# Four-dimensional Šil'nikov-type dynamics in $x'(t) = -\alpha \cdot x(t - d(x_t))$

**(Joint work with Hans-Otto Walther; in progress)**

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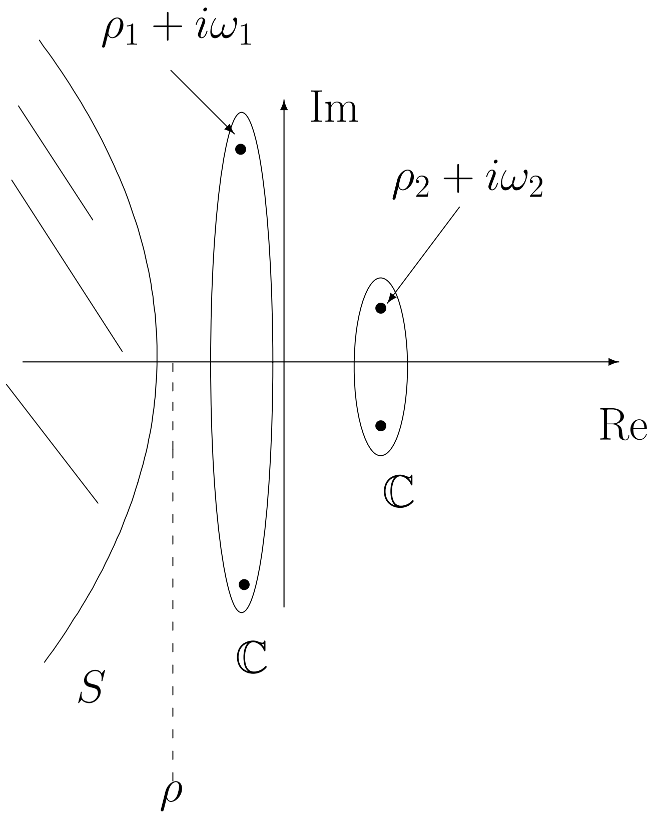
Result of H.-O. Walther:

**Existence of solution homoclinic to 0** for

$$x'(t) = -\alpha \cdot x(t - d(x_t)),$$

if the delay function $d$ is chosen appropriately.

**Spectrum at zero:** $(d = 1, \alpha \approx 5\pi/2)$ $\rho_2 > |\rho_1|,\ 0 > \rho_1 > \rho.$



**Aim of joint work:** Show existence of symbolic dynamics for a return map of the above equation.
(Famous precursor: Result of Šil'nikov in $\mathbb{R}^4$ (1967).)
We describe the essential framework without reference to an equation:

1) $(X, || \, ||)$ Banach space, decomposition $\boldsymbol{X} = \boldsymbol{S} \times \mathbb{C} \times \mathbb{C}$
2) $C^0-$semigroup $T : \mathbb{R}_0^+ \to L_c(X, X)$,

$$T(t)(x_s, z_1, z_2) = (T_S(t)x_s, e^{(\rho_1 + i\omega_1)t} z_1, e^{(\rho_2 + i\omega_2)t} z_2)$$

where $||T_S(t)|| \leq K e^{\rho t}$ for some $K > 0$, and

$$\rho < \rho_1 < 0 < \rho_2, \quad \rho_2 > |\rho_1|.$$

3) Consider the sets

$$S_{r_1,r_2} := \Big\{ (x_S, z_1, z_2) \in X \;\Big|\; ||x_S|| < r_1/K, \boldsymbol{|z_1| = r_1}, 0 < |z_2| < r_2 \Big\},$$
$$\Sigma_{r_1,r_2} := \Big\{ (x_S, z_1, z_2) \in X \;\Big|\; ||x_S|| < r_1/K, |z_1| < r_1, \boldsymbol{|z_2| = r_2} \Big\}.$$

