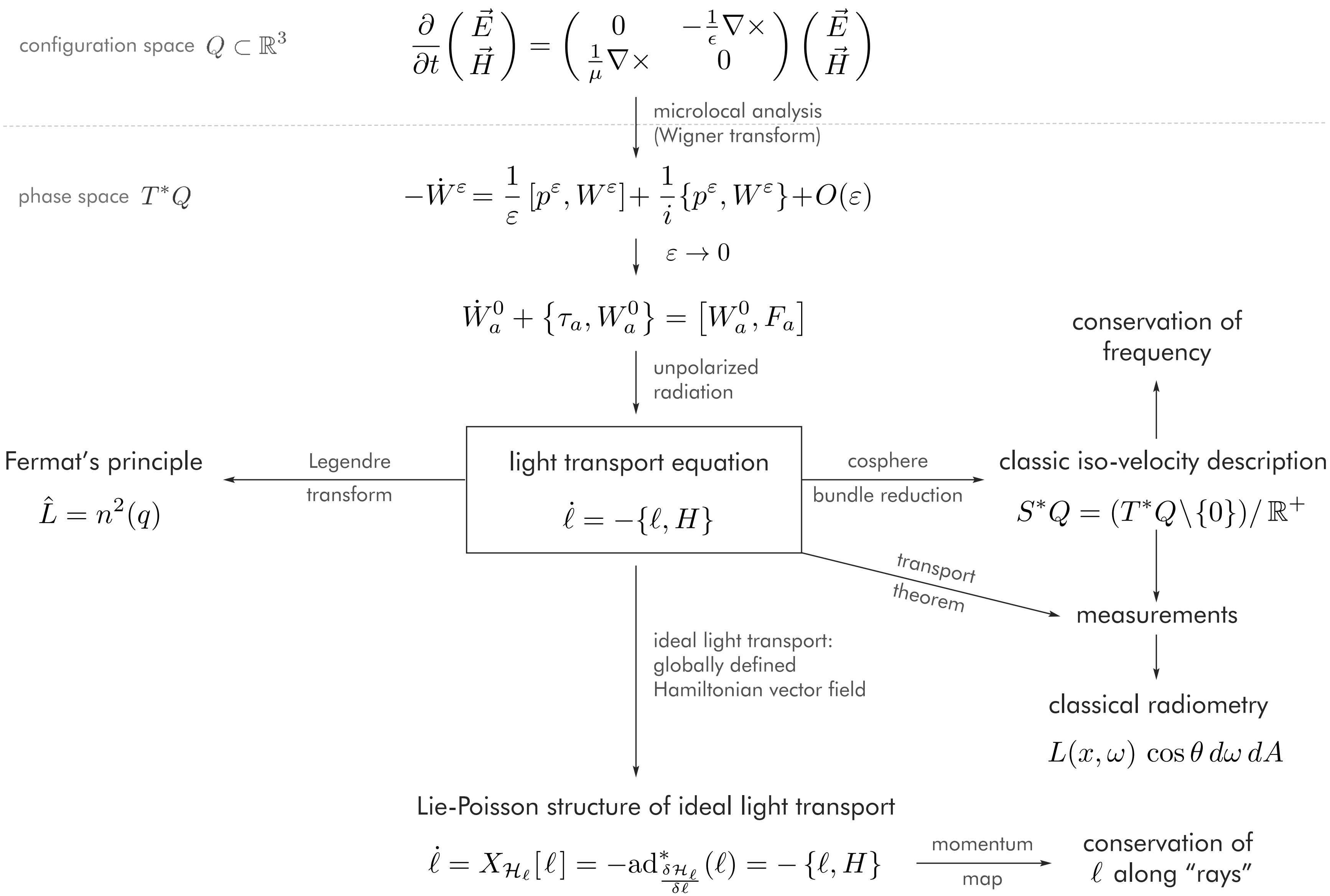

P
$\eta_t(P)$



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momentum map

conservation of ℓ along "rays"

	ideal fluid dynamics	ideal light transport
Lie group		
Lie algebra		
dual Lie algebra		
coadjoint action		
momentum map		

	ideal fluid dynamics	ideal light transport
Lie group	$\mathrm{Diff}_{\mu}(Q)$	$\mathrm{Diff}_{\mathrm{can}}(T^{*}Q)$
Lie algebra		
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momentum map		

