Generic Elementary Embeddings

Fields Institute
November 9, 2012

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Conventional Large Cardinals

Conventional Large Cardinals typically state the existence of an elementary embedding:

$$j: V \to M$$

where M is a transitive class

Conventional Large Cardinals

The axioms can be characterized by two parameters:

- The closure properties of M
- Where j sends the ordinals

Conventional Large Cardinals

Large cardinals are usually expressed in terms of ultrafilters (and systems of ultrafilters).

- Because the formulation is in ZFC
- Because of combinatorial advantages

Ultrafilters

If U is an ultrafilter on X and $f : X \to Z$ is an injection, then we can copy the ultrafilter U onto an ultrafilter V on Z by setting

$$A \in V \text{ if and only if } f^{-1}[A] \in U.$$

Doing this we get a canonical isomorphism between any ultrapower B^X/U and B^Z/V.

Ultrafilters

So we can assume that all of our ultrafilters are on sets of the form $P(Y)$ for some set Y.

(i.e. $U \subseteq PP(Y)$)

If $B \subseteq P(Y)$ and B is in U we will say that

U **concentrates** on B.

Let U be a filter on P(Y)

- We say that A is positive if and only if P(Y)\A not in U.

- A function f with domain A ⊆ P(Y) is regressive if and only if for all a ∈ A, f(a) ∈ a.

- U is normal iff whenever A is positive and f is regressive there is a positive B ⊆ A with f constant on B.

Let U be a filter on P(Y)

- U is countably complete iff U is closed under countable intersections.

- U is fine iff for all $y \in Y$,

  $\{A \subseteq Y: y \in A\} \in U$

- Non-principal for all $y \in Y$ there is a $B \in U$, y is not in B

Types of ultrafilters

Let $\kappa < \lambda$ be cardinals and U a normal countably complete ultrafilter on PP (λ).

measurable cardinal	U concentrates on λ ($\lambda \subset P(\lambda)$)
supercompact	U concentrates on $[\lambda]^{<\kappa}$
huge	U concentrates on $[\lambda]^{\kappa}$

How do we get the embeddings?

We take the ultrapower of V by the corresponding ultrafilter.

The main point is that countable completeness implies that the ultrapower is well-founded.

The problem with large cardinals is that they are "too big" to settle questions about uncountable sets that inherently involve the Axiom of Choice (e.g. the CH).

To settle these questions we need to search for other types of assumptions.

- Strength comes from embeddings
- Embeddings come from ultrapowers
- If the ultrafilter is not countably complete the ultrapower is not well-founded
- If the ultrafilter IS countably complete the critical point is LARGE.

How can we escape this?

Looks pretty bad ...

Idea

Start with a filter U and

generically extend it to an ultrafilter G.

Now take the ultrapower by G

(using functions from V).

What happens?

If $U \subseteq P\,P(\lambda)$ is a normal, fine and countably complete and $G \subseteq P\,P(\lambda)/U$ is generic

then:

- $j``\lambda$ belongs to the ultrapower
- The ultrapower is well-founded up to λ^+
- $P(\lambda)$ is a subset of the ultrapower

Background facts

- U is precipitous if the ultrapower is well-founded

- If the Boolean Algebra $P\,P(\lambda)/U$ is λ^+-c.c. then the ultrapower is closed under λ-sequences.

Type	Ordinary	Generic	Saturated
Supercompact: $[\lambda]^{<\kappa}$	$M^{\lambda} \subseteq M$ κ very large	$P(\lambda) \subseteq M$ κ, λ can be small e.g. ω_1, ω_2	$M^{\lambda} \subseteq M$ κ, λ can be small e.g. ω_1, ω_2
Huge: $[\lambda]^{\kappa}$	$M^{\lambda} \subseteq M$ κ very large	$P(\lambda) \subseteq M$ κ, λ can be small e.g. ω_1, ω_2	$M^{\lambda} \subseteq M$ κ, λ can be small e.g. ω_1, ω_2

For the "huge"-type filters, we also have $j(\kappa) = \lambda$.

Does this help?

Does this help?

Yes: it is consistent to have a saturated generic huge embedding with

- $\kappa = \omega_1$
- $\lambda = \omega_2$

Does this help?

If $\mathbb{P}([\omega_2]^{\omega_1})/U$ has a dense subset of size ω_1, then the CH holds.

(Can settle essentially ALL classical questions in set theory... The problem is OVERDETERMINATION.)

Today's talk

Sometimes natural combinatorial properties give rise to filters. Can the generic elementary embeddings from these filters give a way to get large cardinal strength from these combinatorial properties?

Terminology Shift

Convenient to use

IDEALS

rather than

FILTERS.

Exactly dual notions--just language.

There are lots of natural ideals

The (generalized) non-stationary ideal on $P(\lambda)$ is always available.

