A Couple of Endpoint Restriction Estimates for Eigenfunctions

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Joint work with:

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Restriction estimates: Earlier work

Fix a compact n-dimensional compact boundaryless Riemannian manifold (M, g).
Consider L^2-normalized eigenfunctions

$$-\Delta_g e_\lambda(x) = \lambda^2 e_\lambda(x), \quad \int_M |e_\lambda|^2 \, dV_g = 1.$$

Problem raised by Reznikov and others: If $\Sigma \subset M$ is a d-dimensional submanifold, when do you have bounds of the form

$$\|e_\lambda\|_{L^q(\Sigma, dS)} \leq C(\lambda, n, d)\,?$$

- Answer may depend on curvature assumptions of $\Sigma \subset M$ (better if it is curved).
- We shall focus on simplest case where $\Sigma = \gamma \in \Pi$, where Π is space of unit-length geodesic segments.
- Simplest case (flat, 1-dimensional) and important for applications.

Restriction bounds for geodesics of Burq-Gérard-Tzvetkov and Hu

If Π is space of all unit-length geodesics segments in our n-dimensional Riemannian manifold (M, g)

$$\sup_{\gamma\in\Pi}\Big(\int_\gamma |e_\lambda|^q\, ds\Big)^{\frac{1}{q}} \le C(1+\lambda)^{\sigma(n,q)},$$

where

$$\sigma(2,q) = \begin{cases} \frac{1}{4}, & \text{if } 2 \le q \le 4, \\ \frac{1}{2} - \frac{1}{q}, & \text{if } q \ge 4, \end{cases}$$

and

$$\sigma(n,q) = \frac{n-1}{2} - \frac{1}{q}, \quad \text{if } \ q > 2, \text{ and } \ n = 3, \text{ or } \ q \ge 2, \text{ and } \ n \ge 4.$$

For $n = 3$ they also obtained the endpoint estimate

$$\Big(\int_\gamma |e_\lambda|^2\, ds\Big)^{\frac{1}{2}} \le C\big(\log(2+\lambda)\big)^{\frac{1}{2}}(1+\lambda)^{\frac{1}{2}}.$$

Global Harmonic Analysis: Improved L^p-estimates for eigenfunctions and global dynamics

In 1988 CS proved

$$\|e_\lambda\|_{L^q(M)} \lesssim (1+\lambda)^{\delta(q,n)}.$$

If $n = 2$ and $q = 4$, $\delta = \frac{1}{8}$.
Sharp because of highest weight spherical harmonics on $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$ concentrating on equator (periodic geodesic):

$$Q_\lambda = c(\lambda)(x_1 + ix_2)^\lambda, \quad \lambda = 1, 2, 3, \dots, \quad c(\lambda) \approx \lambda^{\frac{1}{4}}.$$

Question: When can you rule out the existence of such modes ("Gaussian beams")?
Probably want a) not too many periodic geodesics and b) all such are unstable.
Best case: Nonpositive curvature.

$L^4(M)$-norms in 2-d

Bourgain 2010 ($\Rightarrow$) and CS 2011 ($\Leftarrow$) showed that

$$\|e_\lambda\|_{L^4(M)} = o(\lambda^{\frac{1}{8}}) \iff \sup_{\gamma\in\Pi}\Big(\int_\gamma |e_\lambda|^2\, ds\Big)^{\frac{1}{2}} = o(\lambda^{\frac{1}{4}}).$$

CS showed that above are $\iff$ $L^2(dV_g)$-norms over $\lambda^{-\frac{1}{2}}$ tubes about $\gamma \in \Pi$ are $o(1)$ (beating the trivial Kakeya-Nikodym max estimate).

In 2-d Euclidean space $\mathbb{R}^2$, L^4 is a critical space for Bochner-Riesz and oscillatory integral theorems.

