Notes for September 25 and 27

Exercises are to be turned in. Questions are to be discussed.

## 1. Probably all we will get done on September 25

**Definition 1.** For a formula $\varphi(v_1,\dots,v_n)$ and sets $\{\tau_1,\dots,\tau_n\}\subset M$, we say that $p\Vdash\varphi(\tau_1,\dots,\tau_n)$ if for each generic $G\subset\mathbb{P}$, with $p\in G$, we have that $M[G]\models\varphi(val_G(\tau_1),\dots,val_G(\tau_n))$.

**Theorem 2** (A forcing theorem)**.** *For each formula $\varphi(v_1,\dots,v_n)$ (all free variables showing), and each set $\{\tau_1,\dots,\tau_n\}\subset M$, there is a dense set $D$ such that for all $d\in D$, we have that either $d\Vdash\varphi(\tau_1,\dots,\tau_n)$ or $d\Vdash\neg\varphi(\tau_1,\dots,\tau_n)$.*

**Exercise 1** (the maximum principle)**.** *Prove that if $p\Vdash(\exists x)\varphi(x,\tau_1,\dots,\tau_n)$, then there is a $\tau_0$ such that $p\Vdash\varphi(\tau_0,\tau_1,\dots,\tau_n)$.*

Pursuant to our discussion about the difference between forcing with $\omega^{<\omega}$ and $\omega_1^{<\omega}$, let us now consider the Hechler poset $\mathcal{H}=\omega^{<\omega\uparrow}\times\omega^{\omega\uparrow}$ with the ordering $(s,f)<(t,g)$ implies $s(k)>g(k)$ for all $k\in dom(s)\setminus dom(t)$.

**Proposition 3.** *Suppose that $G$ is $\omega^{<\omega}$-generic over $M$ and $H$ is $\mathcal{H}$-generic. Then $M[G]$ is not equal $M[H]$ because $M\cap\omega^\omega$ is unbounded mod finite in $M[G]$.*

**Remark 4.** *It is a reasonable exercise in Kunen to construct for any $\tau$ a function $f_\tau$ so that, for any $p\in\mathbb{P}=\omega^{<\omega}$, if $p\Vdash_{\mathbb{P}}\tau\in\omega^\omega$, then $p\Vdash_{\mathbb{P}}\check{f}_\tau\not<^*\tau$.*

**Exercise 2.** *If $MA_{\mathcal{H}}(\omega_1)$ holds in $M$, then $M\models\mathfrak{b}>\omega_1$? Recall that $\mathfrak{b}$ is the minimum cardinality of a unbounded mod fin set of functions.*

In contrast to Proposition 3 we have the following

**Proposition 5.** *For each $\omega^{<\omega}$-generic filter $G$, there an $\mathcal{H}$-generic filter $H$ so that $M[G]\subset M[H]$. And for each $\mathcal{H}$-generic filter $H$, there is an $\omega^{<\omega}$-generic filter $G$ such that $M[G]\subset M[H]$.*

*Proof.* We choose a partition $\{A_k:k\in\omega\}$ of $\omega$ into infinite sets. Let $\pi$ denote the function from $\omega$ to $\omega$ which sends each $A_k$ to $k$. This is all done in $M$. When we add the generic function $h_{\mathcal{H}}$ for $\mathcal{H}$, we will keep track of the "path" it makes through the sequence of $A_k$'s.

It's an easy step to show that if $H$ is $\mathcal{H}$-generic, then we can define $G=\{\pi\circ s_q : q\in H\}$ and simply check that $G$ is $\omega^{<\omega}$-generic. (very similar to the earlier discussion about a poset being dense in another).

