Consequences of Martin's Maximum and weak square

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1. Introduction

1.1 weak square

<u>Def.</u> (Schimmerling)
For an uncountable cardinal λ and a cardinal $\mu \leq \lambda$,

$\square_{\lambda,\mu} \equiv$ There exists $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+ \rangle$ s.t.

(i) $\mathcal{C}_\alpha$ is a family of club subsets of α of o.t. $\leq \lambda$,

(ii) $1 \leq |\mathcal{C}_\alpha| \leq \mu$,

(iii) $c \in \mathcal{C}_\alpha$ & $\beta \in \mathrm{Lim}(c) \;\Rightarrow\; c \cap \beta \in \mathcal{C}_\beta$.

$\square_{\lambda,<\mu} \equiv$ the statement obtained by replacing (ii) with

(iv) $1 \leq |\mathcal{C}_\alpha| < \mu$.

- $\square_{\lambda,1} \Leftrightarrow \square_\lambda$.
- $\square_{\lambda,\lambda} \Leftrightarrow \square^*_\lambda \Leftrightarrow$ "There is a λ^+-special Aronszajn tree."
- $\lambda^{<\lambda} = \lambda \Rightarrow \square_{\lambda,\lambda}$.

1.2 Martin's Maximum and weak square

Thm. (Cummings-Magidor)
Assume MM. Then we have the following:

(1) $\square_{\omega_1,\omega_1}$ fails.

(2) If $\mathrm{cof}(\lambda) = \omega$, then $\square_{\lambda,\lambda}$ fails.

(3) If $\mathrm{cof}(\lambda) = \omega_1 < \lambda$, then $\square_{\lambda,<\lambda}$ fails.

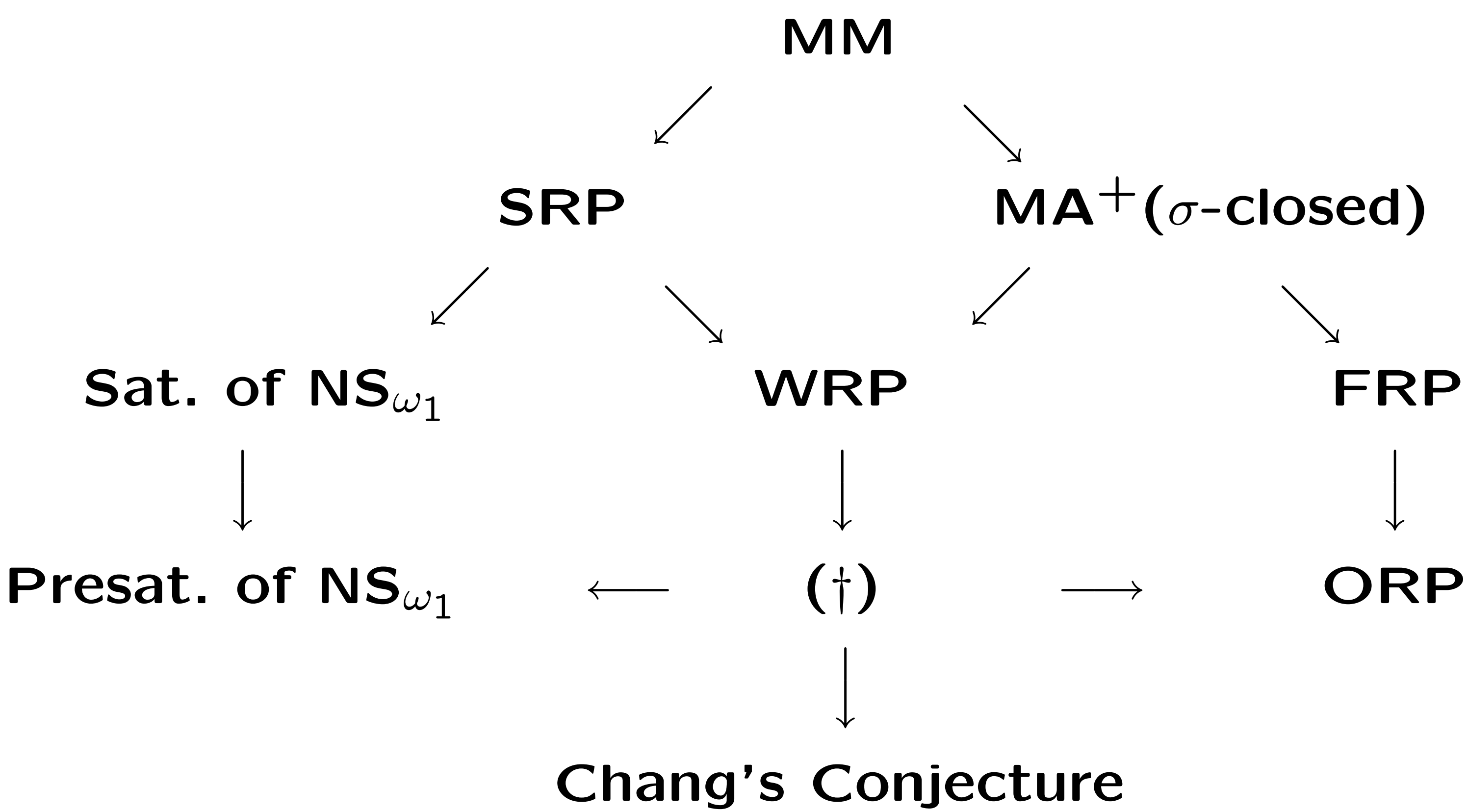
(4) If $\mathrm{cof}(\lambda) > \omega_1$, then $\square_{\lambda,<\mathrm{cof}(\lambda)}$ fails.