For $x = (x_S, z_1, z_2) \in S_{r_1,r_2}$ there exists a unique time $\tau(x) > 0$ such that $T(\tau(x))x \in \Sigma_{r_1,r_2}$, namely

$$\tau(x) := \frac{1}{\rho_2} \log(\frac{r_2}{|z_2|}).$$

## The local map.

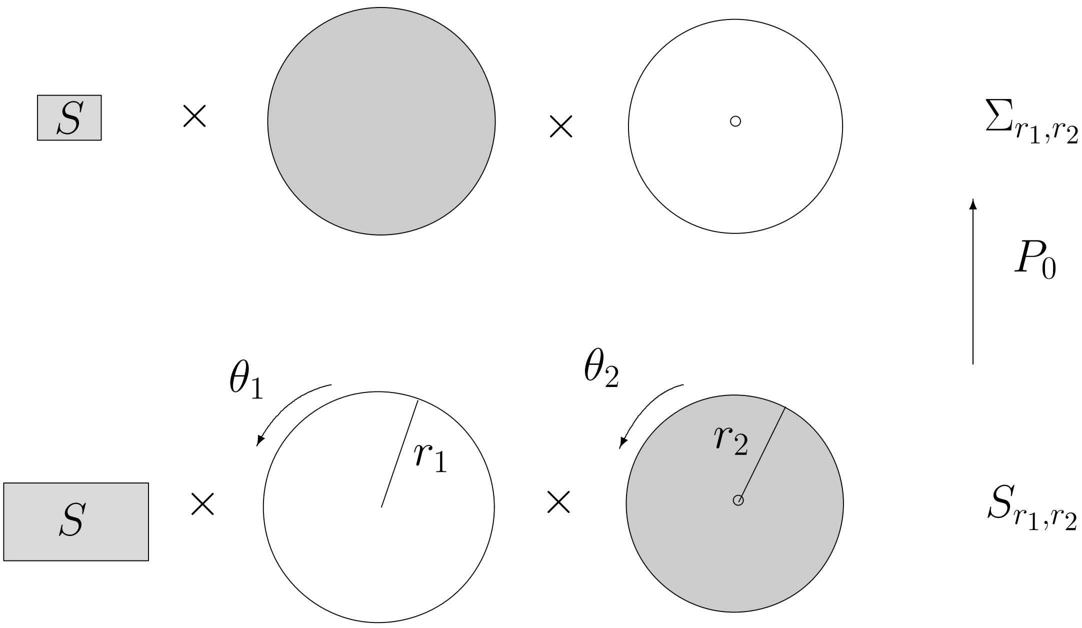
$$\boldsymbol{P_0 : S_{r_1,r_2} \to \Sigma_{r_1,r_2}, \quad P_0(x) := T(\tau(x))x.}$$

Explicitly: For $x = (x_S, z_1, z_2) \in S_{r_1,r_2}$, $z_2 = r_2 e^{i\theta_2}$, $z_1 = r_1 e^{i\theta_1}$,

$$P_0(x) = (y_S, \underbrace{r_1 \left(\frac{r_2}{|z_2|}\right)^{\rho_1/\rho_2} \cdot e^{i(\omega_1\tau(x)+\theta_1)}}_{=:\, w_1}, \underbrace{r_2 e^{i(\omega_2\tau(x)+\theta_2)}}_{=:\, w_2})$$

where $||y_S|| \leq ||x_S|| K e^{\rho\tau(x)} < r_1 e^{\rho\tau(x)}$.

