Eigenfunctions and Nodal sets
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Nodal sets of eigenfunctions

Let (M, g) be a compact C^∞ Riemannian manifold of dimesion n, let φ_λ be an L^2-normalized eigenfunction of the Laplacian,

$$\Delta\varphi_\lambda = -\lambda^2\varphi_\lambda,$$

and let

$$\mathcal{N}_{\varphi_\lambda} = \{x : \varphi_\lambda(x) = 0\}$$

be its nodal hypersurface. The hypersurface volume of the nodal set is denoted

$$|\mathcal{N}_{\varphi_\lambda}| = \mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}).$$

Some Intuition about nodal sets

- Algebraic geometry: Eigenfunctions of eigenvalue λ^2 are analogues on (M, g) of polynomials of degree λ. Their nodal sets are analogues of (real) algebraic varieties of this degree. The $\lambda_j \to \infty$ is the high degree limit or high complexity limit. This analogy is best if (M, g) is real analytic.
- Quantum mechanics: $|\varphi_j(x)|^2 dV_g(x)$ is the probability density of a quantum particle of energy λ_j^2 being at x. Nodal sets are the least likely places for a quantum particle in the energy state λ_j^2 to be. The $\lambda_j \to \infty$ limit is the high energy or semi-classical limit.

Outline of talk

- Recall Nodal volume conjecture and classical results.
- Recent results on lower bounds of nodal volumes for C^∞ metrics.
- Analytic continuation in the real analytic case to the complex domain. Recent results on equi-distribution of intersection points of nodal sets and geodesics.

Yau volume conjecture for C^∞ metrics

S. T. Yau conjecture: for any C^∞ metric, there exist constants $C, c > 0$ depending only on (M, g) and not on λ such that

$$c\lambda \leq \mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}) \leq C\lambda. \tag{1}$$

Both the upper and lower bounds were proved for real analytic C^ω metrics by Donnelly-Fefferman in 1985.

Bounds until recently

There are special bounds on the length of the nodal line for alll C^∞ metrics in dimension 2:

$$c\lambda \leq \mathcal{H}^1(\mathcal{N}_{\varphi_\lambda}) \leq C\lambda^{3/2}.$$

The lower bound was proved by J. Brüning and the upper bound by Donnelly-Fefferman and R. T. Dong.

In dimensions ≥ 3 the bounds were

$$C^{-\lambda} \leq \mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}) \leq \lambda^{C\lambda}. \tag{2}$$

The upper bound was proved by Hardt-Simon, and the lower bound is proved in the book of FH Lin and Q. Han.

New lower bounds on volumes of nodal hypersurfaces: C^∞ case

Theorem

(Colding-Minicozzi, Sogge-Z) In all dimensions,

$$\mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda}) \geq \lambda^{\frac{3-n}{2}}$$

The techniques of Colding-Minicozzi and Sogge-Z are quite different. There was a sequence of results in 2011-2012 by Sogge-Z, Colding-Minicozzi, Hezari-Sogge, Sogge-Zelditch, Mangoubi and others giving lower bounds. The original Sogge-Z result of 2011 was weaker, but recently we noticed that a small tweak of our original gives the same bound as Colding-Minicozzi.

An identity

The proof is based on the following identity, inspired by a closely related identity of R. T. Dong:

Proposition

(Sogge-Z) For any C^∞ Riemannian manifold, we have,

$$\lambda^2 \int_M |\varphi_\lambda| dV = 2 \int_{\mathcal{N}_{\varphi_\lambda}} |\nabla \varphi_\lambda| dS. \tag{3}$$

More generally, for any $f \in C^2(M)$,

$$\int_M \left((\Delta + \lambda^2) f\right) |\varphi_\lambda| dV = 2 \int_{\mathcal{N}_{\varphi_\lambda}} f |\nabla \varphi_\lambda| dS. \tag{4}$$

Application to nodal set volumes

The lower bound on nodal volumes is a simple consequence of the identity the following lemma (which was implicit in our original article)

Lemma

If $\lambda > 0$ *then* $\|\nabla_g \varphi_\lambda\|_{L^\infty(M)} \lesssim \lambda^{1+\frac{n-1}{2}} \|\varphi_\lambda\|_{L^1(M)}$.

