Interspecific strategic effects of mobility in predator-prey systems

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Introduction

We assume both prey and predators adjust their activities to maximize their per capita growth rates.

Prey	Predators
mobile strategy	*mobile (active) strategy*
sessile strategy	*sessile (ambush)*

Introduction

The Model

The underlying predator-prey dynamics is modelled by Gause-type interactions (Gause, 1934; Hofbauer and Sigmund, 1998) as follows. If x is the prey population size (i.e. prey density) and y is predator density, then

$$\begin{aligned} \dot{x} &= xf(x) - yF(x) \\ \dot{y} &= -\mu y + cyF(x). \end{aligned} \tag{1}$$

Here $f(x)$ is the per capita growth rate of the prey population in the absence of predators and $F(x)$ is the number of prey killed by one predator per unit time.
The death rate of predators is μ and c is the conversion factor.

Introduction

α_{MM} Foraging efficiencies of the mobile predator chasing mobile prey

α_{MS} Foraging efficiencies of the mobile predator chasing sessile prey

α_{SM} Foraging efficiencies of a sessile predator catching mobile prey

α_{SS} Foraging efficiencies of a sessile predator catching sessile prey

Introduction

		Prey	
		Mobile	Sessile
Predator	Mobile	α_{MM}	α_{MS}
	Sessile	α_{SM}	α_{SS}

Table 1. Predator foraging efficiency.

Introduction

θ_x and θ_y are the current mobile proportions in the system.

The expected number of prey killed per unit time by a randomly selected predator, $F(x;\theta_x,\theta_y)$, is given by

$$\begin{aligned} F(x;\theta_x,\theta_y) &= \theta_y \left[F_{MM}(\theta_x x) + F_{MS}((1-\theta_x)x)\right] \\ &\quad +(1-\theta_y)\left[F_{SM}(\theta_x x) + F_{SS}((1-\theta_x)x)\right] \end{aligned}$$

where F_{AB} is the functional response of a predator using strategy A when the prey population uses strategy B with $A, B \in \{M, S\}$.

The Model

Both prey and predators use game theory-based strategies to maximize their per capita population growth rates.

Replicator equation:

$$\begin{aligned}\dot{\theta}_x &= u\theta_x\left(1-\theta_x\right)\left[-\frac{F_{MM}(\theta_x x)\theta_y y}{\theta_x x}-\frac{F_{SM}(\theta_x x)(1-\theta_y)y}{\theta_x x}+\frac{F_{MS}((1-\theta_x)x)\theta_y y}{(1-\theta_x)x}+\frac{F_{SS}((1-\theta_x)x)(1-\theta_y)y}{(1-\theta_x)x}\right]\\ \dot{\theta}_y &= v\theta_y\left(1-\theta_y\right)\left[c\left(F_{MM}(\theta_x x)+F_{MS}((1-\theta_x)x)\right)-c\left(F_{SM}(\theta_x x)+F_{SS}((1-\theta_x)x)\right)\right].\end{aligned}$$

The Model

Both prey and predators use game theory-based strategies to maximize their per capita population growth rates.

Smoothed best response:

$$
\begin{aligned}
\dot{\theta}_x &= \frac{\theta_x \exp m\left(-\frac{F_{MM}(\theta_x x)\theta_y y}{\theta_x x} - \frac{F_{SM}(\theta_x x)(1-\theta_y)y}{\theta_x x}\right)}{\theta_x \exp m\left(-\frac{F_{MM}(\theta_x x)\theta_y y}{\theta_x x} - \frac{F_{SM}(\theta_x x)(1-\theta_y)y}{\theta_x x}\right) + (1-\theta_x)\exp m\left(-\frac{F_{MS}((1-\theta_x)x)\theta_y y}{(1-\theta_x)x} - \frac{F_{SS}((1-\theta_x)x)(1-\theta_y)y}{(1-\theta_x)x}\right)} - \theta_x \\
\dot{\theta}_y &= \frac{\theta_y \exp nc(F_{MM}(\theta_x x) + F_{MS}((1-\theta_x)x))}{\theta_y \exp nc(F_{MM}(\theta_x x) + F_{MS}((1-\theta_x)x)) + (1-\theta_y)\exp nc(F_{SM}(\theta_x x) + F_{SS}((1-\theta_x)x))} - \theta_y .
\end{aligned}
$$

The Lotka-Volterra Model

For the L-V model, substituting $F_{AB}(x) = \alpha_{AB}x$, yields the following population density and (replicator) strategy dynamics, respectively.

$$\begin{aligned}\dot{x} = \quad & rx\left(1-\frac{x}{K}\right)-\alpha_{MM}\theta_y y\theta_x x-\alpha_{MS}\theta_y y\left(1-\theta_x\right)x \\ & -\alpha_{SM}\left(1-\theta_y\right)y\theta_x x-\alpha_{SS}\left(1-\theta_y\right)y\left(1-\theta_x\right)x \\ \dot{y} = \quad & -\mu\, y+c\alpha_{MM}\theta_y y\theta_x x+c\alpha_{MS}\theta_y y\left(1-\theta_x\right)x \\ & +c\alpha_{SM}\left(1-\theta_y\right)y\theta_x x+c\alpha_{SS}\left(1-\theta_y\right)y\left(1-\theta_x\right)x\end{aligned}$$

$$\begin{aligned}\dot{\theta}_x &= u\theta_x\left(1-\theta_x\right)\left(-\alpha_{MM}\theta_y y-\alpha_{SM}\left(1-\theta_y\right)y+\alpha_{MS}\theta_y y+\alpha_{SS}\left(1-\theta_y\right)y\right) \\ \dot{\theta}_y &= v\theta_y\left(1-\theta_y\right)\left(c\alpha_{MM}\theta_x x+c\alpha_{MS}\left(1-\theta_x\right)x-c\alpha_{SM}\theta_x x-c\alpha_{SS}\left(1-\theta_x\right)x\right)\end{aligned}$$

The Lotka-Volterra Model

The effect of a dominated strategy

Suppose a mobile predator has higher foraging efficiency than a sessile predator independent of the strategy of the prey

$$\alpha_{MM} > \alpha_{SM}$$

$$\alpha_{MS} > \alpha_{SS}$$

Suppose $\alpha_{MM} > \alpha_{MS}$.

The proportion of mobile predators is increasing

$$\dot{\theta}_y \geq 0$$

and will evolve to 1 unless *x* evolves to 0.

The Lotka-Volterra Model

θ_y is sufficiently close to 1

$\dot{\theta}_x$ is always negative (if $\alpha_{MS} < \alpha_{MM}$)

or

y evolves to 0

x evolves to the carrying capacity K

The Lotka-Volterra Model

θ_y is sufficiently close to 1

$\dot{\theta}_x$ is always negative (if $\alpha_{MS} < \alpha_{MM}$)

⇓

$\theta_x = 0$ and $\theta_y = 1$

⇓

$$\begin{aligned} \dot{x} &= rx\left(1 - \frac{x}{K}\right) - \alpha_{MS}yx \\ \dot{y} &= -\mu y + c\alpha_{MS}yx. \end{aligned}$$

The Lotka-Volterra Model

$$\begin{aligned}\dot{x} &= rx\left(1-\frac{x}{K}\right)-\alpha_{MS}yx\\ \dot{y} &= -\mu y+c\alpha_{MS}yx.\end{aligned}$$