Now suppose that $G$ is a generic for $\omega^{<\omega}$. Remember that $M$ is just a countable model. For any $t\in\omega^{<\omega}$, let $\tilde{t}$ abbreviate $\pi\circ t$. For any $(s,f)\in\mathcal{H}$ and dense subset $D$ of $\mathcal{H}$, show that $E(D,s,f)=\{\tilde{t}:\tilde{t}\perp\tilde{s}$ or $(\exists h)((t,h)<(s,f)$ and $(t,h)\in D)\}$ is a dense subset of $\omega^{<\omega}$. Show that we can inductively define $H$ given $G$. $\square$

This brings up a question (actually Exercise 2 does but we are now better prepared).

**Question 1.** *Does $\mathfrak{b}>\omega_1$ imply that $MA_{\mathcal{H}}(\omega_1)$ holds in $M$?*

No. But here's a very interesting result that makes it feel like we're just a Cohen real away

**Proposition 6** (Truss)**.** *If $f \in \omega^\omega$ dominates mod fin $M \cap \omega^\omega$, and if $c = \bigcup G$ for $G \subset \omega^{<\omega}$ generic over $M[f]$ (actually, given my presentation of $\mathcal{H}$ you should ensure that $c$ is strictly increasing, i.e. force with $\omega^{<\omega\uparrow}$, then $f + c$ is $\mathcal{H}$-generic over $M$.*

I want to use this later in the course. It raised an interesting question though: if $\mathfrak{b}$ is greater than $\omega_1$ and we add a Cohen real, do we get a model of $\mathrm{MA}_{\mathcal{H}}(\omega_1)$?[1]

Recall that if $H$ is $\mathcal{H}$-generic, then $M \cap \mathbb{R}$ is meager. Here's a nice proof that this does not happen with Cohen.

**Proposition 7.** *If $G$ is $\omega^{<\omega}$-generic over $M$, then every dense $G_\delta$ subset of $\mathbb{R}$ in $M[G]$ will hit $M \cap \mathbb{R}$.*

Actually, let us prove a stronger theorem (why is it stronger?). Let $\mathbb{P}$ be the poset

$$Fn(\omega_2, 2) = \{s : (\exists F \in [\omega_2]^{<\omega}) s \in 2^F\}$$

ordered by $s < t$ if $s \supset t$. (If $\{s_\alpha : \alpha \in \omega_1\} \subset Fn(\omega_2, 2)$, then there is a $\lambda < \omega_2$ such that $dom(s_\alpha) \subset \lambda$, and so each $s_\alpha$ corresponds to a clopen subset of the separable space $2^\lambda$ – hence $\mathbb{P}$ is ccc)

**Theorem 8.** *If $G$ is $\mathbb{P}$-generic over $M$, then every dense $G_\delta$ subset of $\mathbb{R}$ in $M[G]$ will hit $M \cap \mathbb{R}$. And, of course, CH fails in $M[G]$.*

*Proof.* Let $\tau$ be such that there is some $p \in \mathbb{P}$ such that $p$ forces that $\tau$ is a countable collection of dense open subsets of $\mathbb{R}$. Using the forcing theorem and the maximum principle we could, but will not, instead simply assume that we have a sequence $\{\tau_n : n \in \omega\}$ so that for each $n$, $p$ forces that $\tau_n$ is a dense open set and that $val_G(\tau) = \{val_G(\tau_n) : n \in \omega\}$.

Let us use an elementary submodel argument (working in $M$). Let $p, \tau \in N$ where $N$ is a countable elementary submodel of $V_\beta$ for some large enough $\beta$ (usually we would use $H(\theta)$ for some large enough regular cardinal $\theta$). As it happens, $\mathbb{P}$ is automatically in $N$ in this case.

Choose any $x \in \mathbb{R}$ such that $x$ is not in any nowhere dense set $K$ which is in $N$. Equivalently, $x$ is in $U$ for each dense open set $U$ which is in $N$.