Proof of lower bound from Lemma:

$$\begin{array}{lll} \lambda^2 \int_M |\varphi_\lambda| \, dV = 2 \int_{\mathcal{N}_{\varphi_\lambda}} |\nabla_g \varphi_\lambda|_g \, dS & \leq & 2|\mathcal{N}_{\varphi_\lambda}| \, \|\nabla_g \varphi_\lambda\|_{L^\infty(M)} \\ & \lesssim & 2|\mathcal{N}_{\varphi_\lambda}| \, \lambda^{1+\frac{n-1}{2}} \|\varphi_\lambda\|_{L^1(M)}. \end{array} \tag{5}$$

The factor $||\varphi_\lambda||_{L^2}$ cancels from the two sides!

Proof of Lemma

The main point is to construct a designer reproducing kernel K_λ for φ_λ:
Let $\rho \in C_0^\infty(\mathbb{R})$ satisfy $\int \rho \, dt = 1$ and let

$$\chi_\lambda f = \int \rho(t) e^{-it\lambda} e^{it\sqrt{-\Delta_g}} f \, dt.$$

Then

$$\chi_\lambda \varphi_\lambda = \varphi_\lambda.$$

Construct ρ further so that $\rho(t) = 0$ for $t \notin [\epsilon/2, \epsilon]$
The kernel $K_\lambda(x, y)$ of χ_λ for ϵ sufficiently small satisfies

$$|\nabla_g K_\lambda(x, y)| \leq C\lambda^{1+\frac{n-1}{2}}. \tag{6}$$

Thus $\|\nabla_g \chi_\lambda f\|_{L^\infty} \leq C\lambda^{1+\frac{n-1}{2}} \|f\|_{L^1}$, which implies the lemma.

Remarks

In our original paper, we used an upper bound from the local Weyl law that omitted the factor of $||\varphi_\lambda||_{L^1}$. So we needed to prove a lower bound on $||\varphi_\lambda||_{L^1}$. It is still quite useful:

Proposition

For any C^∞ Riemannian manifold, there exists constants $C, c > 0$ so that

$$C\,\lambda^{-\frac{n-1}{4}} \leq C\,||\varphi_\lambda||_{L^1}.$$

The lower bound is sharp–it is achieved by the main enemy of nodal volume estimates: Gaussian beams (e.g. highest weight spherical harmonics).

Hezari-Sogge bound

By manipulating the identity,

THEOREM
(Hezari-Sogge, 2011)

$$\mathcal{H}^{n-1}(\mathcal{N}_\lambda) \geq \lambda ||\varphi_\lambda||_{L^1}^2.$$

Thus, the Yau conjectured λ lower bound holds whenever $||\varphi_\lambda||_{L^1} \geq C_0 > 0$. In recent work, Sogge observed that one can proved the λ lower bound unless BOTH of the following hold:

$$||\varphi_\lambda||_{L^\infty} \simeq \lambda^{\frac{n-1}{4}}, \quad ||\varphi_\lambda||_{L^1} \simeq \lambda^{-\frac{n-1}{4}}.$$

Both hold for Gaussian beams!

Proof of identity

It is known that the singular set

$$\Sigma(\varphi_\lambda) = \{x \in \mathcal{N}_{\varphi_\lambda} : \nabla \varphi_\lambda(x) = 0\}$$

satisfies $\mathcal{H}^{n-2}(\Sigma(\varphi_\lambda)) < \infty$. Thus, outside of a codimension one subset, $\mathcal{N}_{\varphi_\lambda}$ is a smooth manifold, and the Riemannian surface measure $dS = \iota_{\frac{\nabla \varphi_\lambda}{|\nabla \varphi_\lambda|}} dV_g$ on $\mathcal{N}_{\varphi_\lambda}$ is well-defined.