Thm. (Cummings-Magidor)
"MM + (1) + (2)" is consistent:

(1) $\square_{\lambda,\lambda}$ holds for all λ with $\mathrm{cof}(\lambda) = \omega_1 < \lambda$.

(2) $\square_{\lambda,\mathrm{cof}(\lambda)}$ holds for all λ with $\mathrm{cof}(\lambda) > \omega_1$.

1.3 consequences of MM



- WRP (Weak Reflection Principle)
  $\equiv$ For any $\lambda \geq \omega_2$ and any stationary $X \subseteq [\lambda]^\omega$
  there is $R \subseteq \lambda$ s.t. $|R| = \omega_1 \subseteq R$ and $X \cap [R]^\omega$ is stationary.

- $(\dagger) \equiv$ Every ω_1-stationary preserving poset is semi-proper.

- Chang's Conjecture
  $\equiv$ For any structure $\mathcal{M} = \langle \omega_2; \ldots \rangle$
  there is $M \prec \mathcal{M}$ s.t. $|M| = \omega_1$ and $|M \cap \omega_1| = \omega$.

- ORP (Ordinal Reflection Principle)
  $\equiv$ For any regular $\lambda \geq \omega_2$ and any stationary $S \subseteq E^\lambda_\omega$
  there is $\alpha \in E^\lambda_{\omega_1}$ s.t. $S \cap \alpha$ is stationary.

$(E^\lambda_\mu = \{\alpha < \lambda \mid \mathrm{cof}(\alpha) = \mu\})$

- FRP (Fodor-type Reflection Principle)

  $\equiv$ For any regular $\lambda \geq \omega_2$, any stationary $S \subseteq E^\lambda_\omega$ and any function f on S with $f(\alpha) \in [\alpha]^\omega$ there is $\alpha \in E^\lambda_{\omega_1}$ s.t.

  $$\{x \in [\alpha]^\omega \mid \sup x \in S \ \wedge \ f(\sup x) \subseteq x\}$$

  is stationary in $[\alpha]^\omega$.