(1) $p$ forces that $\tau = \tau \cap N$ (true – using that $\mathbb{P}$ is ccc – but easier is $p \Vdash_{\mathbb{P}} \tau \subset N$ [not $\check{N}$]) (this is by elementarity and the fact that $p$ forces that $\tau$ is countable)
(2) for each $\sigma \in N$ and each $q \in N$ such that $q \Vdash \sigma$ is a dense open subset of $\mathbb{R}$, $q \Vdash \check{x} \in \sigma$; [$\bigcup\{(a, b) : (a, b \in \mathbb{Q})\ (\exists r < q) r \Vdash \check{(a, b)} \subset \sigma\}$ is a dense set which is $N$; so $x$ is in this union; now to get $q \Vdash \check{x} \in \sigma$ ???]

In fact, $q \Vdash_{\mathbb{P} \cap N} \check{x} \in \sigma$ is easy, and then $q \Vdash_{\mathbb{P}} \check{x} \in \sigma$ because every dense subset of $\mathbb{P} \cap N$ is predense in $\mathbb{P}$ (discussed later). $\square$

**Definition 9.** A sequence $\{a_\alpha : \alpha \in \omega_1\}$ is a $\diamondsuit$-sequence if for all $A \subset \omega_1$, the set $S_A = \{\alpha : A \cap \alpha = a_\alpha\}$ is **stationary**. Of course $\diamondsuit$ implies CH ($2^\omega = \omega_1$) and CH does not imply $\diamondsuit$ (advanced topic).

**Definition 10.** A poset $\mathbb{P}$ is $\omega_1$-closed if each countable descending sequence has a lower bound.

[1]No, Miller shows it can lower the value of $\mathfrak{b}$ to $\omega_1$.

**Exercise 3.** *If $\mathbb{P}$ is $\omega_1$-closed, then every countable set $b$ of ordinals in $M[G]$ is an element of $M$.*

**Proposition 11.** *Let $G$ be $2^{<\omega_1}$-generic over $M$*

1. *$\mathbb{R} \cap M[G] = \mathbb{R} \cap M$, and same for $2^{<\omega_1}$*
2. *CH holds in $M[G]$*
3. *$\diamondsuit$ holds in $M[G]$*
4. *there is a Souslin tree $\mathbb{S} \subset 2^{<\omega_1}$ in $M[G]$ ($\mathbb{S}$ is uncountable, branching, and all antichains are countable)*

*Proof.* (1) is the previous Exercise, and (2) is a simple density argument similar to the proof that forcing with $\omega_1^{<\omega}$ "collapses" $\omega_1$.

for (3), if $G \subset 2^{<\omega_1}$ is generic, and $t_\alpha \in G \cap 2^{\alpha+\alpha}$, let $a_\alpha = \{\beta < \alpha : t_\alpha(\alpha+\beta) = 1\}$. Let $\dot{A}$ (new, but standard, notation for name) be such that $val_G(\dot{A})$ is a subset of $\omega$ and let $\dot{C}$ be a name of a cub. Find increasing sequence $\{\beta_n, \gamma_n, t_n : n \in \omega\}$ where $\beta_n < \gamma_{n+1} < \beta_{n+1}$, $t_n \subset t_{n+1} \in 2^{\beta_{n+1}}$ and $t_{n+1}$ forces $\gamma_{n+1} \in \dot{C}$ and "forces a value on" $\dot{A} \cap \beta_n$.

for (4), the poset $\mathbb{P}$ is forcing isomorphic to making countable sequences of choices $\mathbb{S}_\alpha \in [2^\alpha]^\omega$ so that $\{\mathbb{S}_\beta : \beta < \alpha\}$ is a nice tree topped up by $\mathbb{S}_\alpha$. More precisely, a condition is a sequence $\langle \mathbb{S}_\beta : \beta < \delta \rangle$ for some $\delta \in \omega_1$. For each $\beta < \alpha < \delta$ and each $s \in \mathbb{S}_\alpha$, $s \restriction \beta \in \mathbb{S}_\beta$, and for each $t \in \mathbb{S}_\beta$, there are at least two extensions of $t$ in $\mathbb{S}_\alpha$. Same argument as in $\diamondsuit$ shows it will be Souslin. $\square$

**Question 2.** *For a given $\alpha$, what does a name $\dot{a}_\alpha$ for $a_\alpha$ look like?*

It's important to have an exercise showing that a poset is ccc.

