Non-trivial automorphisms from variants of small $\mathfrak{d}$

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Notation

If A and B are subsets of $\mathbb{N}$ let $\equiv^*$ denote the equivalence relation defined by $A \equiv^* B$ if and only if $A \Delta B$ is finite. Let $[A]$ denote the equivalence class of A modulo $\equiv^*$. Then $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ is the same as $\mathcal{P}(\mathbb{N})/\equiv^*$.

Notation

If f is a function defined on the set A and $X \subseteq A$ then the notation $f(X)$ will be used to denote $\{f(x) \mid x \in X\}$. An automorphism Φ of the Boolean algebra $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ is called trivial *if there is $f : \mathbb{N} \to \mathbb{N}$ such that $f(A) \in \Phi([A])$ for each $A \subseteq \mathbb{N}$.*

Theorem (W. Rudin)

If $2^{\aleph_0} = \aleph_1$ then there is a non-trivial automorphism of the Boolean algebra $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$.

First argument: Let $\mathcal{U}$ and $\mathcal{V}$ be P-points generated by $\{U_\xi\}_{\xi\in\omega_1}$ and $\{V_\xi\}_{\xi\in\omega_1}$. Choose bijections $\psi_\xi : U_\xi \to V_\xi$ forming a (partial) coherent family. Define

$$\Psi([A]) = \begin{cases} \psi_\xi(A) & \text{if } A \subseteq^* U_\xi \\ \psi_\xi(\mathbb{N} \setminus A) & \text{otherwise.} \end{cases}$$

and note that Ψ is a well defined automorphism. How can it be made non-trivial? Ask Hausdorff.

Now use that there are $2^{2^{\aleph_0}}$ P-points assuming $2^{\aleph_0} = \aleph_1$.

Second argument: Use the countable saturation of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ to inductively construct the automorphism. Use $2^{\aleph_0} = \aleph_1$ to diagonalize against all possible trivial automorphisms.

Definition

An automorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ is called somewhere trivial *if there is an infinite $Z \subseteq \mathbb{N}$ and $f : Z \to \mathbb{N}$ such that $f(A) \in \Phi([A])$ for each $A \subseteq Z$. An automorphism that is not somewhere trivial is called* nowhere trivial.

The automorphism constructed by the second method can be made nowhere trivial.

Theorem (Shelah)

It is consistent with set theory that all automorphisms of the Boolean algebra $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ *are trivial.*

Recall that OCA implies that all coherent families are trivial. This play a key role in the following:

Theorem (Velickovic)

OCA and $MA_{\aleph_1}$ *implies that all automorphism of the Boolean algebra* $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ *are trivial.*

On the other hand, the CH P-point argument can be extended to show:

Theorem (Velickovic)

It is consistent with $MA_{\aleph_1}$ *that there is a non-trivial automorphism of* $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$.

Just as in the P-point argument, the automorphism constructed by Velickovic is somewhere trivial. However:

Theorem (Shelah – S.)

It is consistent with MA and $2^{\aleph_0} > \aleph_1$ that there is a nowhere trivial automorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$.

Theorem

In the model obtained by adding $\aleph_2$ Cohen reals to a model of CH there is a nowhere trivial automorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$.

Question

Are there nowhere trivial automorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ in the model obtained by adding $\aleph_3$ Cohen reals to a model of CH?

- The original model of Shelah is obtained by a finite support iteration of oracle-cc partial orders and hence it shares many properties with Cohen real models; for example $\mathfrak{d} = 2^{\aleph_0}$.
- Velickovic's argument for getting all automorphisms trivial uses OCA and recall that OCA implies that $\mathfrak{b} = \aleph_2$.

So one might ask if $\mathfrak{d} > \aleph_1$ is necessary for all automorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ to be trivial. However:

Theorem (Farah – Shelah)

It is consistent with $\mathfrak{d} = \aleph_1$ that all automorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ are trivial.

On the other hand, except for the CH arguments, the other methods mentioned for getting nontrivial automorphisms — Cohen model and consistency with MA — all use methods that yield $\mathfrak{d} > \aleph_1$.

So, is it consistent with $\mathfrak{d} = \aleph_1 < 2^{\aleph_0}$ that all automorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ are trivial?