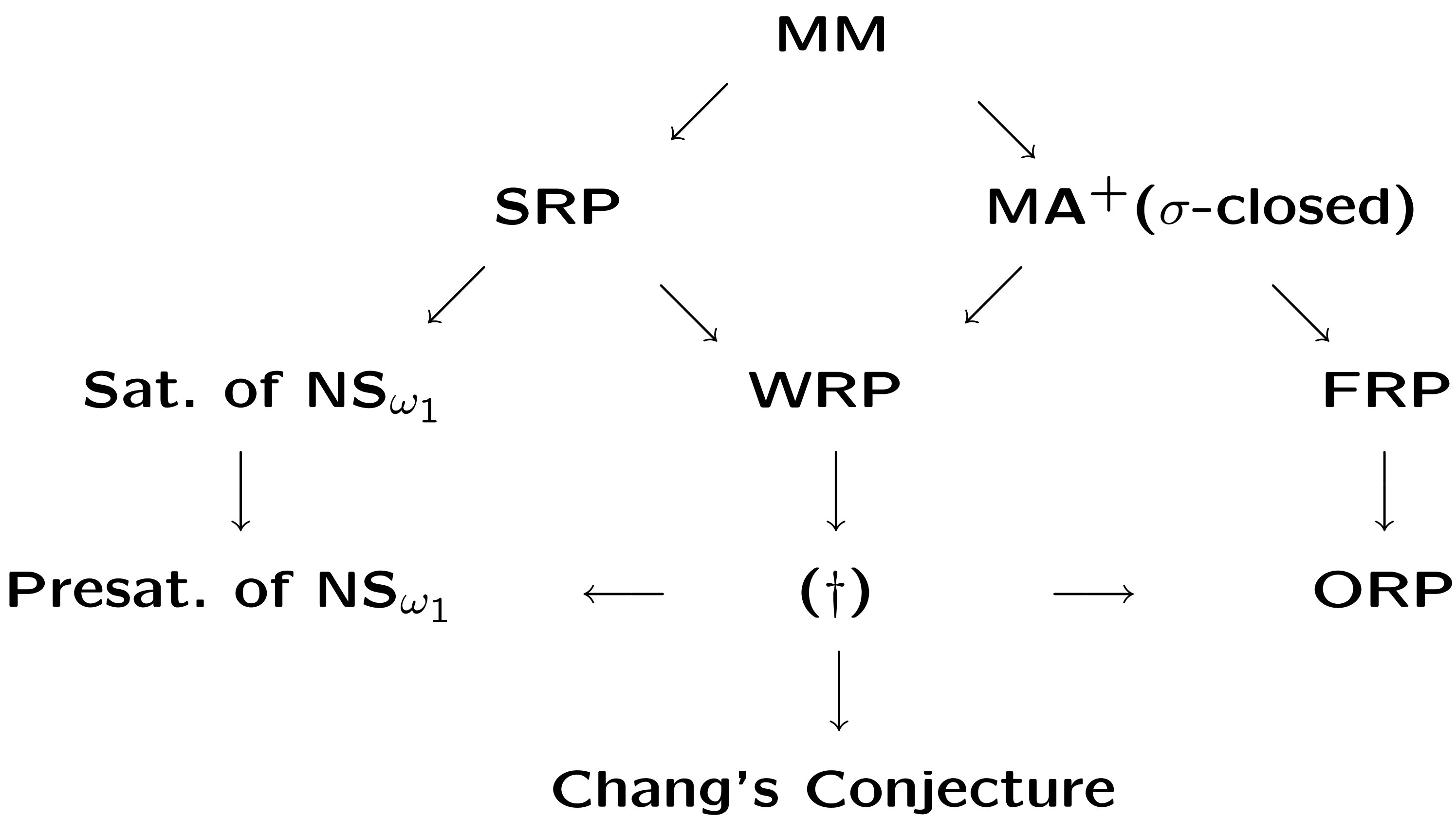
Fact (Fuchino-Juhász-Soukup-S.-Szentmiklóssy-Usuba)
The following are equivalent:

(1) FRP

(2) For every locally countably compact space X, if all subspaces of X of size $\leq \aleph_1$ are metrizable, then X is metrizable.