The system has an interior equilibrium

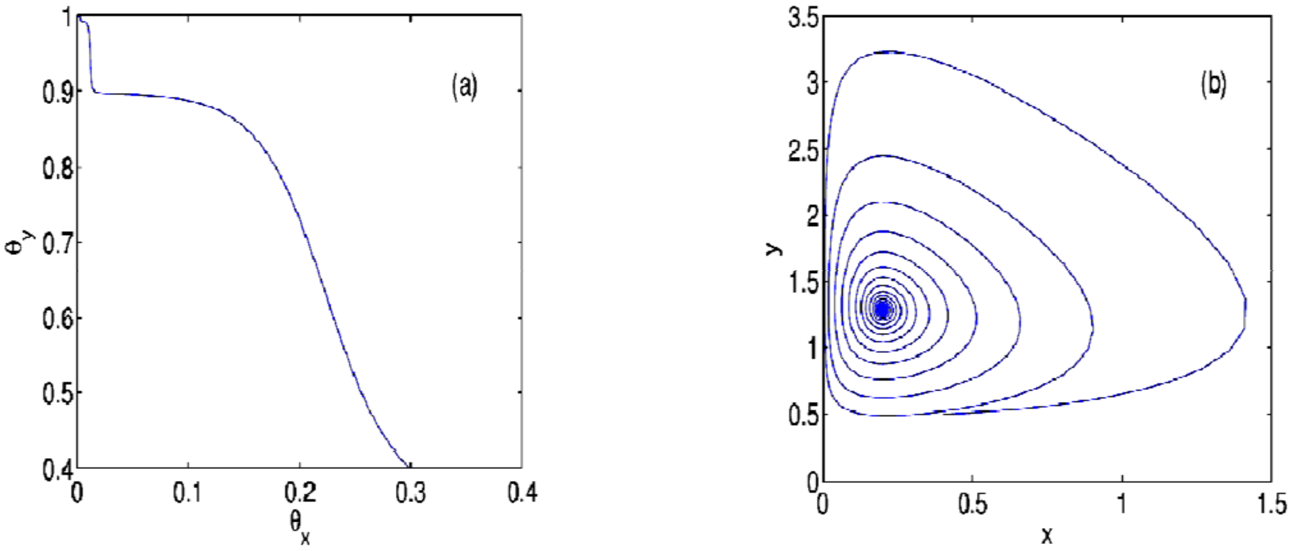
$$(x,y) = \left(\frac{\mu}{c\alpha_{MS}}, \frac{r(c\alpha_{MS}K-\mu)}{c\alpha_{MS}^2 K}\right)$$

if and only if $\frac{\mu}{cK} < \alpha_{MS}$.

The interior equilibrium is globally asymptotically stable if it exists; otherwise,

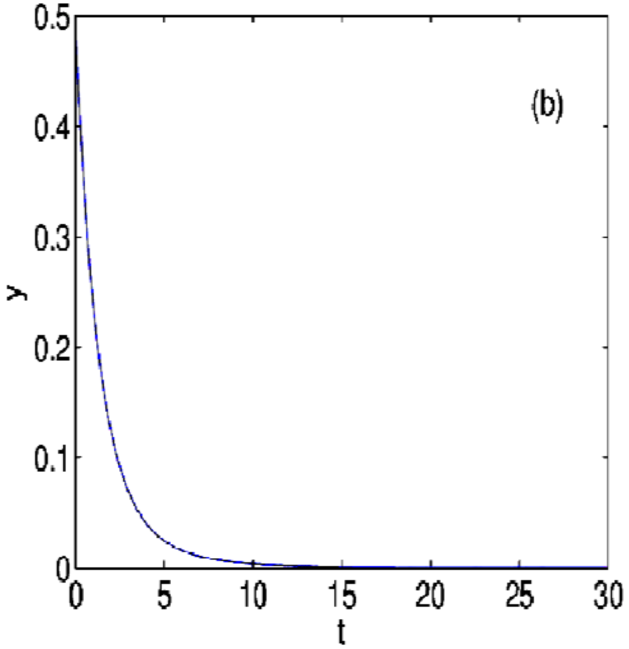
$$(x,y) \text{ evolves to } (K,0).$$

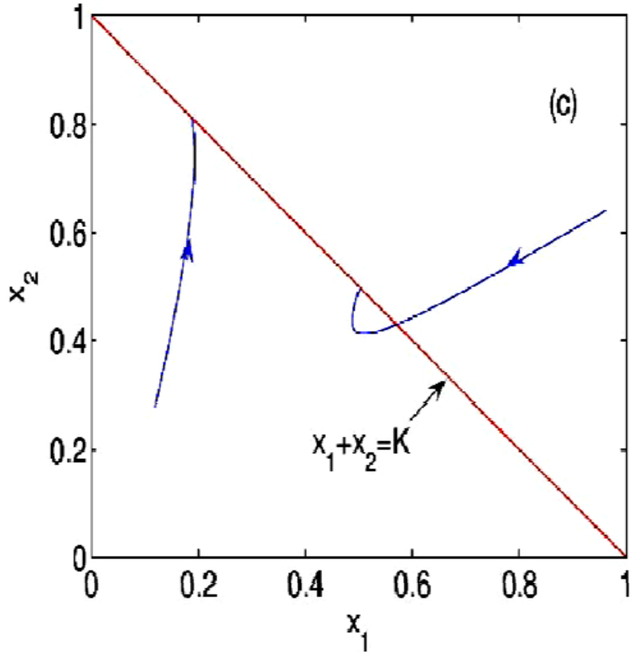
The Lotka-Volterra Model



Replicator equation $\frac{\mu}{cK} < \alpha_{MS}$

The Lotka-Volterra Model





Replicator equation $\frac{\mu}{cK} > \alpha_{MS}$

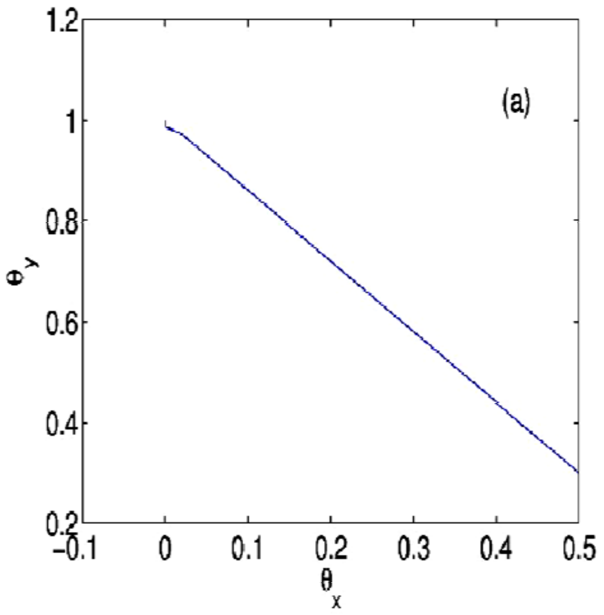
The Lotka-Volterra Model

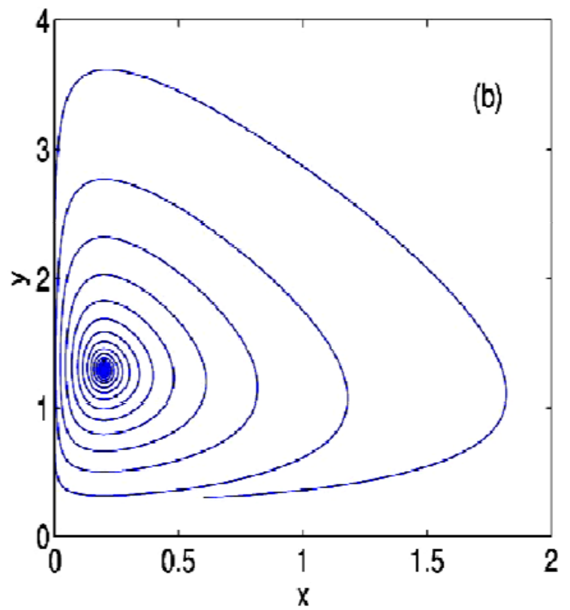
Using the smoothed best response dynamics, we have

$$\dot{\theta}_x = \frac{\theta_x \exp m(-\alpha_{MM}\theta_y y - \alpha_{SM}(1-\theta_y)y)}{\theta_x \exp m(-\alpha_{MM}\theta_y y - \alpha_{SM}(1-\theta_y)y) + (1-\theta_x)\exp m(-\alpha_{MS}\theta_y y - \alpha_{SS}(1-\theta_y)y)} - \theta_x$$

$$\dot{\theta}_y = \frac{\theta_y \exp n(c\alpha_{MM}\theta_x x + c\alpha_{MS}(1-\theta_x)x)}{\theta_y \exp n(c\alpha_{MM}\theta_x x + c\alpha_{MS}(1-\theta_x)x) + (1-\theta_y)\exp n(c\alpha_{SM}\theta_x x + c\alpha_{SS}(1-\theta_x)x)} - \theta_y.$$