The $\Leftarrow$ part was proved by using ideas from the two different proofs of the Carleson-Sjölin theorem for Bochner-Riesz summability for $L^4(\mathbb{R}^2)$, which go back to the 1970s and are due to a) Carleson-Sjölin and Hörmander, and b) Fefferman and Córdoba.

Joint w/ Zelditch: Improved $L^4(M)$ and $L^2(\gamma)$ bounds

If $n = 2$, CS and Zelditch in 2011 showed that if (M, g) has everywhere nonpositive curvature then

$$\sup_{\gamma \in \Pi} \Big(\int_\gamma |e_\lambda|^2 \, ds \Big)^{\frac{1}{2}} = o(\lambda^{\frac{1}{4}}),$$

and hence

$$\|e_\lambda\|_{L^4(M)} = o(\lambda^{\frac{1}{8}}).$$

So *no Gaussian beam* type functions (i.e., max concentration along geodesics) under this curvature assumption.
As shown in CS's 2011 paper, the $L^2(\gamma)$ restriction estimate is easy if γ is not a segment in a periodic orbit.

CS-Zeldtich handled the case when it is part of a periodic orbit, $\tilde{\gamma}$.

Techniques: Lift necessary calculations up to universal cover $(\mathbb{R}^2, \tilde{g})$ where you can use microlocal analysis, Hadamard parametrix and properties of the deck transformations (especially role of stabilizer group for $\tilde{\gamma}$). See below.

Applications: Improved L^1-lower bounds and nodal sets

If $Z_\lambda = \{x \in M : e_\lambda(x) = 0\}$ is the nodal set of a real-valued eigenfunction e_λ of frequency λ, *Yau conjectured that its* $(n-1)$*-dimensional Hausdorff measure should satisfy*

$$|Z_\lambda| \approx \lambda.$$

Fully settled in real analytic case by Donnelly and Fefferman in 1980s.

Until about 2 years ago, for C^∞ case, best lower bounds were $\exp(-c\lambda)$ (and best known upper bounds are doubly exponential).

Lower bounds involving powers of λ were obtained by CS and Zelditch in 2010.

Current best lower bounds (also 2010) are due to Colding and Minicozzi:

$$\lambda^{1-\frac{n-1}{2}} \lesssim |Z_\lambda|.$$

Improved L^1-lower bounds and nodal sets, continued

Using ideas from earlier paper of CS and Zeldich, in 2011 *Hezari and CS showed that* $\lambda\Big(\int_M |e_\lambda|\, dV_g\Big)^2 \lesssim |Z_\lambda|$.

Using earlier $L^1(M)$-lower bounds of CS and Zelditch,

$$\lambda^{-\frac{n-1}{4}} \lesssim \|e_\lambda\|_{L^1(M)},$$

from this you obtain the C-M lower bound

$$\lambda^{1-\frac{n-1}{2}} \lesssim |Z_\lambda|.$$

Problem: *When can you beat the above lower bound for $L^1(M)$-norms?* (Would lead to improved nodal lower bounds.)

This estimate also saturated by highest weight spherical harmonics ("Gaussian beams").

Can do this by Hölder's inequality if you can improve CS's $L^q(M)$ estimates for any $2 < q \leq \frac{2(n+1)}{n-1}$.

For instance, when $n = 2$ get improved $L^1(M)$-lower bounds from improved $L^4(M)$-upper bounds using Hölder:

$$1 = \|e_\lambda\|_{L^2(M)}^{\frac{3}{2}} \leq \|e_\lambda\|_{L^1(M)} \|e_\lambda\|_{L^4(M)}^2,$$

and so if you can beat $\|e_\lambda\|_{L^4(M)} = O(\lambda^{\frac{1}{8}})$, you can beat the $L^1(M)$-lower bound of CS and Zelditch:

$$\lambda^{-\frac{1}{4}} \lesssim \|e_\lambda\|_{L^1(M)}.$$

Conclusion: Several different problems (e.g. nodal sets, $L^q(M)$-bounds for "small q" and restriction estimates) all seem to center on whether you can have eigenfunctions fitting the "Gaussian beam" profile measured in different ways.