Since $d\mu_\lambda := (\Delta + \lambda^2)|\varphi_\lambda| dV = 0$ away from $\{\varphi_\lambda = 0\}$ it is clear that this distribution is supported on $\{\varphi_\lambda = 0\}$.

It turns out that it is the multiple $|\nabla \varphi_\lambda| dS$ times surface measure. The calculation just uses Green's formula.

Proof of identity-II

Since $(\Delta + \lambda^2)|\varphi_\lambda| = 0$ except on the zero set, we have, for all $f \in C^2(M)$,

$$\int_M f(\Delta + \lambda^2)|\varphi_\lambda| dV = \int_{|\varphi_\lambda| \leq \delta} f(\Delta + \lambda^2)|\varphi_\lambda| dV.$$

Almost all δ are regular values of φ_λ by Sard's theorem and so we can apply Green's theorem to such values, to obtain

$$\int_{|\varphi_\lambda| \leq \delta} f(\Delta + \lambda^2)|\varphi_\lambda| dV - \int_{|\varphi_\lambda| \leq \delta} |\varphi_\lambda|(\Delta + \lambda^2) f dV$$

$$= \int_{|\varphi_\lambda| = \delta} (f \partial_\nu |\varphi_\lambda| - |\varphi_\lambda| \partial_\nu f) dS.$$

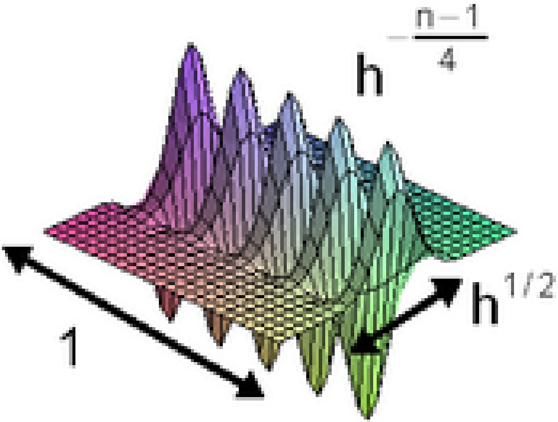
Aside from technicalities (Federer's Gauss-Green formula), the identity follows from the calculation:

$$\nu = \frac{\nabla \varphi_\lambda}{|\nabla \varphi_\lambda|} \text{ on } \{\varphi_\lambda = \delta\}, \quad \nu = -\frac{\nabla \varphi_\lambda}{|\nabla \varphi_\lambda|} \text{ on } \{\varphi_\lambda = -\delta\}. \quad (7)$$

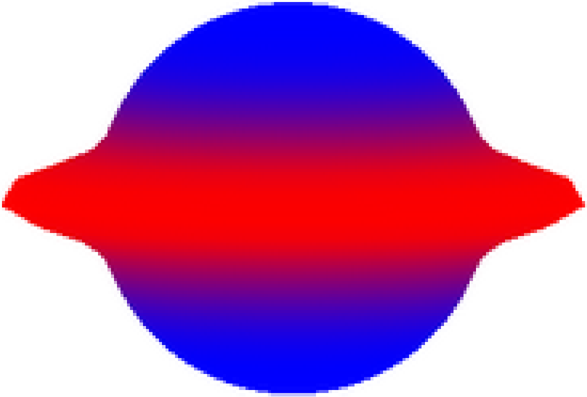
Gaussian beams

Gaussian beams are Gaussian shaped lumps transversal to a closed geodesic which are concentrated on $\lambda^{-\frac{1}{2}}$ tubes $\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)$ around closed geodesics and have height (sup norm) $\lambda^{\frac{n-1}{4}}$. We note that their L^1 norms decrease like $\lambda^{-\frac{(n-1)}{4}}$, i.e. they saturate the L^1 lower bound. In such cases we have
$\int_{\mathcal{N}_{\varphi_\lambda}} |\nabla \varphi_\lambda| dS \simeq \lambda^2 ||\varphi_\lambda||_{L^1} \simeq \lambda^{2-\frac{n-1}{4}}$.
Outside of the tube $\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)$ of radius $\lambda^{-\frac{1}{2}}$ around the geodesic, the Gaussian beam and all of its derivatives decay like $e^{-\lambda d^2}$ where d is the distance to the geodesic. The identity only sees $\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)$.