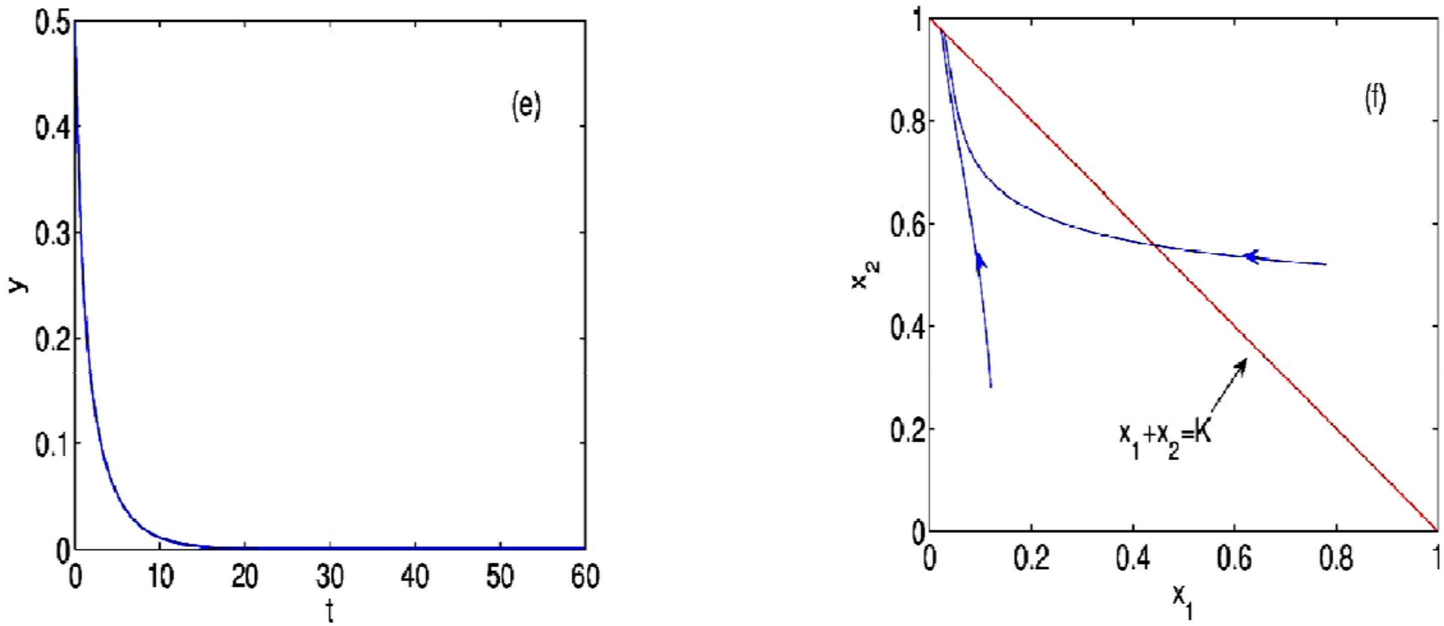
The Lotka-Volterra Model





Smoothed best response strategy dynamics $\frac{\mu}{cK} < \alpha_{MS}$

The Lotka-Volterra Model



Smoothed best response strategy dynamics $\frac{\mu}{cK} > \alpha_{MS}$

The Lotka-Volterra Model

No dominated strategy

We assume that

$$\alpha_{MS} \text{ and } \alpha_{SM} \text{ are both larger than } \alpha_{MM} \text{ and } \alpha_{SS}$$

The predator forages more efficiently if it has the opposite strategy as its prey.

Prey are better able to avoid predation if they have the same strategy as the predator.

Introduction

Nash equilibrium for the frequency-dependent evolutionary game between predators and prey is given by

$$\theta_x^* = \frac{\alpha_{MS} - \alpha_{SS}}{\alpha_{SM} - \alpha_{MM} + \alpha_{MS} - \alpha_{SS}} \text{ and } \theta_y^* = \frac{\alpha_{SM} - \alpha_{SS}}{\alpha_{SM} - \alpha_{MM} + \alpha_{MS} - \alpha_{SS}}.$$

(No individual can increase its fitness by altering its strategy.)

Introduction

With the mobile proportions fixed at their nash equilibrium values, the population dynamics becomes

$$
\begin{aligned}
\dot{x} &= rx\left(1 - \frac{x}{K}\right) - \alpha^* xy \\
\dot{y} &= -\mu y + c\alpha^* xy
\end{aligned}
$$

where

$$\alpha^* \equiv \frac{\alpha_{MS}\alpha_{SM} - \alpha_{SS}\alpha_{MM}}{\alpha_{SM} - \alpha_{MM} + \alpha_{MS} - \alpha_{SS}} > 0$$

Introduction

The equilibrium of most interest now is one where both strategic behaviors are present for each species

$$E_1 = (x^*,\, y^*,\, \theta_x^*,\, \theta_y^*),$$

where

$$x^* = \frac{\mu}{c\alpha^*},$$

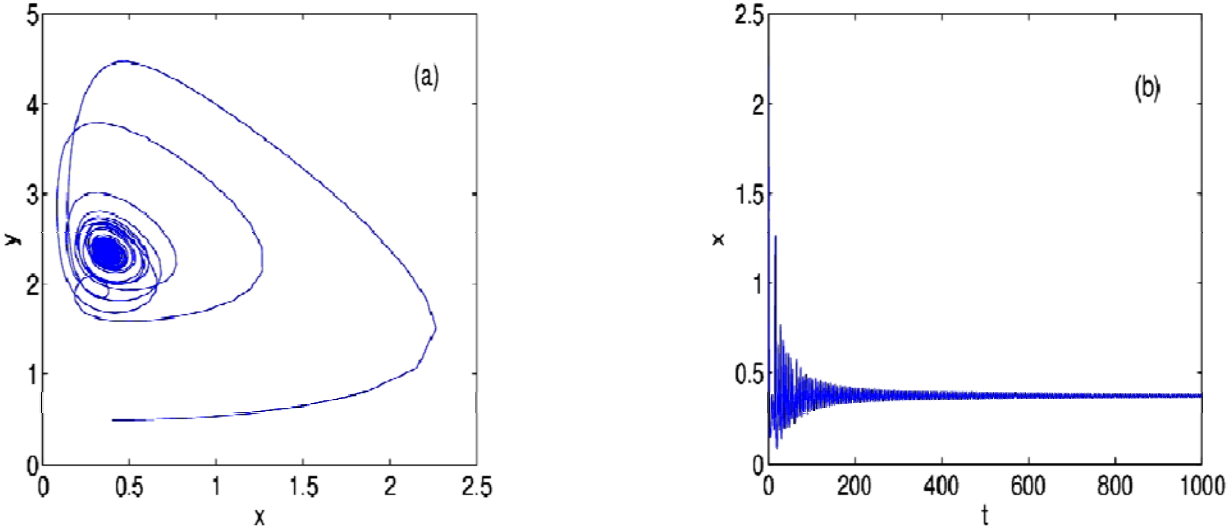
$$y^* = \frac{r}{\alpha^*}\left(1 - \frac{\mu}{cK\alpha^*}\right),$$

$$\theta_x^* = \frac{\alpha_{MS}-\alpha_{SS}}{\alpha_{SM}-\alpha_{MM}+\alpha_{MS}-\alpha_{SS}},$$

$$\theta_y^* = \frac{\alpha_{SM}-\alpha_{SS}}{\alpha_{SM}-\alpha_{MM}+\alpha_{MS}-\alpha_{SS}},$$