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dual Lie algebra		
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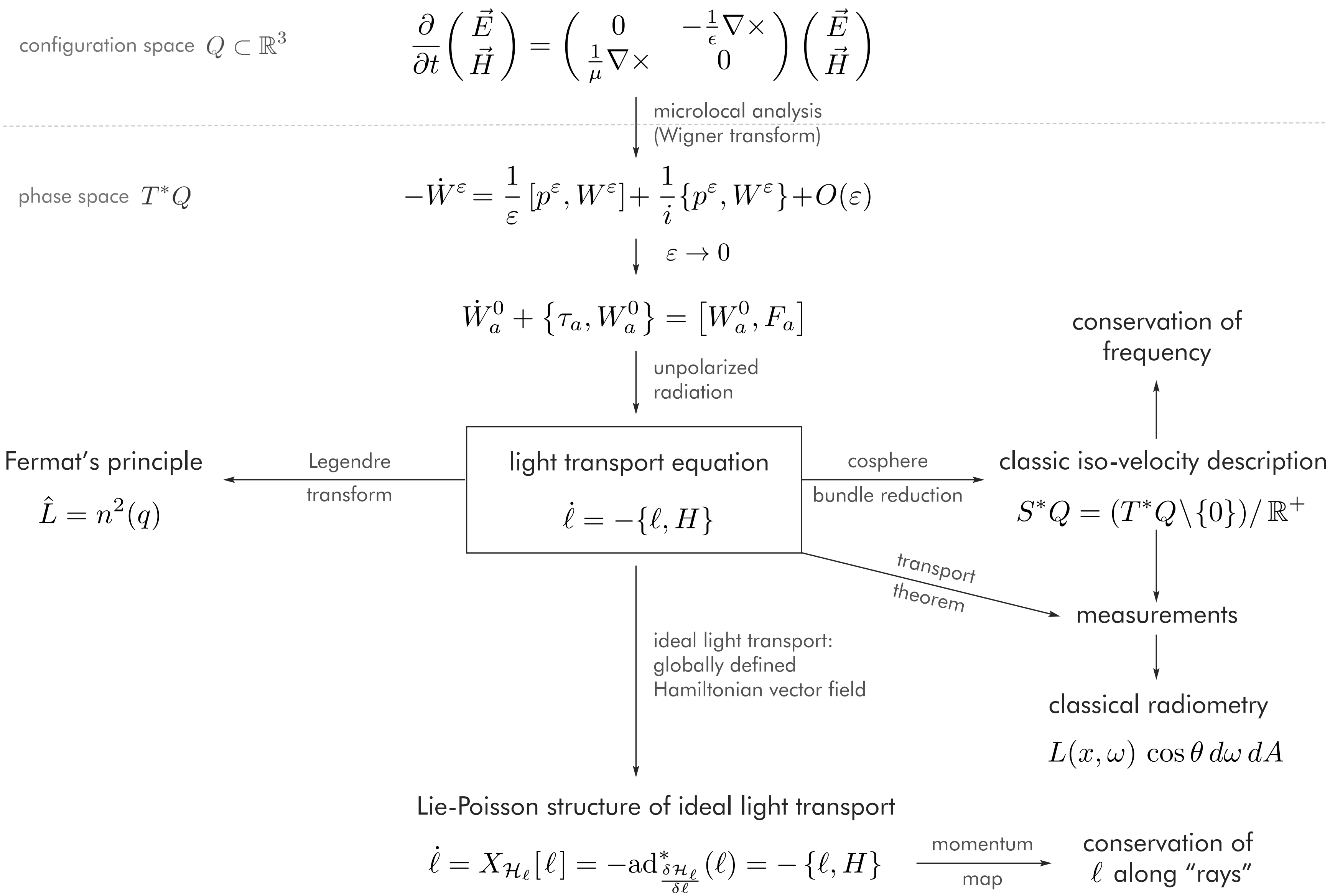
momentum map	Kelvin's circulation theorem	conservation of radiance