Lemma

There is a nontrivial automorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ *provided that there is a partition of* $\mathbb{N}$ *into finite sets* $\{I_n\}_{n\in\omega}$ *such that:*

1. *For each* $\xi \in \omega_1$ *and* $n \in \omega$ *there is a Boolean subalgebra* $\mathfrak{B}_{\xi,n}$ *of* $\mathcal{P}(I_n)$ *and an automorphism* $\Phi_{\xi,n}$ *of* $\mathfrak{B}_{\xi,n}$.
2. *If* $\xi \in \eta$ *then* $\mathfrak{B}_{\xi,n} \subseteq \mathfrak{B}_{\eta,n}$ *and* $\Phi_{\xi,n} = \Phi_{\eta,n} \restriction \mathfrak{B}_{\xi,n}$ *for all but finitely many* $n \in \omega$.
3. *For any one-to-one* $F : \mathbb{N} \to \mathbb{N}$ *there are* $\xi \in \omega_1$ *and infinitely many* $n \in \omega$ *such that there is an atom* $a \in \mathfrak{B}_{\xi,n}$ *and* $j \in a$ *such that* $F(j) \notin \Phi_{\xi,n}(a)$.
4. *For any* $A \subseteq \mathbb{N}$ *there is* $\xi \in \omega_1$ *such that* $A \cap I_n \in \mathfrak{B}_{\xi,n}$ *for all but finitely many* n.

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Define

$$\Phi([A]) = \lim_{\xi \to \omega_1} \left[\bigcup_{n \in \omega} \Phi_{\xi,n}(A \cap I_n) \right]$$

Why is this is well defined?
If $A \Delta B$ is finite there is $\alpha \in \omega_1$ such that for all $\xi > \alpha$ and for all but finitely many n the equation

$$\Phi_{\xi,n}(A \cap I_n) = \Phi_{\xi,n}(B \cap I_n)$$

holds. From Hypothesis 2 it then follows that if ξ and η are greater than α then

$$\bigcup_{n \in \omega} \Phi_{\xi,n}(A \cap I_n) \equiv^* \bigcup_{n \in \omega} \Phi_{\eta,n}(B \cap I_n)$$

and, hence, $\Phi([A])$ is well defined. Since each $\Phi_{\xi,n}$ is an automorphism it follows that Φ is an automorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$.

Why is Φ is nontrivial?
Suppose that there is a one-to-one function $F : \mathbb{N} \to \mathbb{N}$ such that $F(A) \in \Phi([A])$ for all $A \subseteq \mathbb{N}$. Choose $\xi \in \omega_1$ and an infinite $Z \subseteq \mathbb{N}$ and atoms $a_n \in \mathfrak{B}_{\xi,n}$ and $j_n \in a_n$ such that $F(j_n) \notin \Phi_{\xi,n}(a_n)$ for each $n \in Z$.

Let $W \subseteq Z$ be an infinite subset such that for each $n \in W$, if $F(j_n) \in I_k$ and $k \neq n$ then $k \notin W$. Let $A = \bigcup_{n \in W} a_n$. For any $\eta \geq \xi$

$$\{F(j_n) \mid j \in W\} \cap \bigcup_{n \in W} \Phi_{\eta,n}(a_n) \equiv^* \{F(j_n) \mid j \in W\} \cap \bigcup_{n \in W} \Phi_{\xi,n}(a_n) \equiv^* \emptyset$$

and, hence, $F(A) \notin \Phi([A])$.

When are the hypotheses of Lemma 1 satisfied?

Definition

Given functions f and g from ω to ω let $\mathfrak{d}_{f,g}$ be the least cardinal of a family $\mathcal{D} \subseteq \prod_{n\in\omega}[f(n)]^{g(n)}$ such that for every $F \in \prod_{n\in\omega} f(n)$ there is $G \in \mathcal{D}$ such that $F(n) \in G(n)$ for all n.

Given a filter $\mathcal{F}$ on ω define $\mathfrak{d}_{f,g}(\mathcal{F})$ to be the least cardinal of a family $\mathcal{D} \subseteq \prod_{n\in\omega}[f(n)]^{g(n)}$ such that for every $F \in \prod_{n\in\omega} f(n)$ there is $G \in \mathcal{D}$ and $X \in \mathcal{F}$ such that $F(n) \in G(n)$ for all $n \in X$. (So $\mathfrak{d}_{f,g} = \mathfrak{d}_{f,g}(\mathcal{F})$ where $\mathcal{F}$ is the co-finite filter.)