**Exercise 4** (Tennenbaum)**.** *There is a ccc poset $\mathcal{T}$ for adding a Souslin tree: a condition is a pair $\langle t, <_t \rangle$ where $t \in [\omega_1]^{<\omega}$ and $<_t$ is a tree ordering on $t$ so that $\alpha <_t \beta$ implies $\alpha \in \beta$. We define $\langle t_1, <_{t_1} \rangle$ to extend $\langle t_2, <_{t_2} \rangle$ providing $<_{t_1} \cap (t_2 \times t_2) = <_{t_2}$. Show that this poset is ccc (isomorphic conditions are compatible) and forcing with it gives a Souslin tree. [try a countable elementary submodel argument for both]*

**Proposition 12.** *There is a model of $\neg CH$ in which there is a Souslin tree obtained by first forcing with $Fn(\omega_2, 2)$ then, over that model, forcing with Tennenbaum's poset $\mathcal{T}$.*

Not adding uncountable branches (or antichains) to a tree is well studied. That part of this next exercise is due to Kunen and Tall. The other part is what one would call "preserving towers".

**Exercise 5.** *Assume that $M$ is a model in which there is a Souslin tree $\mathbb{S}$ and $\mathcal{A} = \{a_\alpha : \alpha \in \kappa\} \subset [\omega]^\omega$ is a maximal mod finite descending family. Show that if $G$ is $Fn(\omega_2, 2)$-generic, then $\mathbb{S}$ is still Souslin, and that $\mathcal{A}$ is still maximal in $M[G]$.*

## 2. Factoring Forcing

We discussed above how we can view adding a Hechler real as first adding a Cohen real and then adding something else (a Hechler real?). This idea also came up when discussing $\mathbb{P} \cap N$ versus $\mathbb{P}$ in the proof that adding any number of Cohen reals preserves that the ground model is non-meager.

**Definition 13.** Say that a poset $P \subset R$ is completely embedded if every dense $D \subset P$ is a predense subset of $R$. Equivalently, for each $G \subset R$ which is generic for $R$, we have that $G \cap P$ is a generic for $P$.

For a poset $R$ and a set $B \subset R$, $B^\perp$ is the set of $q \in R$ which are not compatible with any element $B$. Of course $r^\perp = \{r\}^\perp$. $B^\perp$ is empty if and only if $B$ is predense.

**Proposition 14.** *If $G \subset P$ is a generic filter, then set $G^+ = \{r \in R : G \cap r^\perp = \emptyset\}$. We can think of $\dot{Q} = G^+$ as a $P$-name of a poset in $M[G]$.*

1. *If $H \subset G^+$ is generic over $M[G]$, then $H$ (as a subset of $R$) is $R$-generic over $M$ and $M[G][H] = M[H]$.*
2. *Similarly, if $H'$ is any $R$-generic, and $G' = H' \cap P$, then $H'$ is $(G')^+$-generic over $M[G']$.*

*Proof.* Here are the key ideas for (1) and (2). Note that $G \subset H$ because $H$ is upward closed.