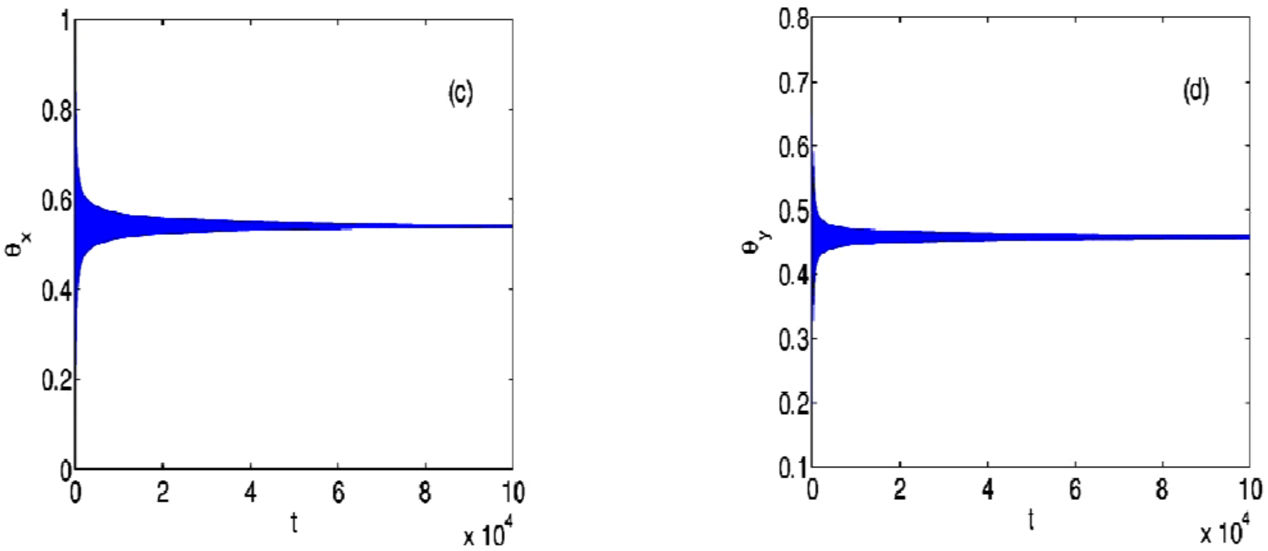
E_1 exists if and only if $\frac{1}{\alpha^*} < \frac{cK}{\mu}$.

The Lotka-Volterra Model



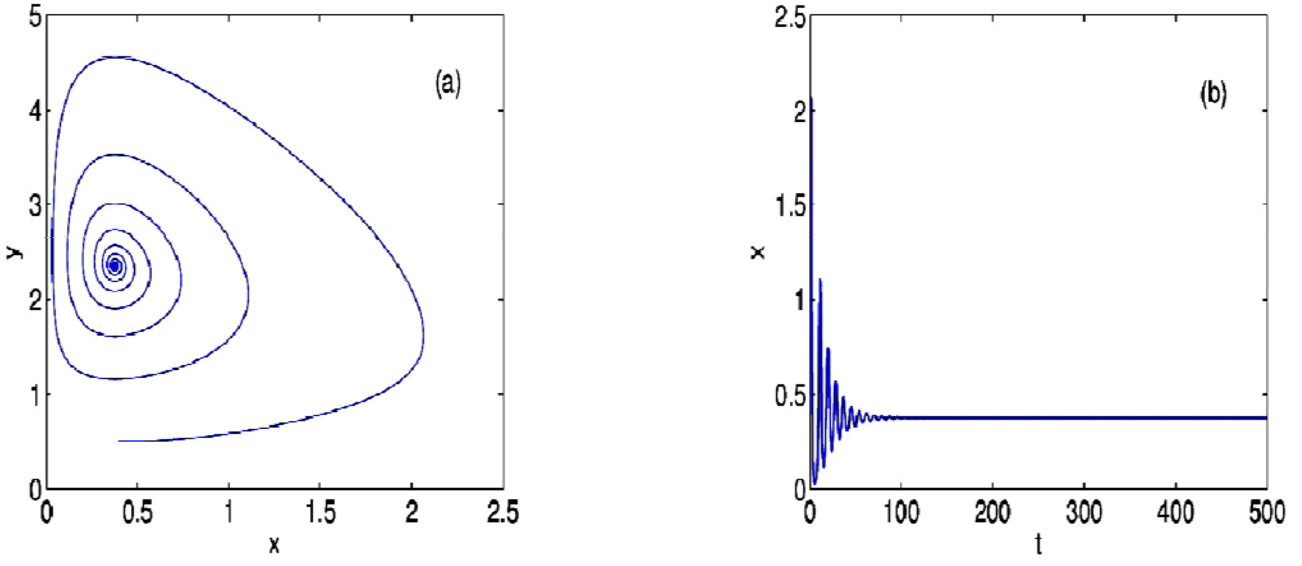
Trajectories of the LV system with strategy dynamics given by the replicator equation when E_1 exists.

The Lotka-Volterra Model



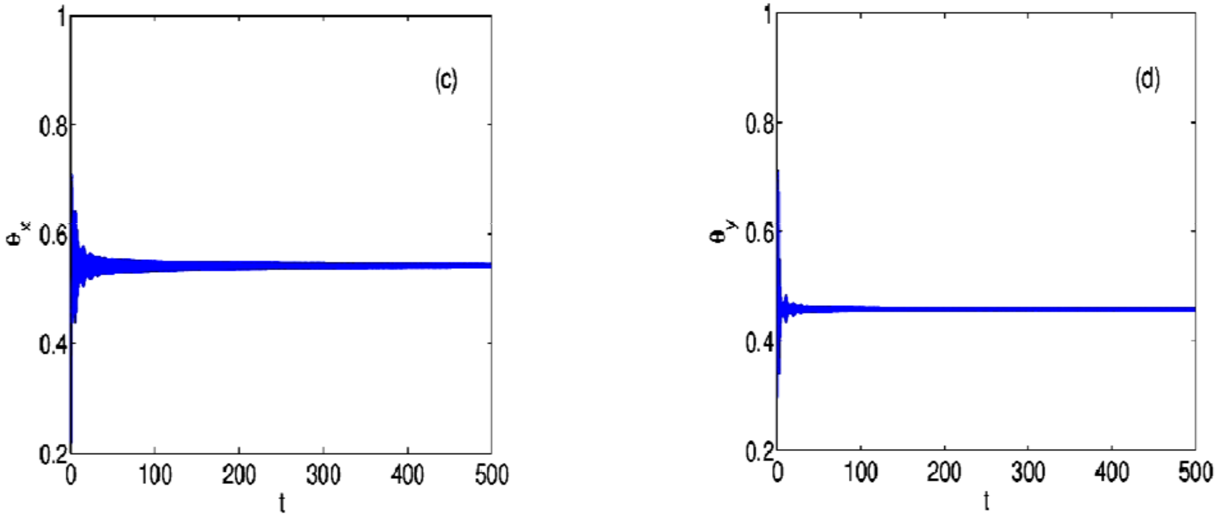
Trajectories of the LV system with strategy dynamics given by the replicator equation when E_1 exists.

The Lotka-Volterra Model



Trajectories of the LV system with strategy dynamics given by the smoothed best response equation when E_1 exists.

The Lotka-Volterra Model



Trajectories of the LV system with strategy dynamics given by the smoothed best response equation when E_1 exists.

The Lotka-Volterra Model

The coupled system may also have equilibria where both predators and prey are present but the populations adopt pure strategies.

$$E_2 = (\frac{\mu}{c\alpha_{SM}}, \frac{r\,(-\mu + c\alpha_{SM}K)}{c\alpha_{SM}^2 K}, 1, 0)$$

$$E_3 = (\frac{\mu}{c\alpha_{MS}}, \frac{r\,(-\mu + c\alpha_{MS}K)}{c\alpha_{MS}^2 K}, 0, 1)$$

$$E_4 = (\frac{\mu}{c\alpha_{SS}}, \frac{r\,(-\mu + c\alpha_{SS}K)}{c\alpha_{SS}^2 K}, 0, 0)$$

$$E_5 = (\frac{\mu}{c\alpha_{MM}}, \frac{r\,(-\mu + c\alpha_{MM}K)}{c\alpha_{MM}^2 K}, 1, 1)$$

E_2, E_3, E_4, E_5 are unstable if they exist.

The Lotka-Volterra Model

The only other equilibria of the coupled system are when the predator population is extinct, and the prey population is at carrying capacity.

$$E_6 = (K,\, 0,\, \hat{\theta}_x,\, 0),$$

$$E_7 = (K,\, 0,\, \hat{\theta}_x,\, 1),$$

$$E_8 = \left(K,\, 0,\, \frac{\alpha_{MS} - \alpha_{SS}}{\alpha_{SM} - \alpha_{MM} + \alpha_{MS} - \alpha_{SS}},\, \hat{\theta}_y\right)$$

where $\hat{\theta}_x$ and $\hat{\theta}_y$ take any value between 0 and 1.