Note that $\mathfrak{d}_{f,g} > \mathfrak{d}$ and $\mathfrak{d}_{f,g} < \mathfrak{d}$ are both possible. Random and Laver reals provide the relevant models.

Lemma

If there are functions $f : \mathbb{N} \to \mathbb{N}$ *and* $g : \mathbb{N} \to \mathbb{N}$ *such that for all* $k \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)k^{g(n)}} = \infty$$

and if $\mathfrak{d}_{f!,g}(\mathcal{F}) = \aleph_1$ *for some filter generated by a* $\subseteq^*$*-descending tower of length* ω_1 *then the hypotheses of Lemma 1 hold.*

First step

Given the hypothesis, it may be assumed that there are $\subseteq^*$-descending sets $\{X_\xi\}_{\xi\in\omega_1} \subseteq \mathcal{F}$ and functions $\{G_\xi\}_{\xi\in\omega_1} \subseteq \prod_{n\in\omega}[f(n)!]^{g(n)}$ such that for every $F \in \prod_{n\in\omega} f(n)!$ there is $\xi \in \omega_1$ such that $F(n) \in G_\xi(n)$ for all but finitely many $n \in X_\xi$.

Why? Reindex so that for all $\xi \in \omega_1$ there are cofinally many $\eta \in \omega_1$ such that $G_\xi = G_\eta$.

Second step

There are functions $h_\xi : \mathbb{N} \to \mathbb{N}$ and $H_\xi : \mathbb{N} \to [f(n)!]^{h_\xi(n)}$ for $\xi \in \omega_1$ such that

1. if $\xi \in \eta \in \omega_1$ then $4h_\xi \leq^* h_\eta \leq g$
2. if $\xi \in \eta \in \omega_1$ then $H_\xi(n) \subseteq H_\eta(n)$ for all but finitely many n
3. if $F \in \prod_{n\in\mathbb{N}} f(n)!$ and $F(n) \in G_\xi(n)$ for all but finitely many $n \in X_\xi$ then also $F(n) \in H_\xi(n)$ for all but finitely many $n \in X_\xi$.

Why? The hypothesis that $\lim_{n\to\infty} f(n)/g(n)k^{g(n)} = \infty$ for all k makes it possible to choose $h : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)h(n)2^{g(n)h(n)}} = \infty$$

and $\lim_{n\to\infty} h(n) = \infty$.

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Third step

Let $H_0(n) = G_0(n)$. Given H_ξ satisfying Conditions 2 and 3, define $H_{\xi+1}(n) = H_\xi(n) \cup G_{\xi+1}(n)$ and note that

$$|H_{\xi+1}(n)| \leq |H_\xi(n)| + |G_{\xi+1}(n)| \leq h_\xi(n) + g(n)/h(n) \leq h_{\xi+1}(n).$$

On the other hand, if η is a limit ordinal and H_ξ satisfying the desired requirements have been chosen for $\xi \in \eta$, then a diagonalization argument yields H_η such that $|H_\eta(n)| = h_\eta(n)$ and $H_\xi(n) \subseteq H_\eta(n)$ for all but finitely many n for each $\xi \in \eta$.

Fourth step: The first sequence of finite algebras

Now let $\{I_n\}_{n\in\omega}$ partition $\mathbb{N}$ such that $|I_n| = f(n)$ and let $\{\theta_{j,n}\}_{j\in f(n)!}$ enumerate all permutations of I_n. Without loss of generality, $f(n)$ is even for each n.

Let $A_{0,n}$ and $A_{1,n}$ partition I_n into two equal sized sets and let $\varphi_{0,n}$ be an involution of I_n interchanging $A_{0,n}$ and $A_{1,n}$. For $n \in X_0$ let $\mathfrak{B}_{0,n} = \{\emptyset, I_n, A_{0,n}, A_{1,n}\}$ and let $\Phi_{0,n}$ be the automorphism of $\mathfrak{B}_{0,n}$ induced by $\varphi_{0,n}$.