Gaussian Beam



Gaussian Beam 2



Distribution of nodal hypersurfaces

As we saw, volumes of nodal sets are a hard problem for C^∞ metrics. We now ask a much harder question: How do nodal hypersurfaces wind around on M.?
We put the natural Riemannian hyper-surface measure $d\mathcal{H}^{n-1}$ to consider the nodal set as a *current of integration* $Z_{\varphi_j}]$: for $f \in C(M)$ we put

$$\langle [\mathcal{N}_{\varphi_\lambda}], f \rangle = \int_{\mathcal{N}_{\varphi_\lambda}} f(x) d\mathcal{H}^{n-1}.$$

Problems:

- How does $\langle [\mathcal{N}_{\varphi_\lambda}], f \rangle$ behave as $\lambda_j \to \infty$.
- If $U \subset M$ is a nice open set, find the total hypersurface volume $\mathcal{H}^{n-1}(\mathcal{N}_{\varphi_\lambda} \cap U)$ as $\lambda_j \to \infty$.
- How does it reflect dynamics of the geodesic flow?

Conjecture on real nodal hypersurface: ergodic case

CONJECTURE

Let (M, g) be a real analytic Riemannian manifold with ergodic geodesic flow, and let $\{\varphi_j\}$ be the density one sequence of ergodic eigenfunctions. Then,

$$\frac{1}{\lambda_j}\langle[\mathcal{N}_{\varphi_\lambda}], f\rangle \sim \frac{1}{Vol(M, g)} \int_M f dVol_g. \tag{8}$$

Evidence: it follows from the "random wave model", i.e. the conjecture that eigenfunctions in the ergodic case resemble Gaussian random waves of fixed frequency (Berry; J. Neuheisel, SZ for proofs that (8) holds for Riemannian random waves) .

Quantum ergodicity

- Classical ergodicity: G^t preserves the unit cosphere bundle $S^*_g M$. Ergodic = almost all orbits are uniformly dense.
- On the quantum level, ergodicity of G^t implies that eigenfunctions become uniformly distributed in phase space (Shnirelman; Z, Colin de Verdière, Zworski-Z) . This is a key ingredient in structure of nodal sets. Namely,

$$\int_E \varphi_j^2 dV_g \to \frac{Vol(E)}{Vol(M)}, \quad \forall E \subset M : Vol(\partial E) = 0.$$

- Equidistribution actually holds in phase space $S^* M$.
- Random wave model: when G^t is chaotic, eigenfunctions of Δ_g behave in some ways like random waves.

Real versus complex nodal hypersurfaces

For rest of talk, we concentrate on equidistribution of nodal hypersurfaces.
The only rigorous results on distribution of nodal sets (and level sets) of eigenfunctions concern the complex zeros of analytic continuations:

$$\mathcal{N}_{\varphi_j^{\mathbb{C}}} = \{\zeta \in M_{\mathbb{C}} : \varphi_j^{\mathbb{C}}(\zeta) = 0\},$$

where $\varphi_j^{\mathbb{C}}$ is the analytic continuation of φ_j to the complexification $M_{\mathbb{C}}$ of M. (The complex zero set is simpler than real zero set.)