Finally, there is also the trivial equilibrium with no prey and predators

$$E_9 = (0,\, 0,\, \hat{\theta}_x,\, \hat{\theta}_y).$$

The Lotka-Volterra Model

An alternative Lotka-Volterra model and the global stability of E_1

$$x_1 = \theta_x x \qquad x_2 = (1-\theta_x)\,x$$

$$y_1 = \theta_y y \qquad y_2 = (1-\theta_y)\,y$$

$$\begin{aligned}
\dot{x}_1 &= r x_1\left(1-\tfrac{x_1+x_2}{K}\right) - \alpha_{SM}x_1y_2 - \alpha_{MM}x_1y_1 + \tfrac{x_1x_2(u-1)(\alpha_{MS}y_1+\alpha_{SS}y_2-\alpha_{SM}y_2-\alpha_{MM}y_1)}{x_1+x_2}\\
\dot{x}_2 &= r x_2\left(1-\tfrac{x_1+x_2}{K}\right) - \alpha_{MS}x_2y_1 - \alpha_{SS}x_2y_2 + \tfrac{x_1x_2(u-1)(\alpha_{SM}y_2+\alpha_{MM}y_1-\alpha_{MS}y_1-\alpha_{SS}y_2)}{x_1+x_2}\\
\dot{y}_1 &= -\mu\, y_1 + c\alpha_{MS}x_2y_1 + c\alpha_{MM}x_1y_1 + \tfrac{y_1y_2(v-1)(c\alpha_{MS}x_2+c\alpha_{MM}x_1-c\alpha_{SM}x_1-c\alpha_{SS}x_2)}{y_1+y_2}\\
\dot{y}_2 &= -\mu\, y_2 + c\alpha_{SM}x_1y_2 + c\alpha_{SS}x_2y_2 + \tfrac{y_1y_2(v-1)(c\alpha_{SM}x_1+c\alpha_{SS}x_2-c\alpha_{MS}x_2-c\alpha_{MM}x_1)}{y_1+y_2}.
\end{aligned}$$

The Lotka-Volterra Model

$$x_1^* = \frac{\mu(\alpha_{MS} - \alpha_{SS})}{c(\alpha_{MS}\alpha_{SM} - \alpha_{SS}\alpha_{MM})},$$

$$x_2^* = \frac{\mu(\alpha_{SM} - \alpha_{MM})}{c(\alpha_{MS}\alpha_{SM} - \alpha_{SS}\alpha_{MM})},$$

$$y_1^* = \frac{r(\alpha_{SM} - \alpha_{SS})(-\mu\alpha_{MS} + \alpha_{MS}c\alpha_{SM}K + \mu\alpha_{MM} - c\alpha_{SS}K\alpha_{MM} - \mu\alpha_{SM} + \mu\alpha_{SS})}{Kc(\alpha_{MS}\alpha_{SM} - \alpha_{SS}\alpha_{MM})^2},$$

$$y_2^* = \frac{r(-\alpha_{MM} + \alpha_{MS})(-\mu\alpha_{MS} + \alpha_{MS}c\alpha_{SM}K + \mu\alpha_{MM} - c\alpha_{SS}K\alpha_{MM} - \mu\alpha_{SM} + \mu\alpha_{SS})}{Kc(\alpha_{MS}\alpha_{SM} - \alpha_{SS}\alpha_{MM})^2},$$

The Lotka-Volterra Model

In particular, in the special case that $u = v = 1$, we obtain

$$\begin{aligned}
\dot{x}_1 &= r x_1 \left(1 - \tfrac{x_1 + x_2}{K}\right) - \alpha_{SM} x_1 y_2 - \alpha_{MM} x_1 y_1 \\
\dot{x}_2 &= r x_2 \left(1 - \tfrac{x_1 + x_2}{K}\right) - \alpha_{MS} x_2 y_1 - \alpha_{SS} x_2 y_2 \\
\dot{y}_1 &= -\mu y_1 + c\alpha_{MS} x_2 y_1 + c\alpha_{MM} x_1 y_1 \\
\dot{y}_2 &= -\mu y_2 + c\alpha_{SM} x_1 y_2 + c\alpha_{SS} x_2 y_2 .
\end{aligned}$$

The global asymptotic stability of the system can the be shown by considering the following Lyapunov function

$$\begin{aligned}
V &= c\left(x_1 - x_1^* - x_1^* \ln\left(\tfrac{x_1}{x_1^*}\right)\right) + c\left(x_2 - x_2^* - x_2^* \ln\left(\tfrac{x_2}{x_2^*}\right)\right) + \left(y_1 - y_1^* - y_1^* \ln\left(\tfrac{y_1}{y_1^*}\right)\right) \\
&\quad + \left(y_2 - y_2^* - y_2^* \ln\left(\tfrac{y_2}{y_2^*}\right)\right),
\end{aligned}$$

The Lotka-Volterra Model

The derivative of *V* is obtained as

$$\begin{aligned}\dot{V} &= c(x_1 - x_1^*)\left(r\left(1 - \frac{x}{K}\right) - \alpha_{SM} y_2 - \alpha_{MM} y_1\right) + c(x_2 - x_2^*)\left(r\left(1 - \frac{x}{K}\right) - \alpha_{MS} y_1 - \alpha_{SS} y_2\right) \\ &\quad + (y_1 - y_1^*)(-\mu y_1 + c\alpha_{MS} x_2 + c\alpha_{MM} x_1) + (y_2 - y_2^*)(-\mu + c\alpha_{SM} x_1 + c\alpha_{SS} x_2) \\ &= -\frac{cr}{K}(x_1 - x_1^* + x_2 - x_2^*)^2 .\end{aligned}$$

Thus, $\dot{V} < 0$ unless $x = x^*$.

The trajectory must converge to an invariant subset of

$$\{(x_1, x_2, y_1, y_2) \mid x_1 + x_2 = x^* \text{ and } y \geq 0\}.$$

It can be shown that E_1 is globally asymptotically stable since all these trajectories converge to E_1.

The Lotka-Volterra Model

Let *M* be the maximal invariant subset

$$\{(x_1, x_2, y_1, y_2) \mid x_1 + x_2 = x^* \text{ and } y \geq 0\}$$

$$\dot{x} = 0$$
$$x = x^*$$

⇨

$$\dot{y} = c\dot{x} + \dot{y} = crx^* \left(1 - \frac{x^*}{K}\right) - \mu y$$

$$crx^* \left(1 - \frac{x^*}{K}\right) = \mu y^* \text{ at } E_1$$

$$\dot{y} = \mu(y^* - y)$$

The Lotka-Volterra Model

Thus, (x_1, x_2, y_1, y_2) converges to an invariant subset of

$$\{(x_1, x_2, y_1, y_2) \mid \quad x_1 + x_2 = x^* \text{ and } y_1 + y_2 = y^*\}.$$

$\dot{x} = \dot{y} = 0$ yields

$$r\left(1 - \frac{x^*}{K}\right)\left(1 - \frac{\alpha_{SM} - \alpha_{MM} + \alpha_{MS} - \alpha_{SS}}{\alpha_{MS}\alpha_{SM} - \alpha_{SS}\alpha_{MM}}\left(\begin{array}{c} \alpha_{SS} + \theta_x\left(\alpha_{MS} - \alpha_{SS}\right) + \theta_y\left(\alpha_{SM} - \alpha_{SS}\right) \\ +\theta_x\theta_y\left(\alpha_{MM} - \alpha_{MS} - \alpha_{SM} + \alpha_{SS}\right) \end{array}\right)\right) = 0$$

which simplifies to

$$\left(\alpha_{SS} - \alpha_{SM} + \theta_y\left(\alpha_{SM} - \alpha_{MM} + \alpha_{MS} - \alpha_{SS}\right)\right)\left(\alpha_{SS} - \alpha_{MS} + \theta_x\left(\alpha_{SM} - \alpha_{MM} + \alpha_{MS} - \alpha_{SS}\right)\right) = 0.$$

The Lotka-Volterra Model

That is, either $\theta_x = \theta_x^*$ or $\theta_y = \theta_y^*$

$\theta_x = \theta_x^*$ ⇨ $x_1 = x_1^*$ and $x_2 = x_2^*$, and $\dot{\theta}_x = 0$

⇩

$$y_1 = \theta_y y = y_1^* \text{ and } y_2 = y_2^*$$

Similarly, $\theta_y = \theta_y^*$

⇩

$$x_1 = x_1^* \text{ and } x_2 = x_2^*$$

Thus, every trajectory that is initially in the interior of M converges to E_1.