I.e., quantitative improvements measuring lack of concentration near periodic geodesics via one of the estimates implies (or should) improvements for the others.

Also related to questions of quantum ergodicity.

Hilbert transform & an improved endpoint estimate in 3-d

Joint work with Xuehua Chen: *If* $n = 3$ *for a general* (M, g) *(no curvature assumptions) get* log*-improvement of the earlier* $L^2(\gamma)$*-restriction estimates of Burq-Gérard-Tzvetkov and Hu:*

$$\Big(\int_\gamma |e_\lambda|^2 \, ds\Big)^{\frac12} \le C(1+\lambda)^{\frac12} \|e_\lambda\|_{L^2(M)}, \quad \gamma \in \Pi.$$

To prove this, note that if $\rho \in \mathcal{S}(\mathbb{R})$ is an even real-valued function satisfying

$$\rho(0) = 1, \quad \text{and} \;\; \hat\rho(\tau) = 0, \quad \text{if} \;\; |\tau| \ge \text{Inj}\,(M)/4,$$

then $\rho(\lambda - \sqrt{-\Delta_g})e_\lambda = e_\lambda$, and so above estimate follows from

$$\|Tf\|_{L^2(\gamma)} \lesssim (1+\lambda)^{\frac12} \|f\|_{L^2(M)}, \quad T = \rho(\lambda - \sqrt{-\Delta_g}).$$

The last statement that $\|T\|_{L^2(M)\to L^2(\gamma)} = O((1+\lambda)^{\frac{1}{2}})$ is equivalent to the statement that the adjoint operator T^* obey

$$\|T^*h\|_{L^2(M)} \lesssim (1+\lambda)^{\frac{1}{2}}\|h\|_{L^2(\gamma)}.$$

But

$$\|T^*h\|^2_{L^2(M)} = \langle T^*h, T^*h\rangle = \int_\gamma TT^*h\,\overline{h}\,ds \le \|TT^*h\|_{L^2(\gamma)}\|h\|_{L^2(\gamma)},$$

Conclude that we would have the log-improvement of BGT and Hu if we could show that

$$\|TT^*h\|_{L^2(\gamma)} \lesssim (1+\lambda)\|h\|_{L^2(\gamma)}.$$

If $\chi(\tau) = (\rho(\tau))^2$, not hard to unravel to see that the kernel of TT^* is the restriction to $\gamma \times \gamma$ of

$$K_\lambda(x,y) = \sum_{j=0}^{\infty} \chi(\lambda - \lambda_j)e_j(x)\overline{e_j(y)}$$

Here $0 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \dots$ are the e.v's listed w/ multiplicity & $\{e_j\}$ the associated o.n. basis of e.f.'s (i.e., $-\Delta_g e_j(x) = \lambda_j^2 e_j(x)$)

Parametrize the geodesic w.r.t arc length $\gamma = \{\gamma(s) : |s| \le \frac{1}{2}\}$ (and assume, WLOG, $\text{Inj}(M) \ge 4$). Then we are left with showing that

$$\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big| \int_{-\frac{1}{2}}^{\frac{1}{2}} K(\gamma(t), \gamma(s)) h(s)\, ds \Big|^2 dt\Big)^{\frac{1}{2}} \le C(1+\lambda) \Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} |h(s)|^2 ds\Big)^{\frac{1}{2}}.$$

To compute $K_\lambda(x, y)$, $x, y \in M$ we use the wave equation:

$$\begin{aligned} K_\lambda(x,y) &= \chi(\lambda - \sqrt{-\Delta_g})(x, y) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\chi}(\tau) e^{i\tau\lambda} \big(e^{-i\tau\sqrt{-\Delta_g}}\big)(x, y)\, d\tau \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \hat{\chi}(\tau) e^{i\lambda\tau} \big(\cos \tau \sqrt{-\Delta_g}\big)(x, y) d\tau - \chi(\lambda + \sqrt{-\Delta_g})(x, y). \end{aligned}$$