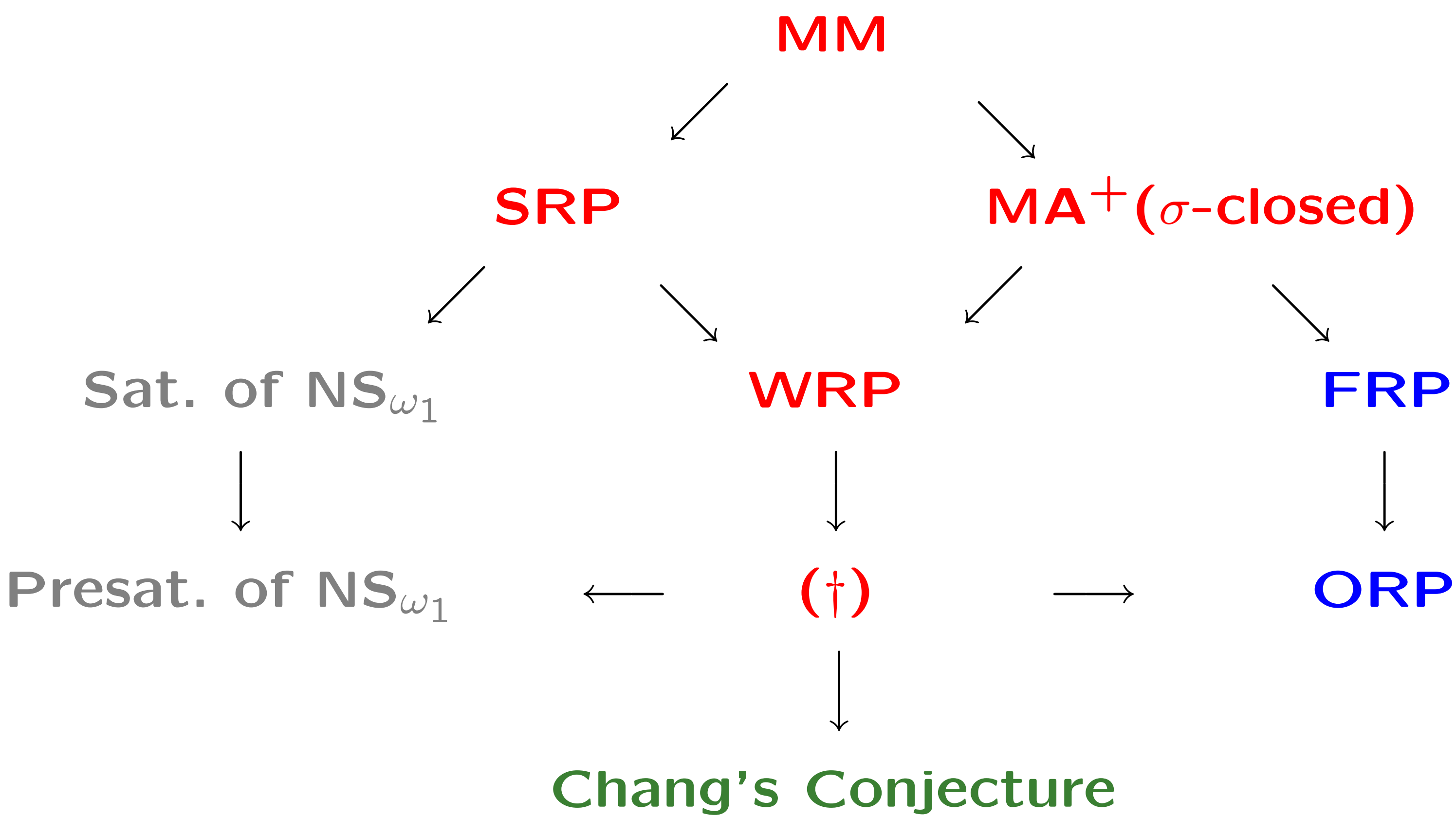
(3) For every graph G, if all subgraphs of G of size $\leq \aleph_1$ have ctble. coloring number, then coloring number of G is ctble.

We discuss how weak square principles are denied
by these consequences of MM.



- Saturation of NS_{ω_1} is consistent with $\square_\lambda$ for any unctble. λ.

- We discuss with partitioning the other principles into 3 groups.



- Saturation of NS_{ω_1} is consistent with $\square_\lambda$ for any unctble. λ.

- We discuss with partitioning the other principles into 3 groups.

2. (†) and stronger principles

Thm. (S., Todorčević-Torres)
Assume (†). Then we have the following:

(1) $\square_{\omega_1,\omega}$ fails. If CH fails in addition, then $\square_{\omega_1,\omega_1}$ fails.

(2) If $\mathrm{cof}(\lambda) = \omega$, then $\square_{\lambda,\lambda}$ fails.

(3) If $\mathrm{cof}(\lambda) = \omega_1 < \lambda$, then $\square_{\lambda,<\lambda}$ fails.

(4) If $\mathrm{cof}(\lambda) > \omega_1$, then $\square_{\lambda,<\mathrm{cof}(\lambda)}$ fails.

- The same as MM except for (1). (So (2)–(4) are optimal.)

- (1) is optimal because CH implies $\square_{\omega_1,\omega_1}$,
  and (†) is consistent with CH.

- WRP and MA^+(σ-closed) deny the same weak square as (†).

- SRP denies the same weak square as MM
  because SRP implies $\neg$CH.

Outline of (1): $(\dagger) \Rightarrow \neg\Box_{\omega_1,<2^\omega}$

- WRP(ω_2) $\equiv$ For any stationary $X \subseteq [\omega_2]^\omega$ there is $\alpha \in \omega_2 \setminus \omega_1$ s.t. $X \cap [\alpha]^\omega$ is stationary.

Fact (Todorčević) $(\dagger)$ implies WRP(ω_2).