The Rosenzweig-MacArthur model

In this section, we consider the game between the prey species and predator species with Holling type II functional (behavioral) response, i.e. $F(x) = \frac{\alpha x}{1+\alpha h x}$.

$$\dot{x} = rx\left(1-\frac{x}{K}\right) - \frac{\alpha_{SM}\theta_x x(1-\theta_y)y}{\alpha_{SM}\theta_x xh+1} - \frac{\alpha_{MS}(1-\theta_x)x\theta_y y}{\alpha_{MS}(1-\theta_x)xh+1} - \frac{\alpha_{MM}\theta_x x\theta_y y}{\alpha_{MM}\theta_x xh+1} - \frac{\alpha_{SS}(1-\theta_x)x(1-\theta_y)y}{\alpha_{SS}(1-\theta_x)xh+1}$$

$$\dot{y} = -\mu y + \frac{c\alpha_{SM}\theta_x x(1-\theta_y)y}{\alpha_{SM}\theta_x xh+1} + \frac{c\alpha_{MS}(1-\theta_x)x\theta_y y}{\alpha_{MS}(1-\theta_x)xh+1} + \frac{c\alpha_{MM}\theta_x x\theta_y y}{\alpha_{MM}\theta_x xh+1} + \frac{c\alpha_{SS}(1-\theta_x)x(1-\theta_y)y}{\alpha_{SS}(1-\theta_x)xh+1}$$

$$\dot{\theta}_x = u\theta_x(1-\theta_x)\left(\frac{\alpha_{MS}\theta_y y}{\alpha_{MS}(1-\theta_x)xh+1} - \frac{\alpha_{SM}(1-\theta_y)y}{\alpha_{SM}\theta_x xh+1} + \frac{\alpha_{SS}(1-\theta_y)y}{\alpha_{SS}(1-\theta_x)xh+1} - \frac{\alpha_{MM}\theta_y y}{\alpha_{MM}\theta_x xh+1}\right)$$

$$\dot{\theta}_y = v\theta_y(1-\theta_y)\left(\frac{c\alpha_{MS}(1-\theta_x)x}{\alpha_{MS}(1-\theta_x)xh+1} - \frac{c\alpha_{SM}\theta_x x}{\alpha_{SM}\theta_x xh+1} + \frac{c\alpha_{MM}\theta_x x}{\alpha_{MM}\theta_x xh+1} - \frac{c\alpha_{SS}(1-\theta_x)x}{\alpha_{SS}(1-\theta_x)xh+1}\right)$$

The Rosenzweig-MacArthur model

We assume $\alpha_{MM} = 0$ and $\alpha_{SS} = 0$

Then the population density and strategy dynamics becomes

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - \frac{\alpha_{SM}\theta_x x(1-\theta_y)y}{\alpha_{SM}\theta_x xh+1} - \frac{\alpha_{MS}(1-\theta_x)x\theta_y y}{\alpha_{MS}(1-\theta_x)xh+1}$$

$$\dot{y} = -\mu y + \frac{c\alpha_{SM}\theta_x x(1-\theta_y)y}{\alpha_{SM}\theta_x xh+1} + \frac{c\alpha_{MS}(1-\theta_x)x\theta_y y}{\alpha_{MS}(1-\theta_x)xh+1}$$

$$\dot{\theta}_x = u\theta_x(1-\theta_x)\left(\frac{\alpha_{MS}\theta_y y}{\alpha_{MS}(1-\theta_x)xh+1} - \frac{\alpha_{SM}(1-\theta_y)y}{\alpha_{SM}\theta_x xh+1}\right)$$

$$\dot{\theta}_y = v\theta_y(1-\theta_y)\left(\frac{c\alpha_{MS}(1-\theta_x)x}{\alpha_{MS}(1-\theta_x)xh+1} - \frac{c\alpha_{SM}\theta_x x}{\alpha_{SM}\theta_x xh+1}\right)$$

The Rosenzweig-MacArthur model

The interior equilibrium is given by $E_{h1} = (x_h^*,\, y_h^*,\, \theta_{xh}^*,\, \theta_{yh}^*)$

where

$$
\begin{aligned}
x_h^* &= \frac{\mu}{\alpha^*(c-\mu h)} \\
y_h^* &= \frac{rc}{\alpha^*(c-\mu h)}\left(1 - \frac{\mu}{(c-\mu h)K\alpha^*}\right) \\
\theta_{xh}^* &= \frac{\alpha_{MS}}{\alpha_{MS}+\alpha_{SM}} \\
\theta_{yh}^* &= \frac{\alpha_{SM}}{\alpha_{MS}+\alpha_{SM}}.
\end{aligned}
$$

The Rosenzweig-MacArthur model

Moreover, with the mobile proportions fixed at the equilibrium values,

$$(\theta^*_{xh}, \theta^*_{yh})$$

the population dynamics becomes

$$\begin{aligned} \dot{x} = & \quad rx\left(1 - \frac{x}{K}\right) - \frac{\alpha^* xy}{\alpha^* xh + 1} \\ \dot{y} = & \quad -\mu\, y + \frac{c\alpha^* xy}{\alpha^* xh + 1}. \end{aligned}$$

That is, equilibrium E_{h1} is biologically feasible if and only if the following conditions hold:

$(\mathcal{H}1)$ $\mu\, h < c$ and $\frac{1}{\alpha^*} < \frac{(c - \mu\, h)K}{\mu}$.

The Rosenzweig-MacArthur model

The system has biologically feasible pure strategic equilibrium $E_{h2} = \left(\frac{\mu}{\alpha_{SM}(c-\mu h)}, \frac{rc}{\alpha_{SM}(c-\mu h)} \left(1 - \frac{\mu}{(c-\mu h)K\alpha_{SM}}\right), 1, 0 \right)$ if and only if the following conditions hold.

$(\mathcal{H}2)$ $\mu h < c$ and $\frac{\mu}{(c-\mu h)K} < \alpha_{SM}$.

The Rosenzweig-MacArthur model

Under the condition

$(\mathcal{H}3)$ $\mu h < c$ and $\frac{\mu}{(c-\mu h)K} < \alpha_{MS}$.

$E_{h3} = \left(\frac{\mu}{\alpha_{MS}(c-\mu h)}, \frac{rc}{\alpha_{MS}(c-\mu h)}\left(1 - \frac{\mu}{(c-\mu h)K\alpha_{MS}}\right), 0, 1\right)$ exists as a biologically feasible pure strategic equilibrium.

The Rosenzweig-MacArthur model

The system has three equilibria where the predator population goes to extinction and the prey population reaches carrying capacity.

These equilibria are

$$E_{h4} = (K,\ 0,\ \hat{\theta}_x,\ 0),$$

$$E_{h5} = (K,\ 0,\ \hat{\theta}_x,\ 1)$$

and

$$E_{h6} = \left(K,\ 0,\ \frac{\alpha_{MS}}{\alpha_{MS} + \alpha_{SM}},\ \hat{\theta}_y\right),$$

where $\hat{\theta}_x$ and $\hat{\theta}_y$ take any value between 0 and 1.