Background on Grauert tubes

A real analytic manifold M can always be complexified (Bruhat-Whitney). Given a real analytic metric g, one can complexify the exponential map

$$\exp_{\mathbb{C}} : B^*_\epsilon M \to M_{\mathbb{C}}, \;\; (x, \xi) \to \exp_x(\sqrt{-1}\xi).$$

Then:

- $\exists \epsilon_0 : \forall \epsilon < \epsilon_0$, $\exp_x \sqrt{-1}\xi$ is a real analytic diffeo to its image. Maximal $\epsilon_0 =$ maximal geometric Grauert tube radius.
- Complexify $r^2(x, y) \to r^2(\zeta, \bar{\zeta})$. Grauert tube function $=$

$$\sqrt{\rho} := \sqrt{-r^2(\zeta, \bar{\zeta})}.$$

Examples: Torus

- Complexification of $\mathbb{R}^n/\mathcal{N}^n_{\varphi_\lambda}$ is $\mathbb{C}^n/\mathcal{N}^n_{\varphi_\lambda}$.
- Grauert tube function: $r(x,y) = |x-y|$ and $r_{\mathbb{C}}(z,w) = \sqrt{(z-w)^2}$. Then

$$\sqrt{\rho}(z) = \sqrt{-(z-\bar{z})^2} = 2|\Im z| = 2|\xi|.$$

- The complexified exponential map is:

$$\exp_{\mathbb{C}x}(i\xi) = x + i\xi.$$

Equi-distribution of complex nodal sets in the ergodic case

THEOREM

(Z, 2007) Assume (M, g) is real analytic and that the geodesic flow of (M, g) is ergodic. Let $\varphi_{\lambda_j}^{\mathbb{C}}$ be the analytic continuation to phase space of the eigenfunction φ_{λ_j}, and let $Z_{\varphi_{\lambda_j}^{\mathbb{C}}}$ be its complex zero set in phase space B^*M. Then for all but a sparse subsequence of λ_j,

$$\frac{1}{\lambda_j} \int_{Z_{\varphi_{\lambda_j}^{\mathbb{C}}}} f \omega_g^{n-1} \to \frac{i}{\pi} \int_{M_\tau} f \bar{\partial} \partial \sqrt{\rho} \wedge \omega_g^{n-1}$$

As usual in quantum ergodicity, we may have to delete a sparse subsequence of exceptional eigenvalues.

Limit distribution of zeros is singular along zero section

- The Kaehler structure on the cotangent bundle is $\overline{\partial}\partial\rho$. But the limit current is $\overline{\partial}\partial\sqrt{\rho}$. The latter is singular along $M = \{\xi = 0\}$ and the associated volume form is not the symplectic one.
- The reason for the singularity is that the zero set is invariant under the involution $\sigma : T^*M \to T^*M$, $(x, \xi) \to (x, -\xi)$, since the eigenfunction is real valued on M. The fixed point set of σ is M and is also where zeros concentrate.

Example: the unit circle S^1

- The (real) eigenfunctions are $\cos k\theta, \sin k\theta$ on a circle.
- The complexification is the cylinder $S^1_{\mathbb{C}} = S^1 \times \mathbb{R}$.
- The complexified configuration space is similar to the phase space T^*S^1. This is always true.
- The holomorphically extended eigenfunctions are $\cos kz, \sin kz$.

Simplest case: S^1

The zeros of $\sin 2\pi kz$ in the cylinder $\mathbb{C}/\mathcal{N}_{\varphi_\lambda}$ all lie on the real axis at the points $z = \frac{n}{2k}$. Thus, there are $2k$ real zeros. The limit zero distribution is:

$$\delta_{S^1} \quad = \quad \tfrac{1}{\pi}\delta_0(\xi)dx \wedge d\xi.$$

On the other hand,

$$\begin{aligned} \tfrac{i}{\pi}\partial\bar{\partial}|\xi| \quad &= \quad \tfrac{i}{\pi}\tfrac{d^2}{4d\xi^2}|\xi| \;\; \tfrac{2}{i}dx \wedge d\xi \\ &= \quad \tfrac{i}{\pi}\;\tfrac{1}{2}\;\delta_0(\xi)\;\; \tfrac{2}{i}dx \wedge d\xi. \end{aligned}$$

Intersections of nodal lines and geodesics

To get closer to real zeros, we "magnify" the singularity along the real domain by intersecting nodal lines and geodesics.