Since $\chi \in \mathcal{S}$ and spectrum of $-\Delta_g$ is nonneg easy to see that last term is $O((1+\lambda)^{-N})$, $N = 1, 2, 3$. So have estimate in blue if same holds with K_λ replaced by

$$\tilde{K}_\lambda(x, y) = \int \hat{\chi}(\tau) e^{i\lambda\tau} (\cos \tau \sqrt{-\Delta_g})(x, y)\, d\tau, \quad x = \gamma(t), y = \gamma(s).$$

Wave kernel

Here $(\cos \tau \sqrt{-\Delta_g})(x, y)$ is the kernel of the operator, $\cos(\tau \sqrt{-\Delta_g})$, which takes an eigenfunction $e_\lambda(x)$ to the function

$$\cos(\tau\lambda)\, e_\lambda(x).$$

Hence, if $f \in C^\infty(M)$ it follows that

$$u(\tau, x) = \big(\cos(\tau \sqrt{-\Delta_g}) f\big)(x)$$

is the unique solution of the Cauchy problem

$$\begin{cases} (\partial_\tau^2 - \Delta_g)\, u(\tau, x) = 0, & \tau \in \mathbb{R}, x \in M \\ u(0, x) = f(x), & \partial_\tau u(0, x) = 0. \end{cases}$$

End of proof: Hadamard parametrix & Hilbert transform

We are assuming that $\text{Inj}(M) \geq 4$. Therefore for $\tau \in \text{supp } \hat{\chi}$, by the Hadamard parametrix

$$(\cos \tau \sqrt{-\Delta_g})(x, y) = w(x, y)(2\pi)^{-3} \int_{\mathbb{R}^3} e^{i d_g(x,y)\xi_1} \cos(\tau|\xi|)\, d\xi + better,$$

for some $w \in C^\infty(M \times M)$ satisfies $w(x, x) \equiv 1$, and where $d_g(x, y)$ is the Riemannian distance. Since $d_g(\gamma(t), \gamma(s)) = |t - s|$ and $\int_{S^2} e^{i\omega\cdot\eta} d\omega = 4\pi \sin|\eta|/|\eta|$, get using polar coordinates

$$\begin{aligned}
\tilde{K}_\lambda(\gamma(t), \gamma(s)) &\approx \frac{w(\gamma(t), \gamma(s))}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \hat{\chi}(\tau) e^{i\lambda\tau} e^{i|t-s|\xi_1} \cos(\tau|\xi|) d\tau d\xi \\
&= \frac{w(\gamma(t), \gamma(s))}{4\pi^2} \int_{\mathbb{R}^3} \chi(\lambda - |\xi|) e^{i|t-s|\xi_1} d\xi + O(1) \\
&= \frac{w(\gamma(t), \gamma(s))}{\pi} \int_0^\infty \chi(\lambda - r) \frac{\sin(t-s)r}{t-s}\, r\, dr + O(1) \\
&= \frac{1}{\pi} \int_0^\infty \chi(\lambda - r) \frac{\sin r(t-s)}{t-s}\, r dr + O(1 + \lambda).
\end{aligned}$$

End of proof continued: Hilbert transform

Want integral operator with kernel $\tilde{K}_\lambda(\gamma(t), \gamma(s))$ to be bounded from $L^2([-1/2, 1/2])$ to itself with norm $O((1+\lambda))$. By the last formula and Minkowski's integral inequality, this would follow from

$$\int_0^\infty \Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big| \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\sin r(t-s)}{t-s} h(s) ds \Big|^2 dt \Big)^{\frac{1}{2}} |\chi(\lambda - r)| r \, dr$$

$$\leq C(1+\lambda) \Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} |h(s)|^2 \, ds \Big)^{\frac{1}{2}}.$$