The Rosenzweig-MacArthur model

In addition, the system has a trivial equilibrium point

$$E_{h7} = (0,\ 0,\ \hat{\theta}_x,\ \hat{\theta}_y)$$

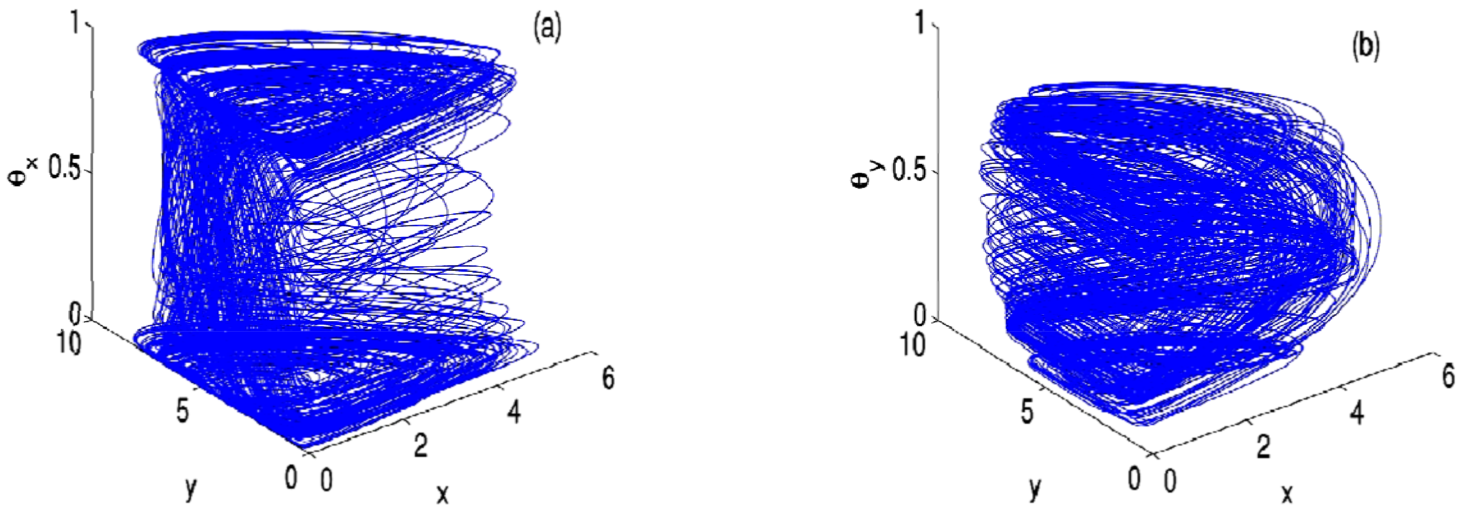
where both prey and predator populations are extinct.

E_{h1} is always unstable.

E_{h2} and E_{h3}, are both unstable.

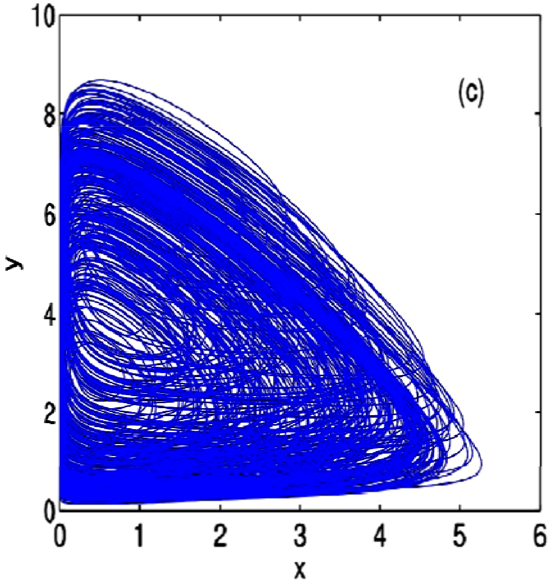
The trivial equilibrium E_{h7}, which corresponds to the extinction of both the prey and predator species is also unstable.

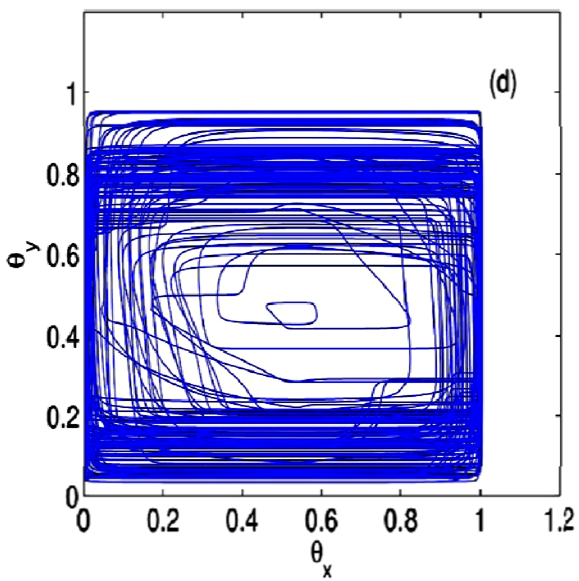
The Rosenzweig-MacArthur model



Trajectories of the RM system with strategy dynamics given by the replicator equation when E_{h1} exists.

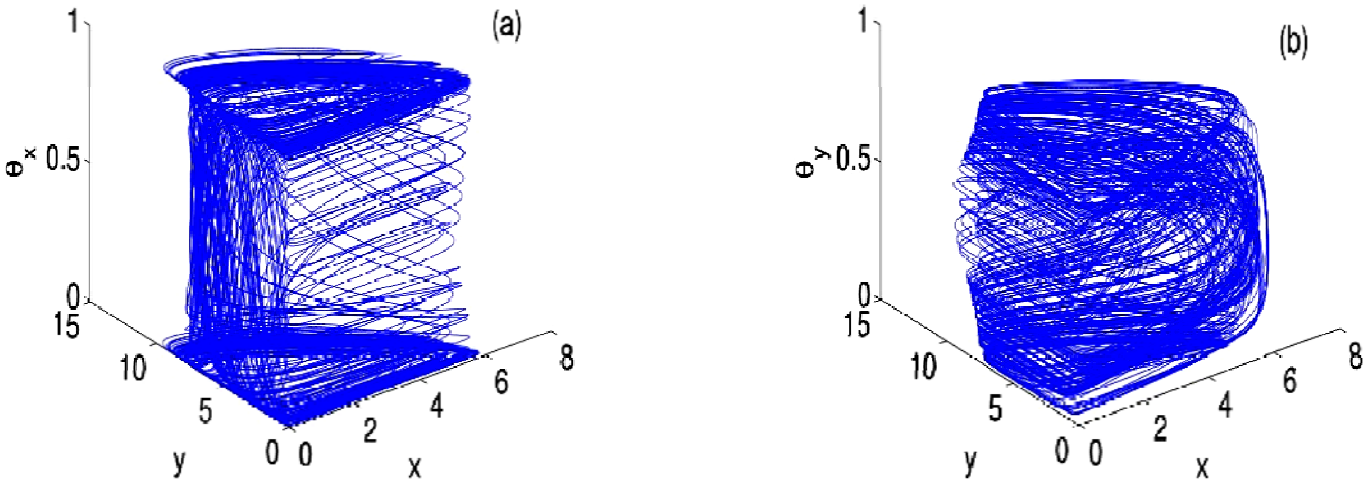
The Rosenzweig-MacArthur model





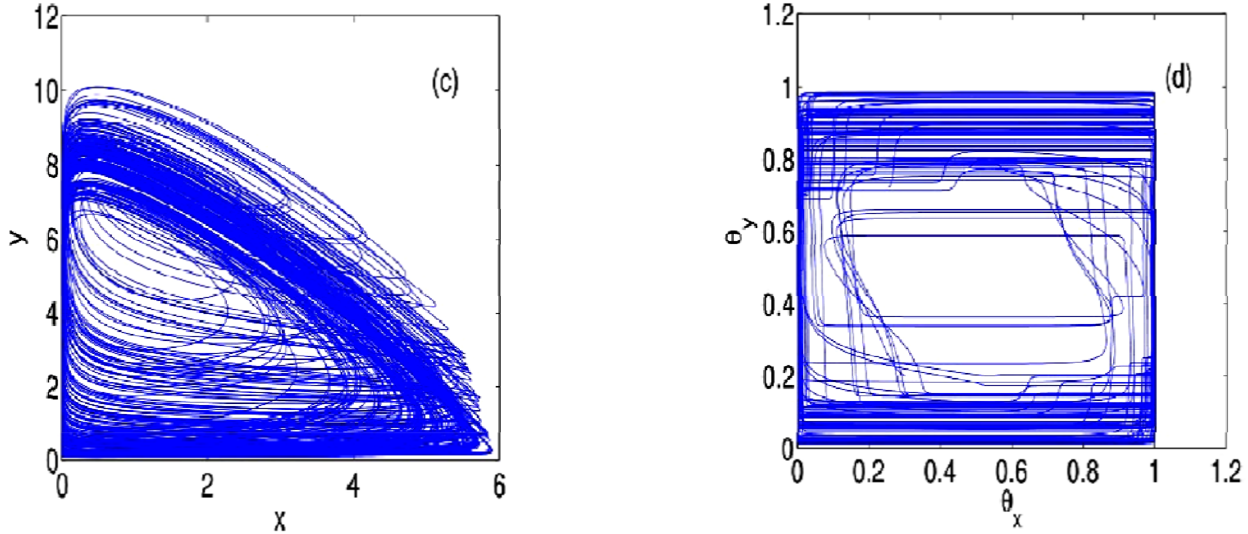
Trajectories of the RM system with strategy dynamics given by the replicator equation when E_{h1} exists.