For $n \in \omega \setminus X_0$ let $\mathfrak{B}_{0,n} = \mathcal{P}(I_n)$ and let $\Phi_{0,n}$ be the identity.

Fifth step: The ξ^{th} sequence of finite algebras

As the induction hypothesis assume Condition 2 of the first lemma holds and that, in addition,

- $\mathcal{A}_{\xi,n}$ are the atoms of $\mathfrak{B}_{\xi,n}$ and that $|\mathcal{A}_{\xi,n}| \leq 2^{4h_\xi(n)}$ provided that $n \in X_\xi$
- for $n \in X_\xi$ there are involutions $\varphi_{\xi,n}$ of I_n that induce $\Phi_{\xi,n}$.

If $\mathfrak{B}_{\xi,n}$, $\mathcal{A}_{\xi,n}$, $\varphi_{\xi,n}$ and $\Phi_{\xi,n}$ have been defined for all ξ less than the limit ordinal η then a standard diagonalization yields $\mathfrak{B}_{\eta,n}$, $\mathcal{A}_{\eta,n}$, $\varphi_{\xi,n}$ and $\Phi_{\eta,n}$.

Sixth step: The successor step

Assume that $\mathfrak{B}_{\xi,n}$, $\mathcal{A}_{\xi,n}$, $\varphi_{\xi,n}$ and $\Phi_{\xi,n}$ have been defined. Let $\mathcal{A}^*_{\xi+1,n}$ be the atoms generated by $\mathcal{A}_{\xi,n}$ and $\{A_n(j), \varphi_{\xi,n}(A_n(j))\}_{j\in H_{\xi+1}(n)}$. Then $|\mathcal{A}^*_{\xi+1,n}| \leq |\mathcal{A}_{\xi,n}|4^{h_{\xi+1}(n)} \leq 2^{g(n)}$.

Since $f(n) > g(n)2^{g(n)}$ there must be some $a_n \in \mathcal{A}^*_{\xi+1,n}$ such that $|a_n| > g(n)$ for each $n \in X_\xi$. Let $\varphi : a_n \to \varphi_{\xi,n}(a_n)$ be any bijection such that for each $n \in X_{\xi+1}$ and each $j \in H_{\xi+1}(n)$ there is some $k_{j,n} \in a_n$ such that $\varphi(k_{j,n}) \neq \theta_{j,n}(k_{j,n})$.

Now for $n \in X_{\xi+1}$ let $\mathcal{A}_{\xi+1,n} = \mathcal{A}^*_{\xi+1,n} \cup \{\{k_{j,n}\} \mid j \in H_{\xi+1}(n)\}$ and let $\varphi_{\xi+1,n}$ be defined by

$$\varphi_{\xi+1,n}(z) = \begin{cases} \varphi_{\xi,n}(z) & \text{if } z \notin a_n \cup \varphi_{\xi,n}(a_n) \\ \varphi(z) & \text{if } z \in a_n \\ \varphi^{-1}(z) & \text{if } z \in \varphi_{\xi,n}(a_n) \end{cases}$$

and let $\Phi_{\xi+1,n}$ be induced by $\varphi_{\xi+1,n}$. Let $\mathfrak{B}_{\xi+1,n}$ be the Boolean algebra whose atoms are $\mathcal{A}_{\xi+1,n}$. On the other hand, for $n \in \omega \setminus X_{\xi+1}$ let $\mathfrak{B}_{\xi+1,n} = \mathcal{P}(I_n)$ and let $\Phi_{\xi+1,n}$ be induced by $\varphi_{\xi,n}$

Then $\mathfrak{B}_{\xi,n} \subseteq \mathfrak{B}_{\xi+1,n}$ and that $\Phi_{\xi+1,n} \restriction \mathfrak{B}_{\xi,n} = \Phi_{\xi,n}$. Moreover,

$$|\mathcal{A}_{\xi+1,n}| \leq |\mathcal{A}^*_{\xi+1,n}| + h_{\xi+1}(n) \leq 2^{3h_{\xi+1}(n)} + h_{\xi+1}(n) \leq 2^{4h_{\xi+1}(n)}$$

for all but finitely many $n \in X_{\xi+1}$ as required.

Why is this non-trivial?

