Unconditional Flocking of the Delayed Cucker-Smale Model

Jianhong Wu

Collaborator: Yicheng Liu, National University of Defense Technology

April 6, 2013

Outline

- The Cucker-Smale Model
- Flocking solutions and unconditional flocking
- Fixed-point theoretic framework for unconditional flocking
- Final flocking velocity and delayed impact
- Connection to dynamical clustering and sub-task performance

Flocking, herding, schooling and self-organization

- Self-organized systems arise very naturally in artificial intelligence, and in physical, biological and social sciences.
- Such systems seem to have remarkable capability to regulate the flow of information from distinct and independent components to achieve a prescribed performance.
- It is of particular interest, in both theories and applications, to understand how self-propelled individuals use only limited environmental information and simple rules to organize into an ordered motion.
- These emerging behaviours such as flocking, herding, and schooling have been observed in many self-organized systems, including fish swimming in schools, birds flying in flocks for the purpose of enhancing the foraging success, and the flight guidance in honeybee swarms.

The Cucker-Smale Model

Cucker F. and Smale S., On the mathematics of emergence. Japan J. Math. 2 (2007),197-227.

Consider a self-organized group with N agents, with each agent i being characterized by its position $\mathbf{x}_i \in R^d$, and velocity $\mathbf{v}_i \in R^d$. The Cucker-Smale model:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \qquad \frac{d\mathbf{v}_i}{dt} = \frac{\alpha}{N}\sum_{j\neq i}^{N} a_{ij}(\mathbf{x})(\mathbf{v}_j - \mathbf{v}_i), \tag{1}$$

where α measures the interaction strength.

- The influence function: $a_{ij}^{CS}(\mathbf{x}) = I(|\mathbf{x}_i - \mathbf{x}_j|)$ is used to quantify the pairwise influence of agent j on the alignment of agent i, as a function of the (metric) distance.
- This influence function I is a strictly positive monotonically decreasing function;
- A prototype given by $I(r) = (1 + r^2)^{-\beta}$ for $r \geq 0$, where β is a constant.

Motsch-Tadmor Non-symmetric Model

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \qquad \frac{d\mathbf{v}_i}{dt} = \frac{\alpha}{N}\sum_{j\neq i}^{N} a_{ij}(\mathbf{x})(\mathbf{v}_j - \mathbf{v}_i),$$

Motsch S., Tadmor E., A New Model for Self-organized Dynamics and Its Flocking Behavior, J Stat Phys 144(2011), 923-947.
A non-symmetric pairwise influence function

$$a_{ij}^{MT}(\mathbf{x}) = \frac{I(|\mathbf{x}_i - \mathbf{x}_j|)}{\sum_{k=1}^{N} I(|\mathbf{x}_i - \mathbf{x}_k|)} \tag{2}$$

is used to emphasize the importance of relative influence.

Acceleration of Flocking

- An agent may receive influence from multiple agents in a specific group, and an agent may also receive influence from another agent indirectly via other agents. [For example, in the bee swarms, only a small minority of informed bees manage to provide guidance to the rest but the entire swarm is able to fly to the new nest intact]
- We suggest that the pairwise influence a_{ij} may take more general form as follows

$$a_{ij}(\mathbf{x}) = \sum_{k=0}^{N-1} \delta_{ij}^{k}(\mathbf{x}),$$

where $\delta_{ij}^{k}(\mathbf{x}(t))$ is defined inductively as follows:

$$\begin{aligned} \delta_{ij}^{0}(\mathbf{x}(t)) &= a_{ij}^{CM}(\mathbf{x}(t)) \text{ or } a_{ij}^{MT}(\mathbf{x}(t)), \\ \delta_{ij}^{k}(\mathbf{x}(t)) &= \sum_{l \neq i,j} \max\{\delta_{il}^{k-1}(\mathbf{x}(t)) - \delta_{lj}^{k-1}(\mathbf{x}(t)), 0\}, \end{aligned}$$

where the matrix (δ^0) givens the pairwise influence of each

Acceleration of Flocking: $a_{ij}(\mathbf{x}) = \sum_{k=0}^{N-1} \delta_{ij}^k(\mathbf{x})$

$\delta_{ij}^k(\mathbf{x}(t))$ is defined inductively as follows:

$$\begin{aligned}
\delta_{ij}^0(\mathbf{x}(t)) &= a_{ij}^{CM}(\mathbf{x}(t)) \ \text{or} \ a_{ij}^{MT}(\mathbf{x}(t)), \\
\delta_{ij}^k(\mathbf{x}(t)) &= \sum_{l \neq i,j} \max\{\delta_{il}^{k-1}(\mathbf{x}(t)) - \delta_{lj}^{k-1}(\mathbf{x}(t)), 0\},
\end{aligned}$$