- Let $\gamma \subset M^2$ be geodesic arc on a real analytic Riemannian surface. We identify it with a a real analytic arc-length parameterization $\gamma : \mathbb{R} \to M$. For small ϵ, $\exists$ analytic continuation

$$\gamma_{\mathbb{C}} : S_\epsilon := \{t + i\tau \in \mathbb{C} : |\tau| \leq \epsilon\} \to M_\tau.$$

- Consider the restricted (pulled back) eigenfunctions

$$\gamma_{\mathbb{C}}^* \varphi_{\lambda_j}^{\mathbb{C}} \text{ on } S_\epsilon.$$

Intersections of nodal lines and geodesics

Let

$$\mathcal{N}_{\lambda_j}^{\gamma} := \{(t + i\tau : \gamma_{\mathbb{C}}^{*}\varphi_{\lambda_j}^{\mathbb{C}}(t + i\tau) = 0\} \tag{9}$$

be the complex zero set of this holomorphic function of one complex variable. Its zeros are the intersection points.
Then as a current of integration,

$$[\mathcal{N}_{\lambda_j}^{\gamma}] = i\partial\bar{\partial}_{t+i\tau} \log \left|\gamma_{\mathbb{C}}^{*}\varphi_{\lambda_j}^{\mathbb{C}}(t + i\tau)\right|^2 . \tag{10}$$

Restrictions of eigenfunctions to geodesics

In the real domain, the restriction

$$\varphi_\lambda(\gamma_{x,\xi}(t)) : t \in \mathbb{R}$$

is a bounded real analytic function on $\mathbb{R}$. It is almost always very "chaotic"– we are sampling a chaotic function using a chaotic trajectory. Wiener developed Generalized Harmonic Analysis based on such functions.

But in special cases, the restriction is not chaotic– it might equal zero. E.g. if γ is the axis of symmetry of an involution and φ_λ is an odd eigenfunction (e.g. the center line of a stadium).

We need conditions to ensure that the restriction is chaotic.

Equidistribution of intersections

Theorem

(Z, 2012) Let (M, g) be real analytic with ergodic geodesic flow. Suppose that γ is either (i) a random geodesic; or (ii) a periodic geodesic satisfying a certain asymmetry condition.
Then there exists a subsequence of eigenvalues λ_{j_k} of density one such that

$$\frac{i}{\pi \lambda_{j_k}} \partial\bar{\partial}_{t+i\tau} \log \left| \gamma_{\mathbb{C}}^* \varphi_{\lambda_{j_k}}^{\mathbb{C}}(t+i\tau) \right|^2 \to \delta_{\tau=0} ds.$$

The convergence is weak convergence on $C_c(S_\epsilon)$.*

Thus, intersections of (complexified) nodal sets and geodesics concentrate in the real domain– and are distributed by arc-length measure on the real geodesic. Almost the physics conjecture.

Image of intersections of nodal lines and geodesics (typist's rendition)

Complex zeros of $\gamma_{\mathbb{C}}^{*}\varphi_{\lambda_j}^{\mathbb{C}}(t+i\tau)$ are labelled z.

——————————- $\Im z = \tau$

z z z

-z - z-z— z z —z— z – –z—-z—z—z—-z—z—z—z—$(\Im z = 0\ )$

z z z

—————————— $\Im z = -\tau$

New ingredient: quantum ergodic restriction theorem for restrictions of eigenfunctions to hypersurfaces

THEOREM

(J. Toth and S. Z 2010-2011) If G^t is ergodic and a hypersurface H is "asymmetric" then the restrictions of $\{\varphi_j\}$ to H are quantum ergodic on H in the sense that

$$\lim_{\lambda_j \to \infty; j \in S} \langle Op_{\lambda_j}(a_0) \varphi_{\lambda_j}|_H, \varphi_{\lambda_j}|_H \rangle_{L^2(H)}$$

$$= c_n \int_{B^* H} a(s, \tau)\, \rho^H_{\partial\Omega}(s, \tau)\, ds d\tau$$

for a certain measure $\rho^H_{\partial\Omega}(s, \tau)\, ds d\tau$.