The Rosenzweig-MacArthur model



Trajectories of the RM system with strategy dynamics given by the replicator equation when E_{h1} exists.

The Rosenzweig-MacArthur model



The Rosenzweig-MacArthur model

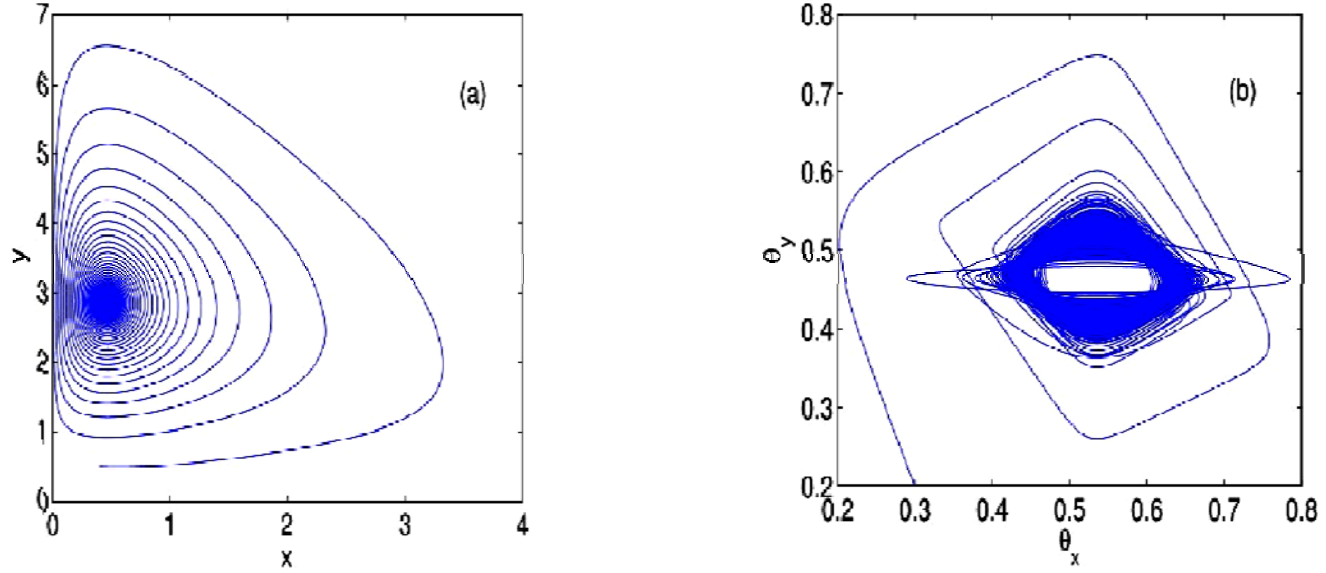
Interestingly, the situation changes when the strategy dynamics is given by the smoothed best response:

$$\dot{\theta}_x = \frac{\theta_x e^{-\frac{m\alpha_{SM}(1-\theta_y)y}{\alpha_{SM}\theta_x x h+1}}}{\theta_x e^{-\frac{m\alpha_{SM}(1-\theta_y)y}{\alpha_{SM}\theta_x x h+1}} + (1-\theta_x)e^{-\frac{m\alpha_{MS}\theta_y y}{\alpha_{MS}(1-\theta_x)x h+1}}} - \theta_x$$

$$\dot{\theta}_y = \frac{\theta_y e^{\frac{nc\alpha_{MS}(1-\theta_x)x}{\alpha_{MS}(1-\theta_x)x h+1}}}{\theta_y e^{\frac{nc\alpha_{MS}(1-\theta_x)x}{\alpha_{MS}(1-\theta_x)x h+1}} + (1-\theta_y)e^{\frac{nc\alpha_{SM}\theta_x x}{\alpha_{SM}\theta_x x h+1}}} - \theta_y.$$

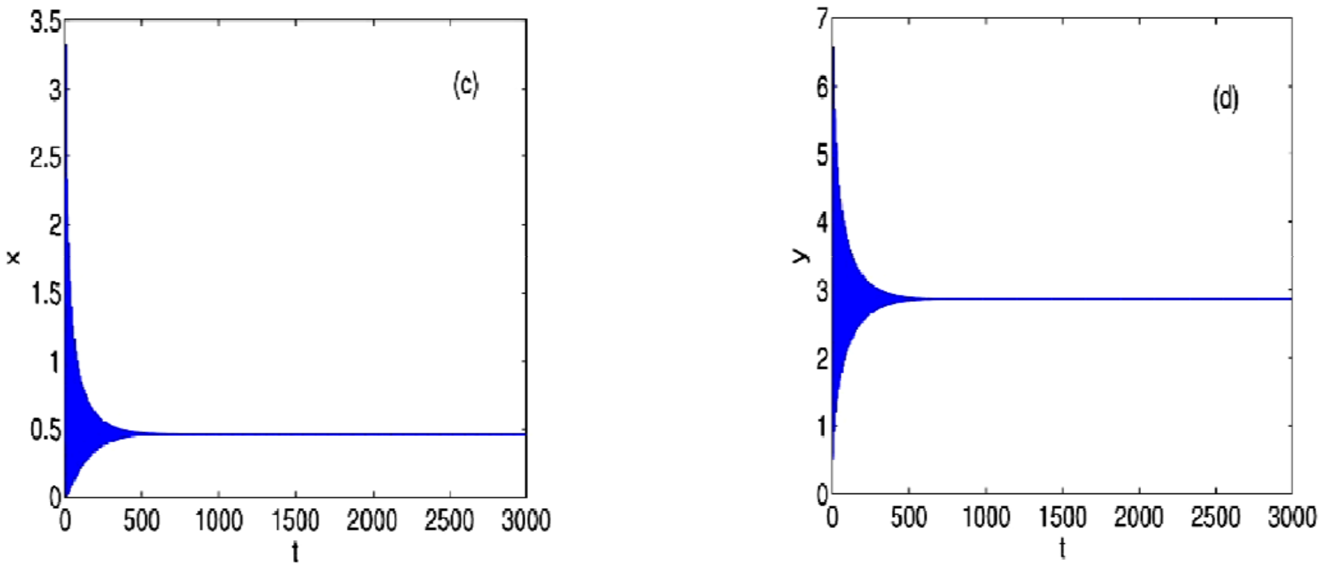
E_{h1} is still unstable.

The Rosenzweig-MacArthur model



Trajectories of the RM system with strategy dynamics given by the smoothed best response when E_{h1} exists.

The Rosenzweig-MacArthur model



Trajectories of the RM system with strategy dynamics given by the smoothed best response when E_{h1} exists.

Conclusions

We investigate the dynamics of a predator-prey system with the assumption that both prey and predators use game theory-based strategies to maximize their per capita population growth rates.

The predators adjust their strategies in order to catch more prey per unit time, while the prey, on the other hand, adjust their reactions to minimize the chances of being caught.

Conclusions

Numerical simulation results indicate that, for some parameter values, the system has chaotic behavior.

Our investigation reveals the relationship between the game theory-based reactions of prey and predators, and their population changes.

Thank you!