Let $F : \mathbb{N} \to \mathbb{N}$ be one-to-one. If there are infinitely many n such that there is $z_n \in I_n$ such that $F(z_n) \notin I_n$ then let $\xi = 0$ and, without loss of generality, it may be assumed that z_n belongs to the atom $A_{0,n}$ of $\mathfrak{B}_{0,n}$ for infinitely many n. Since $\varphi_{0,n}(A_{0,n}) = A_{1,n} \subseteq I_n$ it is clear that $F(z_n) \notin \varphi_{0,n}(A_{0,n})$.

If $F(I_n) \subseteq I_n$ for all but finitely many n then $F \restriction I_n = \theta_{J(n),n}$ for some $J(n)$ also or all but finitely many n. There is some $\xi \in \omega_1$ such that $J(n) \in H_\xi(n)$ for all but finitely many $n \in X_\xi$. By construction, for all but finitely many $n \in X_\xi$ there is a singleton $\{k\} \in \mathcal{A}_{\xi,n}$ such that $\varphi_{\xi,n}(\{k\}) = \{\varphi_{\xi,n}(k)\}$ and $\varphi_{\xi,n}(k) \neq \theta_{J(n),n}(k)$.

Why is the automorphism defined on all of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$? This is the same argument using that $2^n \leq n!$.

Corollary

If $\mathfrak{d}_{f!,g} = \aleph_1$ then there is a nontrivial automorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$.

Let $\mathcal{F}$ be the co-finite filter.

Corollary

If there is an $\aleph_1$-generated filter $\mathcal{F}$ such that $\mathfrak{d}_{f!,g}(\mathcal{F}) = \aleph_1 \neq \mathfrak{d}$ then there is a nontrivial automorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$.

Let $\mathcal{F}$ be generated by $\{X_\xi\}_{\xi\in\omega_1}$. Use Rothberger's argument and $\aleph_1 \neq \mathfrak{d}$ to construct a $\subseteq^*$-descending sequence $\{Y_\xi\}_{\xi\in\omega_1}$ all of whose terms are $\mathcal{F}$ positive and such that $Y_\xi \subseteq X_\xi$. Let $\mathcal{F}'$ be generated by $\{Y_\xi\}_{\xi\in\omega_1}$ and note that $\mathfrak{d}_{f!,g}(\mathcal{F}') = \aleph_1$.

Recall that Farah–Shelah showed it is consistent that $\mathfrak{d} = \aleph_1$ and all automorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ are trivial. Hence the assumption of $\aleph_1 \neq \mathfrak{d}$ in Corollary 2 is essential. (To be precise, one should check that $\mathfrak{u} = \aleph_1$ in their model.)

Question

Is $\aleph_1 \neq \mathfrak{d}$ essential in Corollary 2?

It is worth observing that the automorphism of Lemma 1 is trivial on some infinite sets — indeed, if $\xi \in \omega_1$ and $X \subseteq \mathbb{N}$ are such that $\{x\}$ belongs to some $\mathfrak{B}_{\xi,n}$ for each $x \in X$ then Φ is trivial on $\mathcal{P}(X)$.

However, if $\mathcal{T}(\Phi)$ is defined to be the ideal $\{X \subseteq \mathbb{N} \mid \Phi \restriction \mathcal{P}(X) \text{ is trivial } \}$ then $\mathcal{T}(\Phi)$ is a small ideal in the sense that the quotient algebra $\mathcal{P}(\mathbb{N})/\mathcal{T}(\Phi)$ has large antichains, even modulo the ideal of finite sets — in the terminology of Farah, the ideal $\mathcal{T}(\Phi)$ is not ccc by fin.

One should not, therefore, expect to get a nowhere trivial automorphism by these methods. Hence the following result:

Theorem

Let $\mathbb{P}$ be the product of κ Sacks partial orders. Assuming $2^{\aleph_0} = \aleph_1$, there is an automorphism Θ of $\mathcal{P}(\mathbb{N}/[\mathbb{N}]^{<\aleph_0}$ such that

$$1 \Vdash_{\mathbb{P}} \text{“}\Theta \text{ is a nowhere trivial automorphism of } \mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}\text{”}$$

and note that, in particular, this means Θ has a natural definition in the generic extension by $\mathbb{P}$.