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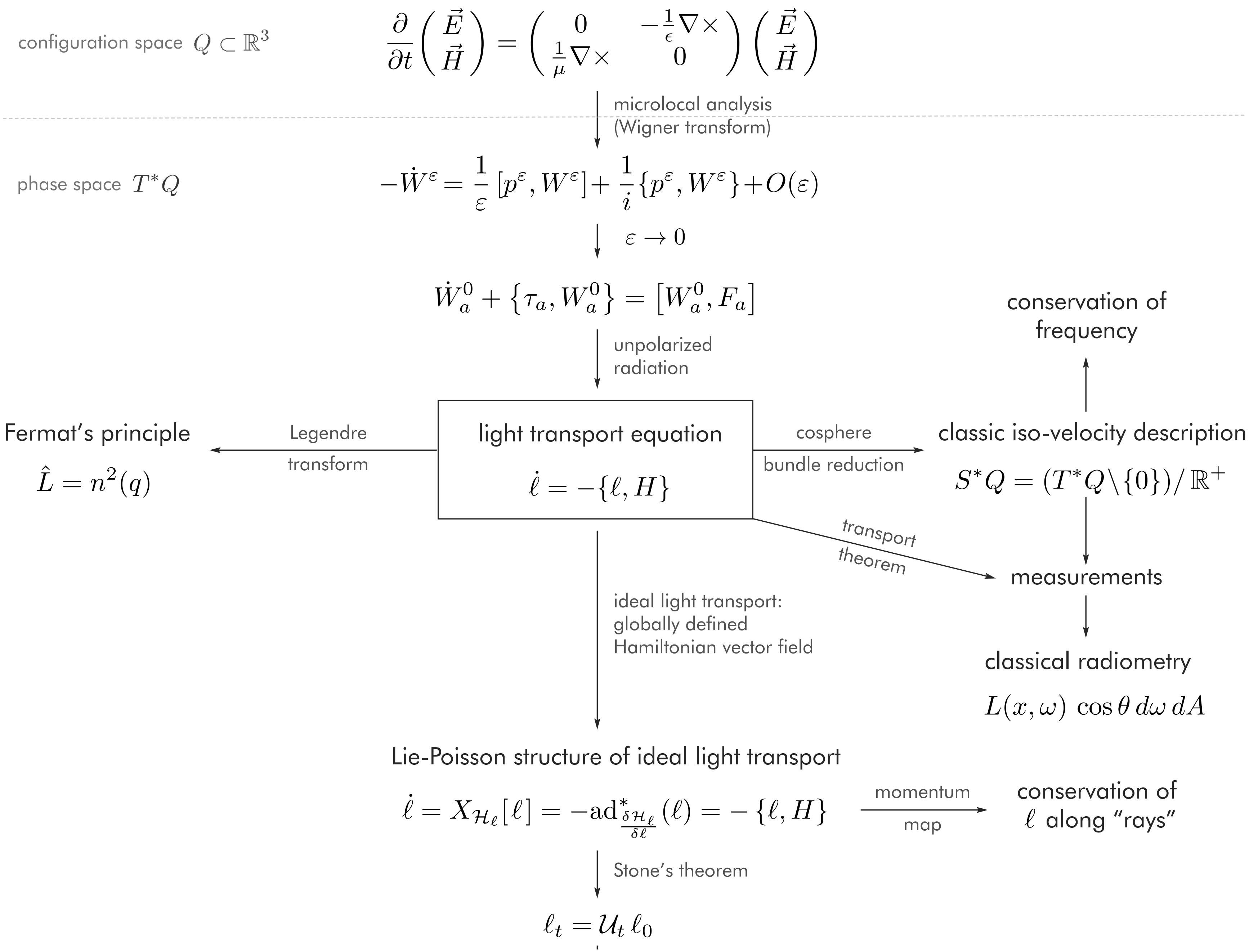