1. Suppose that $D \in M$ is a dense subset of $R$. We have to show that $H \cap D$ is not empty. The key is to show that $D \cap G^+$ is dense in $G^+$. Suppose that $r \in G^+$ and, in $M$, define a set $E$ (and show that it is dense in $P$). $r^\perp \cap P$ will be contained in $E$, and for each $d \in D$ with $d < r$, ensure that $E$ contains the set $(P \cap (d^\perp))^{\perp_P} = \{p \in P : (\forall s \in P)(s \perp d \to s \perp p)\} \neq \emptyset$. This set is non-empty because otherwise there would be a dense subset of $P$ every member of which was incompatible with $d$. Then since we have that $r \in G^+$, for $p \in G \cap E$, it must be that there is some $d \in D$ such that $p \in (P \cap d^\perp)^{\perp_P}$. This ensures that $d \in G^+$.
2. Suppose that $\tau \in M$ and $val_{G'}(\tau)$ is a dense subset of $(G')^+$. We may assume that the base set for $R$ is an ordinal, and so that $\tau$ is of the form $\tau^{SON}$. Notice then that for $(\check{d}, p) \in \tau$, we have that $p \Vdash \check{d} \in G^+$. Define $D$ to be the set $\{d : (\exists p)(\check{d}, p) \in \tau\}$. Check that $D$ is a dense subset of $R$. (okay, kind of standard, I am assuming that $1_P \Vdash \tau$ is dense).

$\square$

**Example 15.** *If $R$ is $Fn(\kappa, 2)$ and $P = Fn(\lambda, 2)$ (for any $\lambda < \kappa$), then for any $P$-generic $G$, $G^+$ is isomorphic to $Fn(\kappa \setminus \lambda, 2)$.*

**Proposition 16.** *If (as above) $\tau$ is an $R$-name and $G$ is $P$-generic, we can (recursively) define $\tau^G = \{(\sigma^G, r) : r \in G^+\}$, i.e. we think of this as getting a $G^+$-name. Then, if $H$ is $G^+$-generic over $M[G]$, $val_H(\tau) = val_H(\tau^G)$.*

It is very useful to be able to factor our forcing. For convenience we start using $V$ to denote the ground model (rather than $M$ a ctm).

**Theorem 17.** *Let $G$ be $Fn(\omega_3, 2)$-generic over $V \models CH$. Let $\mathcal{A} = \{\dot{a}_\alpha : \alpha \in \omega_2\}$ be names of subsets of $\omega$. Let $\mathcal{A} \in M \prec H(\theta)$ such that $M^\omega \subset M$ and $|M| = \omega_1$. The following are true in $V[G]$:*

1. *(Kunen) the family $val_G(\mathcal{A}) = \{val_G(\dot{a}_\alpha) : \alpha \in \omega_2\}$ is not a mod finite chain*
2. *(Miller) the family $val_G(\mathcal{A})$ is not maximal almost disjoint.*
3. *(there are* **many***) interesting results in the Cohen model.*

**Proposition 18.** *If $P, Q$ are posets, and if $R = P \times Q$, then*

1. *$P \times \{1_Q\}$ is completely embeded in $P \times Q$,*

(2) *if $G_P \subset P$ is generic, then $G = G_P \times \{1_Q\}$ satisfies that $G^+$ is isomorphic to $\{1_P\} \times \check{Q} \approx \check{Q}$*
(3) *same as above for $Q \times P$*
(4) *each generic for $P \times Q$ has the form $G_P \times G_Q$, where, simply, $G_P, G_Q$ are generics for $P, Q$ respectively, and yet $G_Q$ is still generic for $Q$ over $V[G_P]$.*
(5) *there is some real substance here:*
    (a) *if $\dot{E}$ is a $P$-name of a dense subset of $\check{Q}$, then there is a dense set $D \subset P$ such that $D^+ = val_G(\dot{E})$ , and*
    (b) *if $D \subset P \times Q$ is dense, then $D^+$ is a dense subset of $\check{Q}$.*

Let us ponder the (psychological) difference between $Q$ and $\check{Q}$

**Question 3.** *Let $G_P$ be $P$-generic. In $V[G_P]$,*
(1) *do we have that $Fn(\omega_2, 2) = (Fn(\omega_2, 2))^V$ ?*
(2) *If $P$ adds a real, then $\mathcal{H}$ is not the same as $(\mathcal{H})^V$*