**Note:** $\boldsymbol{|w_1| \sim |z_2|^{-\rho_1/\rho_2}}$, $0 <$ exponent $< 1$.
(Thus, $1 >> |w_1| >> |z_2|$.)

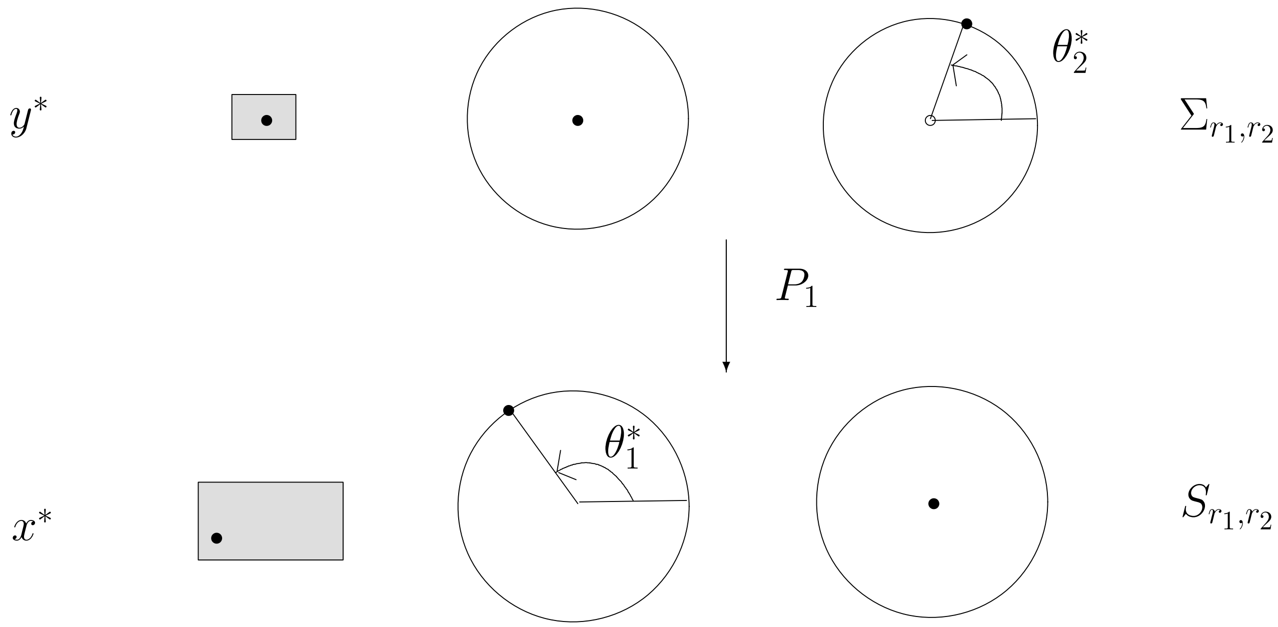


**The global map.** Assume there exists $\theta_1^*, \theta_2^* \in [0, 2\pi)$ and a $C^1$ map $P_1$, with values in $S_{r_1,r_2}$ and defined on the set

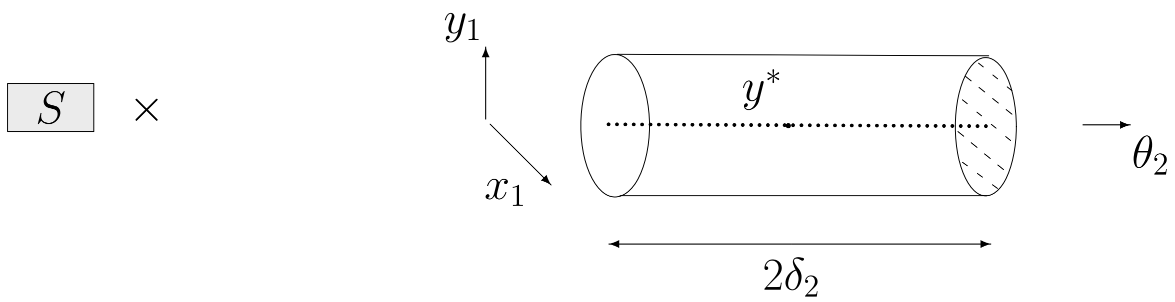
$$\Sigma^*_{r_1,r_2} := \Big\{ y = (y_S, w_1, w_2 = r_2 e^{i\theta_2}) \in \Sigma_{r_1,r_2} \ \Big| \ \max\{||y_S||, |w_1|, |\theta_2 - \theta_2^*|\} < \delta_2 \Big\}$$

such that with $y^* := (0, 0, r_2 e^{i\theta_2^*}) \in \Sigma_{r_1,r_2}$ and $x^* = (x_S^*, r_1 e^{i\theta_1^*}, 0) \in S_{r_1,r_2}$ one has

$$P_1(y^*) = x^*.$$



**Domain of $P_1$:**



**The composition.**

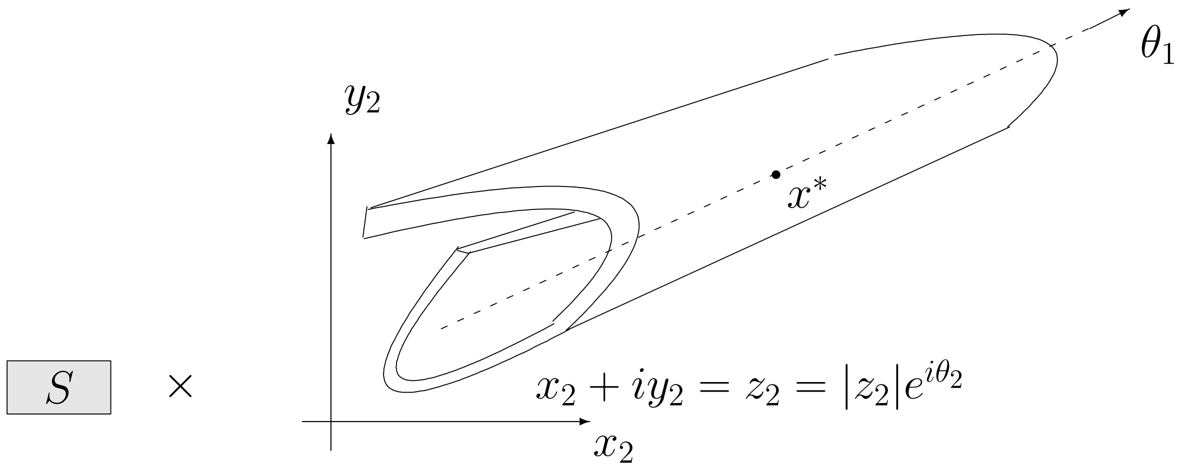
Set $I_2 := [\theta_2^* - \delta_2, \theta_2^* + \delta_2]$. If $\vartheta^{**} > \vartheta^* > 0$ are large enough and $\delta_1 \in (0, \pi/2)$, the set