We prove that $\Box_{\omega_1,<2^\omega}$ denies WRP(ω_2).
Let $\langle \mathcal{C}_\alpha \mid \alpha < \omega_2 \rangle$ be $\Box_{\omega_1,<2^\omega}$-sequence, and let

$$X := \{x \in [\omega_2]^\omega \mid \forall c \in \mathcal{C}_{\sup x},\ c \not\subseteq x\}.$$

We show that X is non-reflecting stationary subset of $[\omega_2]^\omega$.

$X = \{x \in [\omega_2]^\omega \mid \forall c \in \mathcal{C}_{\sup x},\ c \not\subseteq x\}$ is non-reflecting.

Suppose $\alpha \in \omega_2 \setminus \omega_1$.

If $\mathrm{cof}(\alpha) = \omega$, or α is successor, then pick any $c \in \mathcal{C}_\alpha$, and let

$$Y \ := \ \{x \in [\alpha]^\omega \mid \sup x = \alpha \ \wedge \ c \subseteq x\}.$$

Then Y is club in $[\alpha]^\omega$, and $Y \cap X = \emptyset$.

If $\mathrm{cof}(\alpha) = \omega_1$, then choose $c \in \mathcal{C}_\alpha$, and let

$$Z \ := \ \{x \in [\alpha]^\omega \mid \sup x \in \mathrm{Lim}(c) \ \wedge \ c \cap \sup x \subseteq x\}.$$

Then Z is club in $[\alpha]^\omega$.
Moreover $Z \cap X = \emptyset$ because $c \cap \sup x \in \mathcal{C}_{\sup x}$ if $\sup x \in \mathrm{Lim}(c)$.

$X = \{x \in [\omega_2]^\omega \mid \forall c \in \mathcal{C}_{\sup x},\ c \not\subseteq x\}$ is stationary.

Fix a club gessing sequence $\langle d_\alpha \mid \alpha \in E^{\omega_2}_\omega \rangle$.
For each $x \in [\omega_2]^\omega$ with $\sup x$ limit let

$$\mathrm{pat}(x) \ := \ \{n \in \omega \mid x \cap [\delta_n, \delta_{n+1}) \neq \emptyset\} \ \in \ [\omega]^\omega,$$

where $\langle \delta_n \mid n < \omega \rangle$ is the increasing enumeration of $d_{\sup x}$.

Fact (Foreman-Todorčević)
For any seq. $\vec{p} = \langle p_\alpha \mid \alpha \in E^{\omega_2}_\omega \rangle$ of elements of $[\omega]^\omega$, the set

$$X_{\vec{p}} \ := \ \{x \in [\omega_2]^\omega \mid \mathrm{pat}(x) = p_{\sup x}\}$$

is stationary in $[\omega_2]^\omega$.

For each $\alpha \in E^{\omega_2}_\omega$, take $p_\alpha \in [\omega]^\omega$ so that

$$\mathrm{pat}(c) \not\subseteq p_\alpha \text{ for any } c \in \mathcal{C}_\alpha.$$

(We can take p_α because $|\mathcal{C}_\alpha| < 2^\omega$.) Then $X_{\vec{p}} \subseteq X$. $\square$

- Relevant fact and question:

We proved that $\mathrm{WRP}(\omega_2) + \neg\mathrm{CH}$ implies $\neg\Box_{\omega_1,\omega_1}$.
Here recall that

$$\neg\Box_{\omega_1,\omega_1} \Leftrightarrow \text{there is no } \omega_2\text{-special Aronszajn tree.}$$

Fact (Veličković)
$\mathrm{WRP}(\omega_2) + \mathrm{MA}_{\aleph_1}$ implies there are no ω_2-Aronszajn tree.

Question
$\mathrm{WRP}(\omega_2) + \neg\mathrm{CH}$ implies there are no ω_2-Aronszajn tree ?

3. ORP and FRP

3.1 ORP

ORP $\equiv$ For any regular $\lambda \geq \omega_2$ and any stationary $S \subseteq E^{\lambda}_{\omega}$ there is $\alpha \in E^{\lambda}_{\omega_1}$ s.t. $S \cap \alpha$ is stationary.

<u>**Thm.**</u> (Foreman-Magidor, Schimmerling)
Assume ORP. Then we have the following:

(1) $\Box_{\omega_1,\omega}$ fails.

(2) $\Box_{\lambda,<\omega}$ fails for all $\lambda \geq \omega_2$. If $\lambda^{\omega} = \lambda$, then $\Box_{\lambda,\omega}$ fails.