Since $\chi \in \mathcal{S}$, have $\int |\chi(\lambda - r)| r \, dr \lesssim (1+\lambda)$, and so above if

$$\sup_{r>0} \Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big| \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\sin r(t-s)}{t-s} h(s) ds \Big|^2 dt \Big)^{\frac{1}{2}} \leq C \Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} |h(s)|^2 \, ds \Big)^{\frac{1}{2}}.$$

Since $\frac{\sin r(t-s)}{t-s}$ (essentially) the Dirichlet kernel, follows from the uniform L^2 boundedness of the partial summation operators for Fourier series, or since $2i \sin r(t-s) = e^{ir(t-s)} + e^{-ir(t-s)}$ follows from Hilbert transform bounds (replace $\sin r(t-s)$ by 1 in above).

Improved L^4-restriction bounds for nonpositive curvature

Joint with Xuehua Chen: $n = 2$, (M, g) *nonpositive curvature*,

$$\left(\int_\gamma |e_\lambda|^4\, ds\right)^{\frac{1}{4}} = o(\lambda^{\frac{1}{4}}), \quad \gamma \in \Pi.$$

In 2011, CS-Zelditch,

$$\left(\int_\gamma |e_\lambda|^2\, ds\right)^{\frac{1}{2}} = o(\lambda^{\frac{1}{4}}), \quad \gamma \in \Pi.$$

- The XC-CS result implies the CS-Zeldtich one by Hölder's inequality.
- Either gives improved $L^2(M) \to L^q(M)$ estimates for eigenfunctions by 2011 results of CS (see also Bourgain 2010).
- *None of above true on* S^2 because of highest weight spherical harmonics concentrating on periodic geodesics ("Gaussian beams")

Outline of proof

- As in the 2-d case, use wave equation (cosine-transform).
- To exploit curvature hypothesis, lift the calculation up to the universal cover.
- Obtain a sum with a great deal of terms.
- Terms corresponding to stabilizer group for geodesic can handle as before (no oscillation). Also, very few terms.
- Exponentially growing number of terms not arising from stabilizer group.
- Get improved bounds for each using oscillator integral theorems of Hörmander, Greenleaf-Seeger, Phong-Stein, which allows one to estimate the sum favorably.

To prove the new $o(\lambda^{\frac{1}{4}})$ result, need to show that given $\varepsilon > 0$ $\exists \Lambda < \infty$ so that

$$\Big(\int_\gamma |e_\lambda|^4 \, ds\Big)^{\frac{1}{4}} \le \varepsilon \lambda^{\frac{1}{4}}, \quad \lambda \ge \Lambda.$$

Take $T \approx \varepsilon^{-4}$ and if $\rho \in \mathcal{S}(\mathbb{R})$ is as above with $\rho(0) = 1$, $\hat{\rho}(\tau) = 0$, $|\tau| \ge 1/2$, have

$$e_\lambda = \rho(T(\lambda - \sqrt{-\Delta_g}))e_\lambda,$$

and so it suffices to show that

$$\Big(\int_\gamma |\rho(T(\lambda - \sqrt{-\Delta_g}))f|^4 \, ds\Big)^{\frac{1}{4}} \le \big(C T^{-\frac{1}{4}} \lambda^{\frac{1}{4}} + C_T \lambda^{\frac{3}{16}}\big) \, \|f\|_{L^2(M)}.$$

By earlier TT^* argument, above if when $\chi(\tau) = (\rho(\tau))^2$ have

$$\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big| \int_{-\frac{1}{2}}^{\frac{1}{2}} \chi\big(T(\lambda - \sqrt{-\Delta_g})\big)(\gamma(t), \gamma(s)) \; h(s) ds \Big|^4 dt\Big)^{\frac{1}{4}}$$
$$\le \big(CT^{-\frac{1}{2}}\lambda^{\frac{1}{2}} + C_T \lambda^{\frac{3}{8}}\big) \|h\|_{L^{\frac{4}{3}}}.$$