$$\begin{aligned} D_{\vartheta*,\vartheta^{**}} := \Big\{ (x_S, z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}) \in S_{r_1,r_2} \ \Big| \\ |\theta_1 - \theta_1^*| < \delta_1, \\ -\vartheta^{**} < \theta_2 \leq -\vartheta^*, \\ |z_2| \in r_2 \cdot \exp[-\frac{\rho_2}{\omega_2}(I_2 - \theta_2)] \Big\} \end{aligned}$$

satisfies $P_0(D_{\vartheta*,\vartheta^{**}}) \subset \Sigma^*_{r_1,r_2}$, and hence one can define the composition

$$\boldsymbol{P := P_1 \circ P_0 : D_{\vartheta*,\vartheta^{**}} \longrightarrow S_{r_1,r_2}.}$$

A typical domain $D_{\vartheta*,\vartheta^{**}}$ :



**Explicit formulas.**

Describe $P_1$ in the form $y = (y_S, x_1 + iy_1, r_2 e^{i\theta_2}) \mapsto (\tilde{x}_S, r_1 e^{i\tilde{\theta}_1}, \tilde{z}_2)$, with $C^1$ functions $\tilde{x}_S, \tilde{\theta}_1, \tilde{z}_2$, and partial derivatives $\frac{\partial}{\partial \theta_2}\Big|_{y^*}\tilde{z}_2, \frac{\partial}{\partial x_1}\Big|_{y^*}\tilde{\theta}_1$, etc.

For $x = (x_S, r_1 e^{i\theta_1}, z_2) \in D_{\vartheta *}$, set

$$\tau := \tau(x) \text{ (as above) }, r_1' := r_1(r_2/|z_2|)^{\rho_1/\rho_2},$$
$$x_1 := r_1' \cos(\omega_1\tau + \theta_1), y_1 := r_1' \sin(\omega_1 \tau + \theta_1),$$
$$y_S := T(\tau)x_S, ||y_S|| \leq r_1 e^{\rho\tau} \sim |z_2|^{|\rho/\rho_2|}$$

$$\text{Then } P(x) = \Big(0, r_1 \exp\{i[\theta_1^* + < \nabla_3 \tilde{\theta}_1\Big|_{y^*}, (x_1, y_1, \theta_2 - \theta_2^*) > + E_1]\},$$
$$< \nabla_3 \tilde{z}_2\Big|_{y^*}, (x_1, y_1, \theta_2 - \theta_2^*) > + E_2\Big) + E_3 + E_4,$$

where $E_1, E_2 = o(r_1' + r_2(\omega_2\tau + \theta_2 - \theta_2^*))$, $E_3 = O(||y_S||), E_4 = (\tilde{x}_S, 0, 0)$, and $||\tilde{x}_S|| = O(r_1' + \delta_2 r_2)$.

(Briefly: Taylor expansion of first order w.r.t. 3d-Variables, but only to zero order w.r. to $S$.)

$$\text{Set } Y_3 := \text{ span}(\frac{\partial}{\partial\theta_2}, \quad \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1})\big|_{y^*}, \quad X_3 := \text{ span}(\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2})\big|_{x^*},$$

$$\text{then } T_{y^*}\Sigma_{r_1,r_2} = S \oplus Y_3, \quad T_{x^*}S_{r_1,r_2} = S \oplus X_3$$

with a corresponding projection $\mathrm{pr}_3$ to $X_3$.