Moreover lots of combinatorial properties can be restated as saying that various sets are stationary

Burke (1997): Suppose that I is a normal fine ideal on $P(\mu)$ and $\lambda \gg \mu$.

Then there is a stationary set $Z \subseteq P(\lambda)$ such that I is the projection of NS restricted to Z to $P(\mu)$.

If there is a κ that is κ^+ supercompact then

there is a stationary set $A \subseteq P(2^{\kappa^+})$ such that:

$L[NS|A, A] \models \kappa$ is κ^+ supercompact.

Negative Evidence

If there is an ω_2-saturated ideal on ω_1 then the closure of the generic ultraproduct is similar to the closure from an almost-huge cardinal.

But the statement

"NS_{ω_1} is ω_2-saturated"

is much weaker than an almost huge cardinal.

Why?

The embeddings associated with ideals that come from the collapses of large cardinals agree with the large cardinal embeddings on the original ground model.

Is this enough to see that they come from large cardinals? Can we figure out how much agreement is needed?

The set up:

- $Z \subset P(X)$
- J an ideal on Z.
- $X' \subset X$ and I is the projection of J to an ideal on $P(X')$ via the map

$$\pi(z) = z \cap X'$$

- A is in the dual of I and has a canonical well-ordering.

The hypothesis

There are A', O' and I' such that for all generic $G \subset P(Z)/J$:

- An initial segment of the ordinals of V^Z/G are well founded and isomorphic to $(|A'|^+)^V$ and

2. if $j : V \to M$ is the canonical elementary embedding then

$j(A)=A'$, $j``|A|=O'$, $I' = j(I) \cap P(A')^V$.

In this situation we say that

J decides I

The theorem

Suppose that J decides I. Then either:

- $L[A, I] \models \check{I}$ is an ultrafilter on A or
- for some generic $G \subset P(Z)/J$ if

$$j: V \to V^Z/G$$

is the ultrapower embedding,

then $L[j(A), I'] \models (\check{I'})$ is an ultrafilter on $j(A)$

Corollaries

The following are equiconsistent:

- For $1 < n < m \in \omega$ there is normal, fine, decisive ideal on $[\omega_m]^{<\omega_n}$

- There is a $\kappa^{+(m-n)}$ supercompact cardinal κ.

Corollaries

The following are equiconsistent:

- For $1 < n < m \in \omega$, there is normal, fine, decisive ideal on $[\omega_m]^{\omega_n}$.

- There is a huge cardinal.

Takeaways

- There is a class of ideals on the ω_n's with consistency strength as high as huge cardinals

- If there is an inner model for a supercompact or huge cardinal then there is one of the form

$$L[A, NS|A]$$

Go back in time ..

1970's style model theory was partly concerned with generalizations of the Lowenheim-Skolem theorem that involve second order properties. One successful version are variants of Chang's Conjecture.

Go back in time ..

Let L be a countable language with a distinguished unary predicate R. An L-structure $\mathfrak{A}$ has type (κ, λ) if and only if $|A| = \kappa$ and $|R^A| = \lambda$.

Go back in time ..

We say

$$(\kappa,\lambda) \to (\kappa',\lambda')$$

if and only if for all $\mathfrak{A}$ of type (κ,λ) there is an elementary substructure

$\mathfrak{B} \prec \mathfrak{A}$ of type (κ', λ').

If the definition were made today it would look more like this:

In a simple special case

$$(\omega_{n+k}, \omega_{n+1}) \twoheadrightarrow (\omega_{n+k-1}, \omega_n)$$

if and only if there is a $\theta \gg \omega_{n+2}$ and an $N < \langle H(\theta), \in \Delta\rangle$