Equidistribution = growth saturation

It is immediate from the Poincare-Lelong formula

$$[\mathcal{N}_{\lambda_j}^{x,\xi}] = i\partial\bar{\partial}_{t+i\tau} \log \left| \gamma_{x,\xi}^{*} \varphi_{\lambda_j}^{\mathbb{C}}(t+i\tau) \right|^2 \tag{11}$$

that the equidistribution result follows if the restricted eigenfunctions have maximal growth:

Proposition

(Growth saturation) If $\gamma_{x,\xi}$ is (i) a periodic geodesic which satisfies the QER asymmetry assumption, or else (ii) if it is a random geodesic then there exists a subsequence $\mathcal{S}_{x,\xi}$ of density one so that, for all $\tau < \epsilon$,

$$\lim_{k\to\infty} \frac{1}{\lambda_{j_k}} \log \left| \gamma_{x,\xi}^{\tau *} \varphi_{\lambda_{j_k}}^{\mathbb{C}}(t+i\tau) \right|^2 = |\tau| \quad \text{in } \; L^1_{loc}(S_\tau).$$

The subsequence $\mathcal{S}_{x,\xi}$ is the ergodic sequence along $\gamma_{x,\xi}$ given by Theorem **??**.

Ideas of proof for periodic geodesics

Study the Fourier series

$$\varphi_{\lambda_j}(\gamma_{x,\xi}(t)) = \sum_{n \in \mathcal{N}_{\varphi_\lambda}} \nu_{\lambda_j}^{x,\xi}(n) e^{\frac{2\pi i n t}{L_\gamma}}. \tag{12}$$

Its analytic continuation is given by

$$\varphi_{\lambda_j}^{\mathbb{C}}(\gamma_{x,\xi}(t+i\tau)) = \sum_{n \in \mathcal{N}_{\varphi_\lambda}} \nu_{\lambda_j}^{x,\xi}(n) e^{\frac{2\pi i n(t+i\tau)}{L_\gamma}}. \tag{13}$$

The growth rate of $\varphi_{\lambda_j}^{\mathbb{C}}(\gamma_{x,\xi}(t+i\tau))$ is thus intimately related to the joint asymptotics of the Fourier coefficients $\nu_{\lambda_j}^{x,\xi}(n)$ in (λ_j, n).

QER and Fourier coefficients

We use the QER hypothesis in the following way:

Lemma

Suppose that $\{\varphi_{\lambda_j}\}$ *is QER along the periodic geodesic* $\gamma_{x,\xi}$*. Then for all* $\epsilon > 0$*, there exists* $C_\epsilon > 0$ *so that*

$$\sum_{n:|n|\geq(1-\epsilon)\lambda_j} |\nu_{\lambda_j}^{x,\xi}(n)|^2 \geq C_\epsilon.$$

The lemma implies that for any $\epsilon > 0$,

$$\sum_{n:|n|\geq(1-\epsilon)\lambda_j} |\nu_{\lambda_j}^{x,\xi}(n)|^2 e^{-2n\tau} \geq C_\epsilon e^{2\tau(1-\epsilon)\lambda_j}.$$

In essence, we prove "growth saturation" in the ergodic case by showing that all of the Fourier coefficients in the allowed energy region $|n| \leq \lambda_j$ are of uniformly large size. Since the top frequency term dominates and its Fourier coefficient is large, $\gamma_{x,\xi}^* \varphi_j^{\mathbb{C}}$ must have maximal growth.