Similar to the 3-d case, write for $x, y \in M$

$$\chi(T(\lambda - \sqrt{-\Delta_g}))(x,y) = \sum_{j=0}^{\infty} \chi(T(\lambda - \lambda_j)) e_j(x) \overline{e_j(y)}$$

$$= \frac{1}{2\pi T} \int_{-\infty}^{\infty} \hat{\chi}(\tau/T) e^{it\lambda} \big(\cos \tau \sqrt{-\Delta_g}\big)(x,y)\, d\tau + O((1+\lambda)^{-N}).$$

Can ignore "error term": much better bounds.

To understand integral operator coming from blue kernel restricted to $\gamma \times \gamma$ need to understand kernel of $\cos \tau \sqrt{-\Delta_g}$ for $|\tau| \le T$ (large times).

Also need that there is some sort of "dispersion" for this wave kernel that allows us to have small restriction bounds. This comes from our curvature hypothesis.

Hadamard and the universal cover of (M, g)

Hadamard (1898): *For* any *point $P \in M$, the exponential map at P,*

$$\kappa = \exp_P : T_P M \simeq \mathbb{R}^2 \to M,$$

is a covering map. Take $P = \gamma(0)$, the center of our geodesic segment.

Pullback metric g on M to get metric $\tilde{g}$ on the universal cover, $\mathbb{R}^2$, of M.

The set of all diffeomorphisms $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

$$\kappa \circ \alpha = \kappa$$

forms a group, Γ, of "deck transformations" for the covering map.

Each is an isometric diffeomorphism of $(\mathbb{R}^2, \tilde{g})$ and $M \simeq \mathbb{R}^2/\Gamma$.

Fundamental domains and the wave equation on M

Let $D \subset \mathbb{R}^2$ be a fundamental domain for (M, g). Then $M \simeq D$ and smooth solutions $u(\tau, x)$ of wave equations on (M, g) are in one-to-one correspondence with smooth periodic ones $\tilde{u}(\tau, \tilde{x})$ on $(\mathbb{R}^2, \tilde{g})$ with respect to Γ, i.e.,

$$\tilde{u}(\tau, \tilde{x}) = \tilde{u}(\tau, \alpha(\tilde{x})), \quad \alpha \in \Gamma.$$

Thus if for $x \in M$, $\tilde{x}$ is the unique point in D such that $\kappa(\tilde{x}) = \tilde{x}$, we have the following formula relating wave kernel on (M, g) to that on its universal cover:

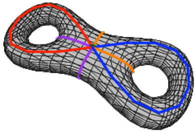
$$\left(\cos \tau \sqrt{-\Delta_g}\right)(x, y) = \sum_{\alpha \in \Gamma} \left(\cos \tau \sqrt{-\Delta_{\tilde{g}}}\right)(\tilde{x}, \alpha(\tilde{y})).$$

This formula is analogous to the *classical Poisson summation formula* for Fourier series (i.e., $M = \mathbb{T}^2$)

Fundamental domain for the double torus

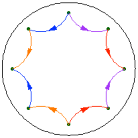
Recall a fundamental domain of $\mathbb{T}^2 = S^1 \times S^1$ is "square", $[-\pi, \pi)^2$. Comes from cutting the $\mathbb{T}^2$ along each S^1 & unraveling.

Consider the double (two-holed) torus of constant negative curvature and genus two:



Hard to visualize since cannot be embedded in $\mathbb{R}^3$.