- $\delta_{ij}^0(\mathbf{x}(t))$ represents the direct impact from j to i;
- $\delta_{ij}^1(\mathbf{x}(t))$ represents the impact via an intermediate agent.
- Similarly, $\delta_{ij}^k(\mathbf{x}(t))$ denotes the impact from agent i to agent j via k intermediate agents.
- In particular, in the case where $k = 1$, we have

$$\begin{aligned}
a_{ij}(\mathbf{x}(t)) &= \delta_{ij}^0(\mathbf{x}(t)) + \delta_{ij}^1(\mathbf{x}(t)) \\
&= \phi_{ij}(\mathbf{x}(t)|) + \sum_{l \neq i,j} \max\{\phi_{il}(\mathbf{x}(t)) - \phi_{lj}(\mathbf{x}(t)), 0\}.
\end{aligned}$$

- This further increases the acceleration of agent i using the difference of velocities between agent i and agent j should agent i is further away from agent j than other intermediate

Delayed Cucker-Smale Model: motivation

- We are interested in a more general setting by incorporating delay arguments in the pairwise influence due to the finite speed in processing the influence.
- This addition of time lags seem to be very natural for most self-organized systems, and we will show that these time lags will not change the unconditional flocking property qualitatively, but alter the flocking velocity in a nonlinear way.
- In general, the influence of agents on each other is realized in various fashions including smell, sound and vision. For examples, the influence among honey bees is transferred mainly by a certain chemical material, while the influence among geese is mainly made through vision. As such, the influence of an agent on another is naturally transferred with a finite speed.
- We will focus on the case of processing the information about the location and velocity of neighbouring agents.

Delayed Cucker-Smale Model

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \qquad \frac{d\mathbf{v}_i}{dt} = \alpha \sum_{j\neq i}^{N} a_{ij}(\mathbf{x}(t-\tau_{ij}))(\mathbf{v}_j(t-\tau_{ij}) - \mathbf{v}_i(t)), \quad (3)$$

where τ_{ij} denotes the communication time between agents i and j.

- In general, the time delay τ_{ij} is non-symmetric so that $\tau_{ij} \neq \tau_{ji}$. In what follows, we assume also $\tau_{ii} = 0$ for all i.
- To specify a solution for the self-organized system (3), we need to specify the initial conditions

$$\mathbf{x}_i(\theta) = \mathbf{f}_i(\theta), \mathbf{v}_i(\theta) = \mathbf{g}_i(\theta) \quad \text{for } \theta \in [-\tau, 0], \qquad (4)$$

  where $\mathbf{f}$ and $\mathbf{g}$ are given continuous vector-value functions, $\tau = \max_{i,j}\{\tau_{i,j}\}$.
- It will be shown that the flocking velocity will depend not only on the size of the time lags, but also the variation of the agent positions at the initial time intervals.

Flocking solutions

Let d_X and d_V denote the diameters in position and velocity spaces, namely,

$$d_X(\mathbf{x}) = \max_{i,j}\{|\mathbf{x}_j - \mathbf{x}_i|\}, \quad d_V(\mathbf{v}) = \max_{i,j}\{|\mathbf{v}_j - \mathbf{v}_i|\}.$$

A solution $\{\mathbf{x}_i(t), \mathbf{v}_i(t)\}_{i=1}^N$ of system (3) subject to the initial condition (4) is called a flocking solution if it converges to a flock in the sense that

$$\sup_{t\geq 0} d_X(\mathbf{x}(t)) < +\infty \text{ and } \lim_{t\to+\infty} d_V(\mathbf{v}(t)) = 0.$$

Flocking solution candidates

The key in our fixed-point theoretic argument to establish sufficient conditions for unconditional flocking is to identify candidate flocking solutions of the self-organized system by imposing constraints on the bound, decaying rates at infinite, and variation at maximum delay interval of the flocking velocity.

We define the following set (3) and define the following set

$$
\begin{aligned}
E \quad = \quad & \{(\mathbf{x},\mathbf{v}) : \mathbf{x} = \{\mathbf{x}_i\}_{i=1}^N, \mathbf{v} = \{\mathbf{v}_i\}_{i=1}^N, \mathbf{x}_i, \mathbf{v}_i \in C([-\tau,+\infty), R^d); \\
& \qquad \mathbf{x}_i(s) = \mathbf{f}_i(s), \mathbf{v}_i(s) = \mathbf{g}_i(s) \text{ for } s \in [-\tau, 0]; \\
& \qquad \sup_{t\geq 0,i,j} |\mathbf{x}_i(t) - \mathbf{x}_j(t)| < +\infty; \\
& \qquad \sup_{t\geq 0,i} |\mathbf{v}_i(t)| \leq \sup_i |\mathbf{g}_i(0)| e^{\tau}; \\
& \qquad \sup_{\theta\in[0,\tau],t\geq 0,i,j} e^{\frac{\alpha}{2}t} |\mathbf{v}_i(t) - \mathbf{v}_j(t+\theta)| < +\infty\}.
\end{aligned}
$$