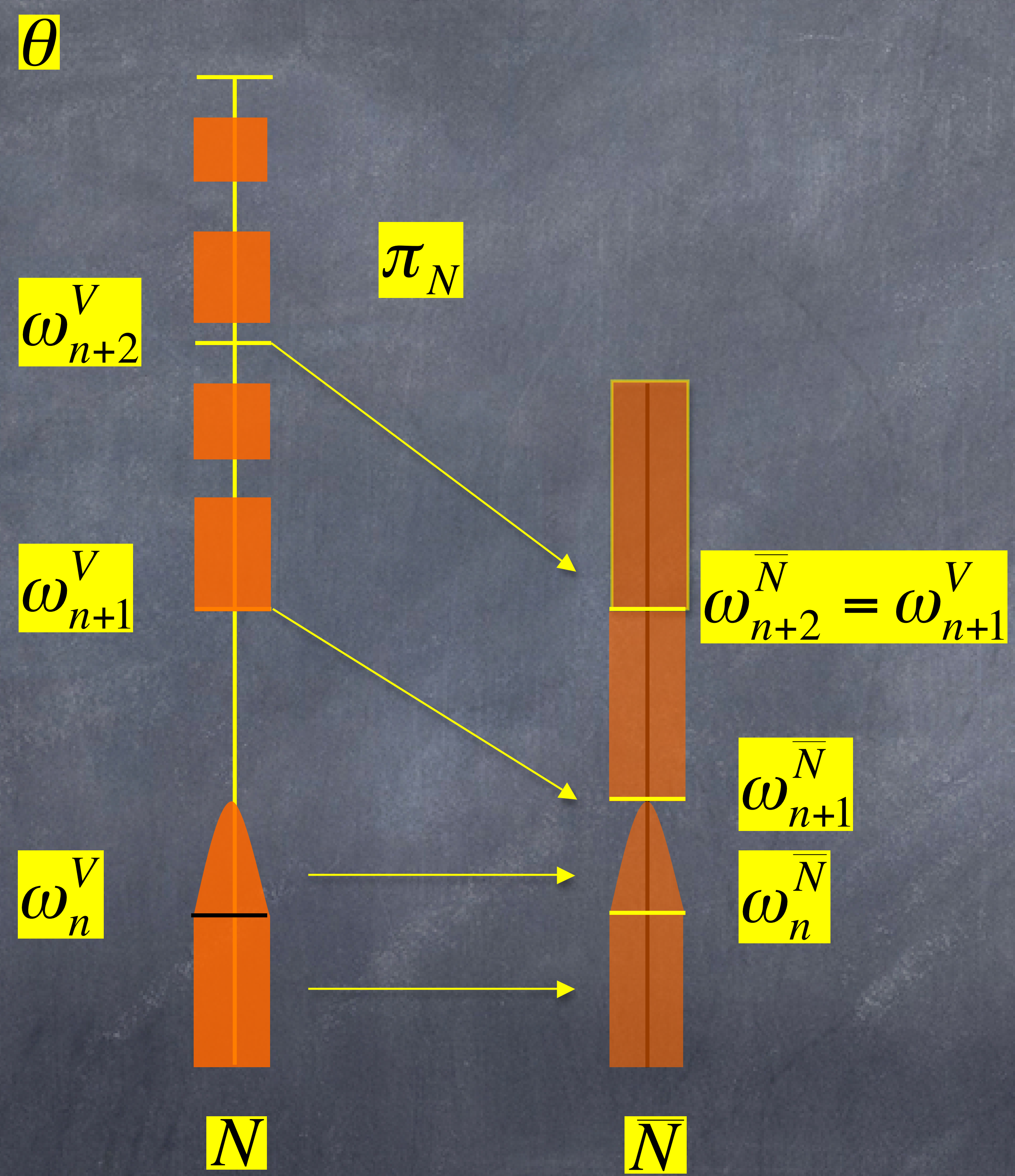
such that

if $\pi : N \rightarrow \bar{N}$ is the transitive collapse then

- $\pi|\omega_n = id$

and

- $\pi(\omega_{n+2}) = \omega_{n+1}$.


θ
π_N
ω_{n+2}^V
ω_{n+1}^V
ω_n^V
$\omega_{n+2}^{\overline{N}} = \omega_{n+1}^V$
$\omega_{n+1}^{\overline{N}}$
$\omega_n^{\overline{N}}$
N
$\overline{N}$

Correctness

Let $N < \langle H(\theta), \in \Delta\rangle$. Then N is correct for A iff $A \in N$ and there are $A', I' \in N$ such that

$$\pi(I' | A') = (NS|A) \cap \bar{N},$$

where π is the transitive collapse of N to $\bar{N}$.

Strong Chang Reflection

Strong Chang reflection holds for (ω_{n+3}, ω_n) if and only if for all large enough θ there is an canonically well ordered $A \in [\omega_{n+2}]^{\omega_{n+1}}$ such that for some

$$N < \langle H(\theta), \in, \Delta, A \rangle$$

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we have:

1. $N \cap \omega_{n+2} \in A$ and $|N \cap \omega_{n+3}| = \omega_{n+2}$,

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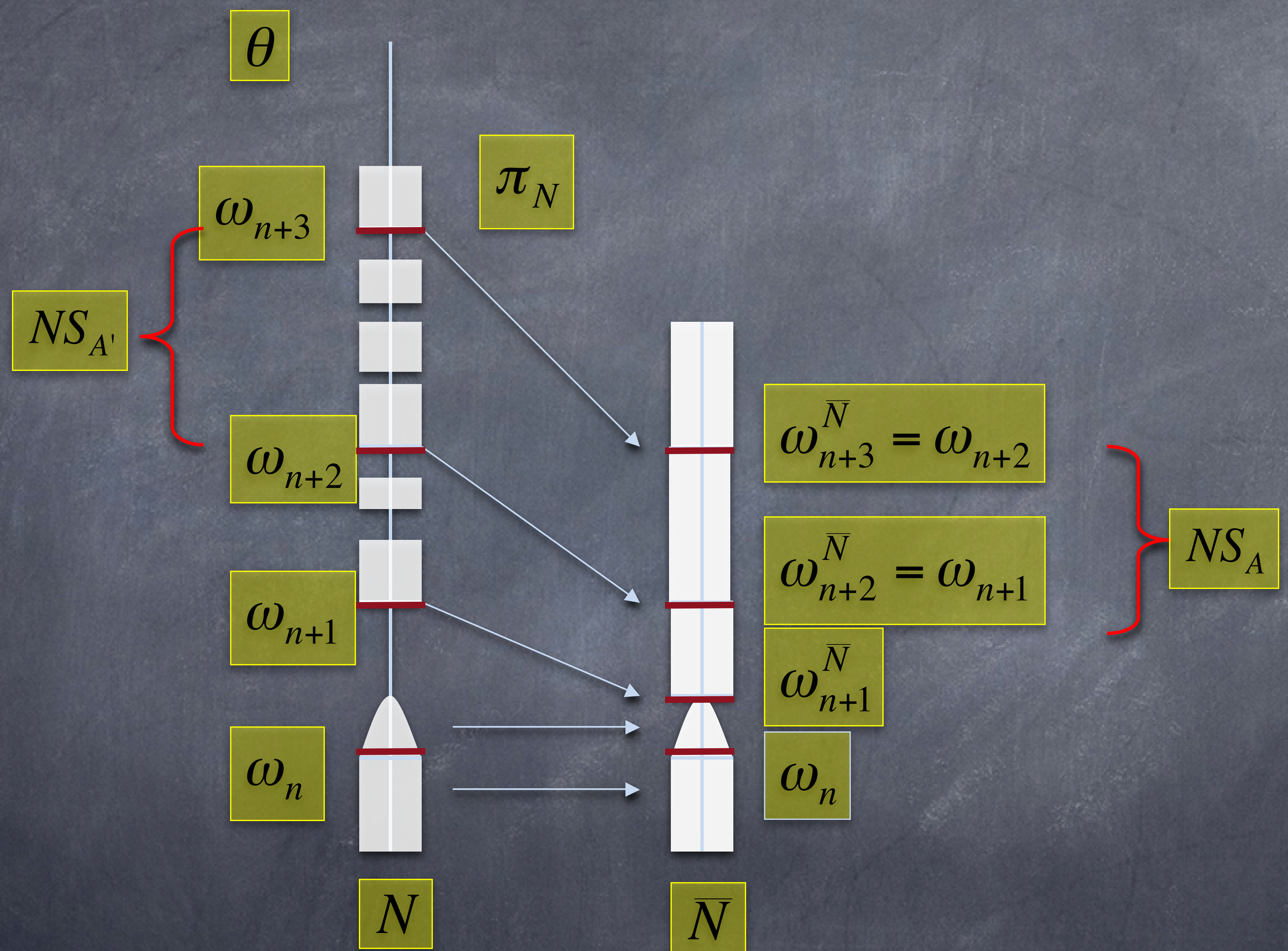
we have:

1. $N \cap \omega_{n+2} \in A$ and $|N \cap \omega_{n+3}| = \omega_{n+2}$,
2. $N \cap \omega_{n+2} \in \bar{N}$,
3. N is correct for NS|A.



θ
π_N
ω_{n+3}
$NS_{A'}$
ω_{n+2}
ω_{n+1}
ω_n
N
$\omega^{\bar{N}}_{n+3} = \omega_{n+2}$
$\omega^{\bar{N}}_{n+2} = \omega_{n+1}$
$\omega^{\bar{N}}_{n+1}$
ω_n
NS_A
$\bar{N}$

So you made a definition …

Now what?

Recall:

The decisive ideals were those that were constructed by collapsing large cardinals and taking the remnants of the large cardinal filters

No surprise:

Strong Chang reflection is what happens in the models where Chang's Conjecture is witnessed by the remains of a large cardinal embedding.

Stated as a Theorem

Suppose there is a 2-huge cardinal. Then for each n there is a forcing extension in which SCR holds for

$$(\omega_{n+3}, \omega_n).$$

Point of the talk

Suppose Strong Chang Reflection holds for (ω_{n+3}, ω_n). Then

there is a transitive inner model for "ZFC + there is a huge cardinal."

Upshot:

Some of the natural properties of ideals arising combinatorially when we collapse some large cardinals to be the ω_n's imply inner models of large cardinals that are essentially as strong as those we started with.

And what about MM? PFA?

Thank you!