To obtain the fundamental domain, cut along the colored loops and unravel. Obtain octagonal fundamental domain (shown as subset of hyperbolic disk (not plane)):



To finish we are trying to estimate the $L^{\frac{4}{3}}(|t| \le 1/2) \to L^4(|t| \le 1/2)$ norm of operator with kernel

$$K_\lambda(t,s) = \frac{1}{2\pi T} \sum_{\alpha \in \Gamma} \int_{-T}^{T} \hat{\chi}(\tau/T) e^{i\tau\lambda} \big(\cos \tau \sqrt{-\Delta_{\tilde{g}}}\big)(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s)))\, d\tau.$$

Here $\tilde{\gamma}(t)$ denotes the lift of the geodesic $\gamma(t)$, $t \in \mathbb{R}$ we are considering.

$\tilde{\gamma}$ is a geodesic in $(\mathbb{R}^2, \tilde{g})$, since $\tilde{g}$ is the pullback of metric g on M. $\alpha(\tilde{\gamma})$ is also a geodesic when $\alpha \in \Gamma$ since every deck transformation preserves distances and angles.

Let $\mathrm{Stab}(\gamma)$ be all α so that $\alpha(\tilde{\gamma}) = \tilde{\gamma}$. This is a cyclic subgroup of Γ, which is just trivial subgroup (i.e. *Identity*) unless γ is a periodic geodesic with some minimal period $\ell > 0$.

If $\alpha \in \mathrm{Stab}(\gamma)$, have for some $k \in \mathbb{Z}$, $\alpha(\tilde{\gamma}(t)) = \tilde{\gamma}(t + k\ell)$ (shift).

Let $K_\lambda^{\mathrm{Stab}(\gamma)}(t,s)$ denote the analog of above where we sum over $\alpha \in \mathrm{Stab}(\gamma)$ (typically just one term)

Contribution of stabilizer group

$(\mathbb{R}^2, \tilde{g})$ has infinite injectivity radius. Also $(\cos\tau\sqrt{-\Delta_{\tilde{g}}})(x,y)$ vanishes if $d_{\tilde{g}}(x,y) > \tau$. So can use Hadamard parametrix and arguments from 3-d case to see that

$$\Big|\frac{1}{2\pi T}\int_{-T}^{T} \hat{\chi}(t/T)e^{i\tau\lambda}(\cos\tau\sqrt{-\Delta_{\tilde{g}}})(\tilde{\gamma}(t), \tilde{\gamma}(s+\ell k))\,d\tau\Big| \\ \le CT^{-1}\lambda^{\frac{1}{2}}|t-s-\ell k|^{-\frac{1}{2}} + C_T.$$

Since we are assuming that the injectivity radius is ≥ 4, it follows that either $\ell = 0$ (nonperiodic base geodesic) or $\ell \ge 4$. Also, by Huygens each term in the left is $= 0$ if $k > 2T$.

So if we sum over all $\alpha \in \mathrm{Stab}(\gamma)$ we conclude that

$$|K^{\mathrm{Stab}(\gamma)}(t,s)| \le C\lambda^{\frac{1}{2}}(T^{-1}|t-s|^{-\frac{1}{2}} + T^{-\frac{1}{2}}) + C_T, \quad |t|, |s| \le 1/2,$$

and hence, by Hardy-Littlewood, the operator with this kernel maps $L^{\frac{4}{3}}(|t| \le 1/2) \to L^4(|t| \le 1/2)$ with norm $O(T^{-\frac{1}{2}}\lambda^{\frac{1}{2}}) + C_T$, as desired.

Oscillatory integrals and non-stabilizers

Let $K_\lambda^{\mathrm{Osc}}(t,s)$ be equal to the remaining piece,

$$\frac{1}{2\pi T}\sum_{\alpha\notin \mathrm{Stab}(\gamma)}\int_{-T}^{T}\hat{\chi}(\tau/T)e^{i\tau\lambda}\left(\cos\tau\sqrt{-\Delta_{\tilde{g}}}\right)(\tilde{\gamma}(t),\alpha(\tilde{\gamma}(s))d\tau.$$