Unconditional Flocking

(LipInf) There exists a constant L such that, for all $\mathbf{x}, \mathbf{y} \in R^{dN}$ and all $1 \leq i, j \leq N$, we have

$$|a_{ij}(\mathbf{x}) - a_{ij}(\mathbf{y})| \leq L|\mathbf{x} - \mathbf{y}|.$$

With this assumption, we can introduce the constant

$$c = (\alpha + 1)N(L \sup_i |\mathbf{g}_i(0)|e^{\tau} + 1) + 1,$$

and define a metric D on the the set E by

$$D((\mathbf{x}, \mathbf{v}), (\mathbf{p}, \mathbf{q})) = \sup_{t \geq 0}\{e^{-ct} \max\{|\mathbf{x}(t) - \mathbf{p}(t)|, |\mathbf{v}(t) - \mathbf{q}(t)|\}\},$$

for $(\mathbf{x}, \mathbf{v}), (\mathbf{p}, \mathbf{q}) \in R^{dN} \times R^{dN}$. (E, D) is a complete metric space.
Theorem: If the global Lipschitz condition condition (**LipInf**) holds, then the self-organized system (3) with the initial value (4) has a unique flocking solution $\{\mathbf{x}_i(t), \mathbf{v}_i(t)\}_{i=1}^{N}$ in E.

Unconditional Flocking: Idea of the Proof

By using the variation-of-constants formula, we see that the solution of system (3) with the initial value (4) can be translated as a fixed point of operator $T : R^{dN} \times R^{dm} \to R^{dN} \times R^{dm}$ given by

$$T \begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \Phi(\mathbf{x}, \mathbf{v}) \\ \Psi(\mathbf{x}, \mathbf{v}) \end{pmatrix}, \text{ for } (\mathbf{x}, \mathbf{v}) \in E,$$

where $\Phi(\mathbf{x}, \mathbf{v}) = (\phi_1, \phi_2, \cdots, \phi_N)^T, \Psi(\mathbf{x}, \mathbf{v}) = (\psi_1, \psi_2, \cdots, \psi_N)^T$,

$$\begin{aligned} \phi_i(\mathbf{x}, \mathbf{v})(t) \quad &= \alpha^{-1}(1 - e^{-\alpha t})\mathbf{g}_i(0) + \mathbf{f}_i(0) \\ &+ \int_0^t (1 - e^{-\alpha(t-s)}) \sum_{j=1}^N a_{ij}\mathbf{v}_j(s - \tau_{ij})ds \end{aligned}$$

and

$$\psi_i(\mathbf{x}, \mathbf{v})(t) = e^{-\alpha t}\mathbf{g}_i(0) + \alpha \int_0^t e^{-\alpha(t-s)} \sum_{j=1}^N a_{ij}\mathbf{v}_j(s - \tau_{ij})ds$$

for $t \geq 0$.

Corollaries

Corollary If $a_{ij}(\mathbf{x}) = \frac{l(|\mathbf{x}_i - \mathbf{x}_j|)}{\sum_{k=1}^{N} l(|\mathbf{x}_i - \mathbf{x}_k|)}$ and $l(r) = (1 + r^2)^{-\beta}$ for $r \geq 0$ and $\beta > 0$, then the self-organized system (3) with initial value (4) has a unique flocking solution for all $\beta > 0$.

Remark: The case of $a_{ij}(\mathbf{x}) = \frac{l(|\mathbf{x}_i - \mathbf{x}_j|)}{\sum_{k=1}^{N} l(|\mathbf{x}_i - \mathbf{x}_k|)}$ without delay was considered by Motsch and Tadmor, and it was concluded that the model has a flocking solution if $\int^{+\infty} \phi^2(r)dr = +\infty$ holds. This, for the influence function $\phi(r) = (1 + r^2)^{-\beta}$, requires that $\beta \in (0, \frac{1}{4}]$. Our result shows that unconditioning flocking is guaranteed for the general Cucker-Smale model even with delay for all $\beta > 0$.

Asymptotic flocking velocity and delayed impact

Theorem: Assume the condition (**LipInf**) holds and $\{(\mathbf{x}_i(t), \mathbf{v}_i(t))\}_{i=1}^N$. Then

$$\lim_{t\to+\infty} \mathbf{v}_i(t) = \frac{\mathbf{g}_i(0) + \alpha \mathbf{w}_i}{1+\alpha\tau_i} + \frac{\alpha}{1+\alpha\tau_i}[\mathbf{f}_i(0) - \mathbf{f}_i(-\tau_i)] \equiv \mathbf{v}_\infty,$$

where $\mathbf{v}_\infty$ given above is independent of i, $\tau_i = \max_j\{\tau_{ij}\}$ and

$$\mathbf{w}_i = \lim_{t\to\infty} \int_0^t \sum_{j=1}^N a_{ij}(\mathbf{x}(s-\tau_{ij}))(\mathbf{v}_j(s-\tau_{ij}) - \mathbf{v}_i(s-\tau_i))ds.$$

Corollary: If $a_{ij} = a_{ji}$ and $\tau_{ij} = \tau$ for all i, j, then

$$\mathbf{v}_\infty = \frac{\sum_{i=1}^N \mathbf{g}_i(0)}{N(1+\alpha\tau)} + \frac{\alpha}{N(1+\alpha\tau)} \sum_{i=1}^N [\mathbf{f}_i(0) - \mathbf{f}_i(-\tau)].$$

Remark We give a positive answer to the problem posed by Motsch and Tadmor. We also note that the time delay and the variation of the initial position during the delay interval may impact on the final flocking velocity.