<u>Proof of the first statement of (2)</u>

Suppose there is a $\Box_{\lambda,<\omega}$-sequence $\langle \mathcal{C}_\alpha \mid \alpha < \lambda^+ \rangle$.
Let $f : E^{\lambda^+}_{\omega} \to [\lambda]^{<\omega}$ be s.t. $f(\alpha) = \{\mathrm{otp}(c) \mid c \in \mathcal{C}_\alpha\}$.
There is $x \in [\lambda]^{<\omega}$ s.t. $S := \{\alpha \mid f(\alpha) = x\}$ is stationary.
For each $\alpha < \lambda^+$ of cof. ω_1, taking $c \in \mathcal{C}_\alpha$, we have

$$|\mathrm{Lim}(c) \cap S| \leq |x| < \omega.$$

So $S \cap \alpha$ is non-stat. for each $\alpha < \lambda^+$ of cof. ω_1. $\Box$

Thm. (Cummings-Foreman-Magidor)
ORP is consistent with $\square_{\lambda,\mathrm{cof}(\lambda)}$ for any λ.

Outline of $\mathrm{Con}(\mathrm{ORP} + \square_{\lambda,\mathrm{cof}(\lambda)})$ **for singular** λ

From a model of MM, first add a "nice" $\square_{\lambda,\mathrm{cof}(\lambda)}$-sequence.

Then, by an iteration of club shootings, destroy all non-reflecting stationary subsets of $E^{\lambda^+}_{\omega}$. $\square$

- Different from MM at singular cardinals of cof. ω and ω_1:
  - MM denies $\square_{\lambda,\lambda}$ for singlar cardinals λ of cof. ω.
  - MM denies $\square_{\lambda,<\lambda}$ for singlar cardinals λ of cof. ω_1.

- $\mathrm{ORP} + \mathrm{MA}_{\aleph_1}$ is consistent with $\square_{\omega_1,\omega_1}$
  because both ORP and $\square_{\omega_1,\omega_1}$ are preserved by c.c.c. forcings.

Question
Does ORP ($+\lambda^{\omega_1} = \lambda$) deny $\square_{\lambda,\omega_1}$ for λ of cof. $> \omega_1$?

3.2 FRP

- FRP $\equiv$ For any regular $\lambda \geq \omega_2$, any stationary $S \subseteq E^\lambda_\omega$ and any function f on S with $f(\alpha) \in [\alpha]^\omega$ there is $\alpha \in E^\lambda_{\omega_1}$ s.t.

$$\{x \in [\alpha]^\omega \mid \sup x \in S \ \wedge \ f(\sup x) \subseteq x\}$$

is stationary in $[\alpha]^\omega$.

Thm. (Fuchino-Juhász-Soukup-Szentmiklóssy-Usuba)
Assume FRP. Then we have the following:

(1) $\square_{\lambda,\omega}$ fails for all λ.

(2) If $\mathrm{cof}(\lambda) = \omega$, then $\square_{\lambda,\lambda}$ fails.

Thm. (S.)
FRP is consistent with $\square_{\lambda,\mathrm{cof}(\lambda)}$ for any λ with $\mathrm{cof}(\lambda) > \omega$.

- The same as MM at singular cardinals of cof. ω,
  but different at singular cardinals of cof. ω_1.

- $\mathsf{FRP} + \mathsf{MA}_{\aleph_1}$ is consistent with $\square_{\omega_1,\omega_1}$
  because both FRP and $\square_{\omega_1,\omega_1}$ are preserved by c.c.c. forcings.

Question

Does FRP deny $\square_{\lambda,\omega_1}$ for λ of cof. $> \omega_1$?

4. Chang's Conjecture

Thm. (Todorčvić)
Chang's Conjecture implies the failure of $\square_{\omega_1}$.

Thm. (S.)
Chang's Conjecture is consistent with $\square_{\omega_1,2}$.

- Chang's Conjecture $+$ $\mathrm{MA}_{\aleph_1}$ is consistent with $\square_{\omega_1,2}$ because c.c.c. forcings preserve Chang's Conjecture and $\square_{\omega_1,2}$.

Outline of Con(**Chang's Conjecture** $+ \square_{\omega_1,2}$)

Let κ be a measurable cardinal. We prove

$$\Vdash_{\mathsf{Col}(\omega_1,<\kappa)*\dot{\mathbb{P}}} \text{"Chang's Conjecture} + \square_{\omega_1,2}\text{"},$$

where $\mathbb{P}$ is the poset adding a $\square_{\omega_1,2}$-seq. by initial segments:

- $\mathbb{P}$ consists of all $p = \langle \mathcal{C}_\alpha \mid \alpha \leq \delta \rangle$ $(\delta < \omega_2)$ which is an approximation of a $\square_{\omega_1,2}$-seq.
- $p \leq q$ iff p is an end-extension of q.