Would be done if

$$\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}}\Big|\int_{-\frac{1}{2}}^{\frac{1}{2}}K_\lambda^{\mathrm{Osc}}(t,s)\,h(s)ds\Big|^4 ds\Big)^{\frac{1}{4}}\le C_T\lambda^{\frac{1}{2}}\lambda^{-\frac{1}{8}}\|h\|_{L^{\frac{4}{3}}}. \qquad (1)$$

If $\alpha\notin \mathrm{Stab}(\gamma)$ then $\alpha(\tilde{\gamma})$ is a different geodesic than $\tilde{\gamma}$ and using Hadamard parametrix have for $|t|,|s|\le 1/2$,

$$\frac{1}{2\pi T}\int_{-T}^{T}\hat{\chi}(\tau/T)e^{i\tau\lambda}\left(\cos\tau\sqrt{-\Delta_{\tilde{g}}}\right)(\tilde{\gamma}(t),\alpha(\tilde{\gamma}(s))d\tau$$

$$\approx T^{-1}\lambda^{\frac{1}{2}}\left(d_{\tilde{g}}(\tilde{\gamma}(t),\alpha(\tilde{\gamma}(s)))\right)^{-\frac{1}{2}}\sum_{\pm}e^{\pm i\lambda d_{\tilde{g}}(\gamma(t),\alpha(\tilde{\gamma}(s)))}+O_T(1).$$

Oscillatory integrals and non-stabilizers, continued

If $\alpha \notin \mathrm{Stab}(\gamma)$ and we set

$$\phi(t,s) = d_{\tilde{g}}(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s))),$$

can show that mixed-Hessian satisfies

$$|\phi''_{ts}(t,s)| + |\nabla_{t,s}\, \phi''_{ts}(t,s)| \neq 0.$$

Thus, by oscillatory integral theorems of Hörmander, Greeleaf-Seeger and Phong-Stein,

$$\Big(\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big| \int_{-\frac{1}{2}}^{\frac{1}{2}} \big(d_{\tilde{g}}(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s)))\big)^{-\frac{1}{2}} \sum_{\pm} e^{\pm i\lambda d_{\tilde{g}}(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s)))}\, h(s)\, ds \Big|^4 dt\Big)^{\frac{1}{4}}$$

$$\leq C\lambda^{-\frac{1}{8}} \|h\|_{L^{\frac{4}{3}}},$$

and so by summing over the $O(e^{c|T|})$ nonzero terms in the sum defining K_λ^{Osc}, we get (1) which completes proof.

Problem: Improvements for the $L^2(\gamma)$ restriction estimate when $n = 3$ and curvature ≤ 0?

We were able to obtain endpoint restriction estimates in 2-d for geodesics due to the simple fact that if $\gamma_1(t)$ and $\gamma_2(s)$ are two different geodesics parameterized by arc-length and

$$\phi(t, s) = d_{\tilde{g}}(\gamma_1(t), \gamma_2(s)),$$

then the associated canonical relation has at worst one-sided folds whenever $\gamma_1(t) \neq \gamma_2(s)$.

This is true without any assumptions on the curvature if $(\mathbb{R}^2, \tilde{g})$ has no conjugate points.

This is no longer true when $n = 3$, even for the flat metric. Indeed there are pairs of lines for which

$$|\phi''_{ts}(t,s)| + |\nabla_{t,s}\,\phi''_{ts}(t,s)|$$

may vanish at a certain point (t, s).

On the other hand if $(\mathbb{R}^3, \tilde{g})$ is the universal cover for (M, g) with nonpositive curvature, perhaps this quantity never vanishes when, as before,

$$\gamma_2 = \alpha(\gamma_1), \quad \text{with } \alpha \in \Gamma \backslash \mathrm{Stab}(\gamma_1).$$

This fact is true for $\mathbb{T}^3$, but we can't show it presently for general 3-dimensional compact manifolds with nonpositive sectional curvatures.

If we could, we would obtain $o(\lambda)$ restriction estimates for 3-d manifolds with nonpositive curvature.