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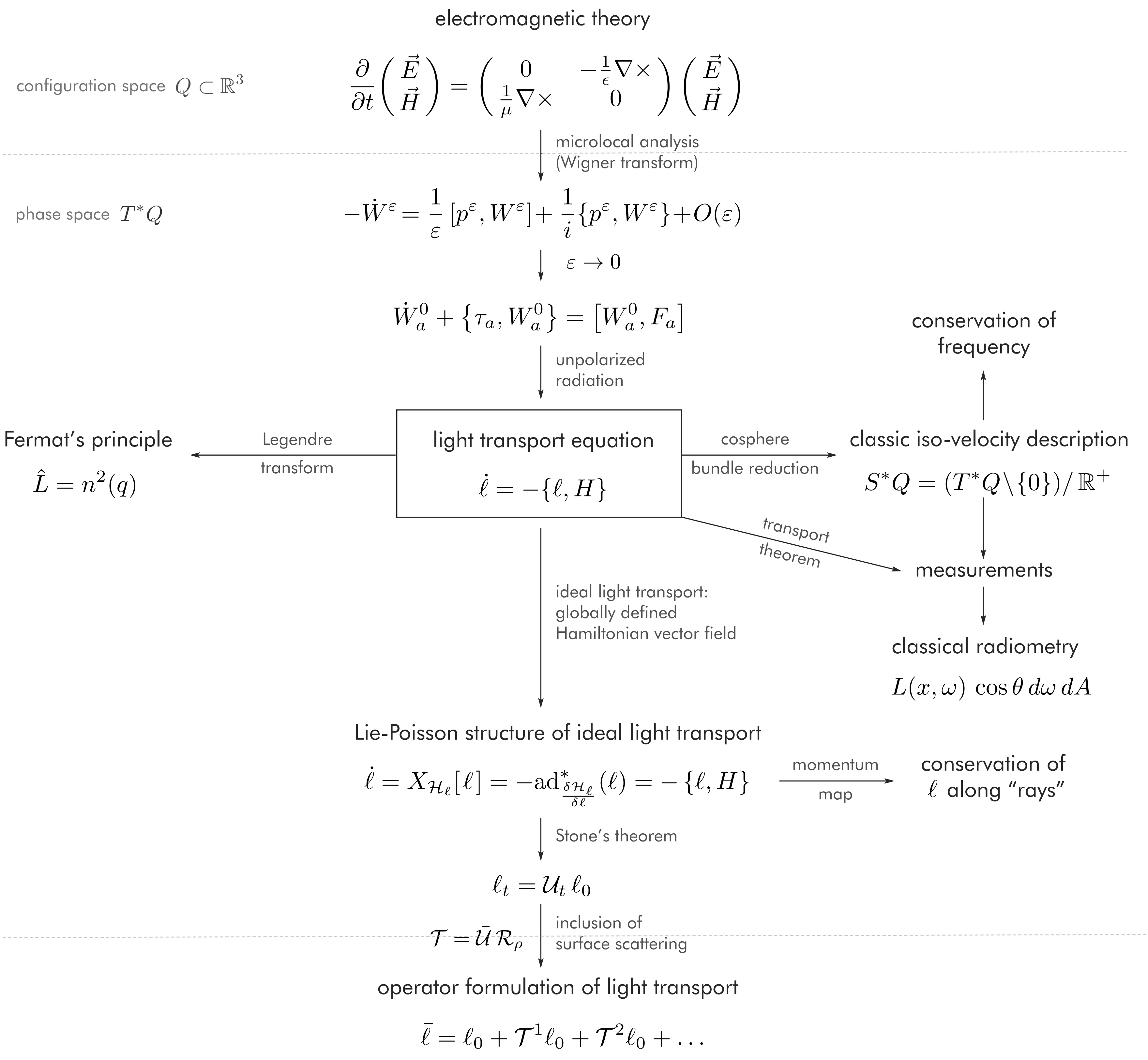
$$\ell_t = \mathcal{U}_t\, \ell_0$$



electromagnetic theory

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$$\mathcal{T} = \bar{\mathcal{U}}\,\mathcal{R}_\rho$$

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$$\bar{\ell} = \ell_0 + \mathcal{T}^1\ell_0 + \mathcal{T}^2\ell_0 + \ldots$$