($\mathbb{P}$ is $<\omega_2$-distributive and forces $\square_{\omega_1,2}$.)

We must prove $\mathsf{Col}(\omega_1,<\kappa) * \dot{\mathbb{P}}$ forces Chang's Conjecture.

In $V^{\mathrm{Col}(\omega_1,<\kappa)}$ suppose

$p \in \mathbb{P}$,
$\dot{\mathcal{M}}$ is a $\mathbb{P}$-name for a structure on ω_2,
$\mathcal{N} := \langle \mathcal{H}_\theta, \in, p, \dot{\mathcal{M}} \rangle$.

It suffices to prove that in $V^{\mathrm{Col}(\omega_1,<\kappa)}$ there is $p^* \leq p$ and $N^* \prec \mathcal{N}$ s.t

- p^* is N^*-generic,
- $|N^* \cap \omega_2| = \omega_1$ & $|N^* \cap \omega_1| = \omega$.

(p^* forces that $N^* \cap \omega_2$ witnesses Chang's Conjecture for $\dot{\mathcal{M}}$.)

We construct a $\subseteq$-increasing seq. $\langle N_\xi \mid \xi < \omega_1 \rangle$ of ctble. elem. submodels of $\mathcal{N}$ and a descending seq. $\langle p_\xi \mid \xi < \omega_1 \rangle$ in $\mathbb{P}$ below p s.t.

- $N_0 \cap \omega_1 = N_1 \cap \omega_1 = \cdots = N_\xi \cap \omega_1 = \cdots$,
- p_ξ is N_ξ-generic, and $p_\xi \in N_{\xi+1}$,
- $\{p_\xi \mid \xi < \omega_1\}$ has a lower bound,

using some modification of the Strong Chang's Conjecture.

Then $N^* := \bigcup_{\xi < \omega_1} N_\xi$ and a lower bound p^* of $\{p_\xi \mid \xi < \omega_1\}$ are as desired.

Modification of the Strong Chang's Conjecture:

Lem. (In $V^{\mathrm{Col}(\omega_1,<\kappa)}$)
If $N \prec \mathcal{N}$ is ctble. and $\langle q_n \mid n < \omega \rangle$ is an $(N, \mathbb{P})$-generic seq., then

$\forall c \subseteq \sup(N \cap \omega_2)$: club, threads $\bigcup_{n<\omega} q_n$

$\exists d \subseteq \sup(N \cap \omega_2)$: club, threads $\bigcup_{n<\omega} q_n$

$\exists q^* \leq \bigcup_{n<\omega} q_n {}^\frown \langle \{c, d\} \rangle$ s.t.

$\mathrm{sk}^{\mathcal{N}}(N \cup \{q^*\}) \cap \omega_1 = N \cap \omega_1$.