**Transversality conditions:**

1) $\mathrm{pr}_3 \circ DP_1(y^*)$ is invertible on $Y_3$;

2) $\zeta_2 := \dfrac{\partial \tilde{z}_2}{\partial\theta_2}\big|_{y^*} \neq 0$, or equivalently: $DP_1(y^*)\frac{\partial}{\partial\theta_2}\big|_{y^*} \notin \mathbb{R}\frac{\partial}{\partial\theta_1}\big|_{x^*}$.

(Geometric meaning:

The image of $D_{\vartheta*}$ under $P$ is not coaxial with $D_{\vartheta*,\vartheta**}$.)

**Consequences:**

a) With $U_1 := \mathrm{pr}_3 DP_1(y^*)\ \mathrm{span}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1})\big|_{y^*}$, one has $X_3 = U_1 \oplus \mathbb{R}\cdot\zeta_2$.

b) Let $H \subset X_3$ be a plane containing $\zeta_2$ and such that $\frac{\partial}{\partial\theta_1} \notin H$; then $\mathrm{pr}_{x_2,y_2}$ is an isomorphism on $H$.

(particularly convenient choice possible).

**Choice of $N_0, N_1$.**

With suitably chosen numbers $\vartheta^0, \vartheta^{00}, \vartheta^1, \vartheta^{11}$ and $\varepsilon_1 > 0$, the sets

$$N_0 := D_{\vartheta^0,\vartheta^{00}}, N_1 := D_{\vartheta^1,\vartheta^{11}} \text{have the properties below:}$$

a) (their images lie on different sides of the plane $x^* + H$).

b) For fixed $\bar{\theta}_1$ and $j \in \{1, 2\}$, the map

$$N_j \ni (0, r_1 e^{i\bar{\theta}_1}, z_2) \mapsto \mathrm{pr}_{x_2,y_2} \mathrm{pr}_{X_3} P((0, r_1 e^{i\bar{\theta}_1}, z_2))$$

is homeomorphic. (Easier to see for $\mathrm{pr}_H$; then use that $\mathrm{pr}_{x_2,y_2}$ is isomorphic on $H$.)

**Main Theorem.** $\forall(\ldots s_{-2}s_{-1}s_0s_1s_2\ldots) \in \{0,1\}^{\mathbb{Z}}$ $\exists$ trajectory $(x_j)_{j\in\mathbb{Z}}$ of $P$ with $x_j \in N_{s_j}$ for all $j \in \mathbb{Z}$.

**Proof (ideas):**

**1)** For a finite, periodic symbol sequence $\alpha = (s_0, s_1, \ldots, s_k = s_0) \in \{0,1\}^{k+1}$ and a map $f$ defined on $N_0 \cup N_1$, define

$$N_{\alpha,f} := N_{s_0} \cap f^{-1}(N_{s_1}) \cap \ldots \cap f^{-k}(N_{s_k}).$$

**Lemma** (Zgliczyński). If $f, g$ are homotopic maps and the invariant set is disjoint to $\partial N_0 \cup \partial N_1$ throughout the homotopy, then

$$\mathrm{ind}(f^k, N_{\alpha,f}) = \mathrm{ind}(g^k, N_{\alpha,g}).$$

**2)** Three homotopies as in the lemma:

a) $P \sim P_3 := \mathrm{pr}_{X_3} \circ P$; (eliminate $S-$component from image of $P$)

b) $P_3 \sim \tilde{P}_3$; (eliminate $\theta_1-$dependence)

c) $\tilde{P}_3 \sim P_2 := \mathrm{pr}_{x_2,y_2} \circ \tilde{P}_3$ (project values to $x_2, y_2$-space).

**3)** With the Lemma and the reduction property of fixed point index:

$$\mathrm{ind}(P^k, N_\alpha) = \mathrm{ind}(P_2^k, N_\alpha) = \mathrm{ind}(P_2^k, N_\alpha \cap (x_2, y_2) - \text{space}).$$

**4)** $(N_0 \cup N_1) \cap (x_2, y_2)$ – space consists of two sets homeomorphic to a ball in $\mathbb{R}^2$, mapped by $P_2$ homeomorphically to a larger ball containing both.

**5) Lemma.** For a map $f$ as in the situation of 4), $\mathrm{ind}(f^k, N_\alpha) = \pm 1$.

**6) Corollary.** There is a periodic orbit of $P$ obeying $\alpha$.

**7)** The main theorem now follows with a standard compactness argument, using that $P$ is compact and that periodic symbol sequences are dense in the space of all symbol sequences (with the product topology).

**Thank you**

**for your attention!**

## References

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(In particular, Section 13.7, pp. 574 - 578.)

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