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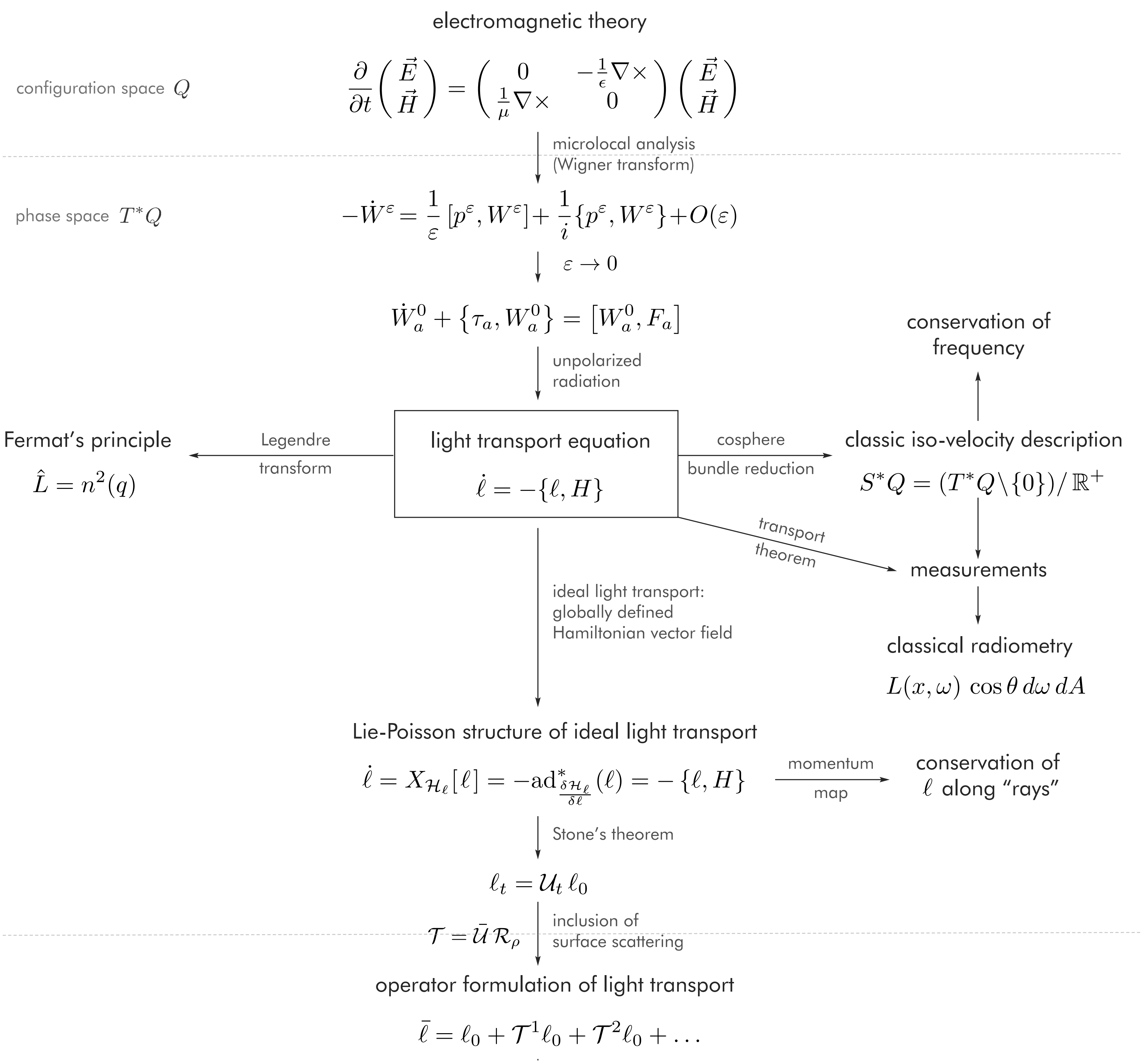
lessig@caltech.edu

http://users.cms.caltech.edu/~lessig/dissertation/
http://arxiv.org/abs/1206.3301



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