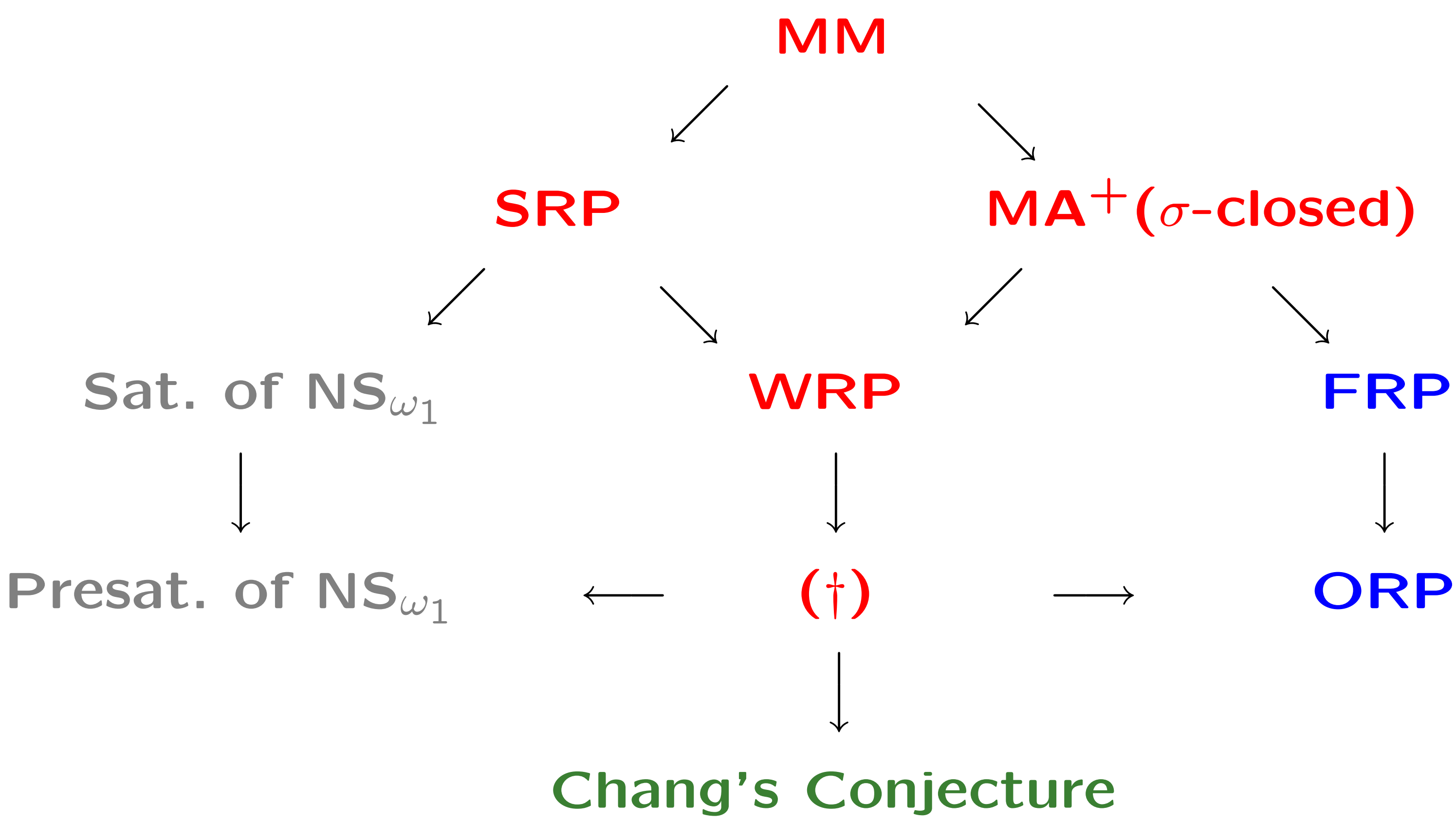
$\square$

We used a measurable cardinal to construct a model of Chang's Conjecture and $\square_{\omega_1,2}$. On the other hand, recall:

Fact (Silver, Donder)
Con (ZFC + Chang's Conjecture)
$\Leftrightarrow$ Con (ZFC + $\exists\omega_1$-Erdös cardinal).

Question
What is the consistency strength of
"Chang's Conjecture + $\square_{\omega_1,2}$" ?

5. Summary



- Saturation of NS$_{\omega_1}$ is consistent with $\square_\lambda$ for all unctble. λ.

- (†) and stronger principles:

  Almost the same as MM. With ¬CH the same as MM.

- ORP:

  Different from MM at singular cardinals of cof. ω and ω_1.

  FRP:

  Different from MM at singular cardinals of cof. ω_1.

- Chang's Conjecture:

  Consistent with $\square_{\omega_1,2}$.

6. Consequences of PFA

Thm. (Magidor, Todorčević)
PFA denies $\square_{\lambda,\omega_1}$ for all λ.

Thm. (Magidor)
PFA is consistent with $\square_{\lambda,\omega_2}$ for all λ.

Thm. (Magidor)
PDFA denies $\square_{\lambda,\omega}$ for all λ.

Thm. (Magidor)
PDFA is consistent with $\square_{\lambda,\omega_1}$ for all λ.

Thm. (Strullu)

(1) MRP denies $\square_{\lambda,\omega}$ for all λ.

(2) $\mathrm{MRP} + \mathrm{MA}_{\aleph_1}$ denies $\square_{\lambda,\omega_1}$ for all λ.

Thm. (Raghavan)

(1) PID denies $\square_{\lambda,\omega}$ for all λ.

(2) $\mathrm{PID} + \mathrm{MA}_{\aleph_1}$ denies $\square_{\lambda,\omega_1}$ for all λ with $\mathrm{cof}(\lambda) > \omega_1$.

Question

Does $\mathrm{PID} + \mathrm{MA}_{\aleph_1}$ deny $\square_{\lambda,\omega_1}$ for all λ ?

In particular, does it deny $\square_{\omega_1,\omega_1}$?