It is interesting to be able to reverse the order

**Exercise 6.** *Assume, for convenience, that $2^{\omega_1} = \omega_2$. Let $P = Fn(\omega_2, 2)$ and $Q = 2^{<\omega_1}$ and let $G_P, G_Q$ be generics.*
(1) *in $V[G_P]$, the poset $Q$ (i.e. $\check{Q}$) is not $\omega_1$-closed, but it does still preserve $\omega_1$ and it adds no new countable sets*
(2) *(Easton) again in $V[G_P]$, if $D$ is a dense subset of $Q$, then there is an $E \in V$ such that $D \supset E$ (try for a combinatorial proof using names).*
(3) *CH fails in $V[G_P \times G_Q]$, but "a weak $\diamondsuit$ for clubs" holds (for each club $C$, there is a stationary $S = \{\alpha : C \cap \alpha \supset a_\alpha$ cofinal in $\alpha\}$; show/use that a ccc poset "adds no new clubs")*

And confusing to reverse the order

**Proposition 19.** *The measure algebra on $2^I$ will be denoted $\mathcal{M}_I$; let $G_\kappa$ be a generic. If $\omega \leq \lambda < \kappa$ then $\mathcal{M}^\kappa_\lambda = \{b \times 2^{\kappa \setminus \lambda} : b \in \mathcal{M}_\lambda\}$ is isomorphic to $\mathcal{M}_\lambda$. It is completely embedded of course.*

*Then if $G_\lambda = G_\kappa \cap \mathcal{M}^\kappa_\lambda$, we have that $V[G_\lambda] \models G^+_\lambda \approx \mathcal{M}_{\kappa\setminus\lambda}$, and the set $V \cap \mathbb{R}$ is now meager and has outer measure 1 (dual to the Cohen results). Also, $V \cap \omega^\omega$ is dominating.*

*Proof.* The most direct is to show that $V \cap 2^\omega$ is meager. From the generic $G_\lambda$, we get a function $g \in 2^\omega$ given by $g(n) = 1$ if and only if the clopen set $[(n, 1)]$ is in $G$. For each $n \in \omega$, define the open set $U_n \subset 2^\omega$ given by $[\ g \restriction [2^n, 2^{n+1}\ )\ ]$. For each infinite $I \subset \omega$, the set $\bigcup_{n\in I} U_n$ is dense. Let $\dot{U}_n$ denote the canonical name for $U_n$. Back in $V$, we notice that for $x \in 2^\omega$, the set $D_x = \{b \in \mathcal{M}_\kappa : \{n \in \omega : b \cap [x \restriction [2^n, 2^{n+1})] \neq \emptyset\}$ is finite$\}$ is dense.

It is a nice "question" to show that $\mathcal{M}_\kappa$ is $\omega^\omega$-bounding (adds no unbounded real) and we skip.

Now we check that $V \cap 2^\omega$ has outer measure 1. Suppose that $\tau$ is a name of an open subset of $2^\omega$ and that $b \in \mathcal{M}_\lambda$ forces that $\tau$ is open, contains $V \cap 2^\omega$ and has measure less than $1 - \frac{1}{k}$. Working in $V$, define $W$ to be a subset of $2^\omega \times b$ given by $\bigcup\{(a, b) \times c : a, b \in [\mathbb{Q}]^2$ and $(\exists c \in \mathcal{M}_\lambda)\ c \Vdash (a, b) \subset \tau\}$. Since $\mathcal{M}_\lambda$ is ccc (every set has a measure), $W$ is a Borel set. Use Fubini's theorem to calculuate the measure of $W$. For each $x \in 2^\omega$, the measure of $W \cap (\{x\} \times b)$ is equal to $b$ (since $b \Vdash x \in \tau$). Choose a compact subset $K$ of $W$ with measure greater than

$(1-\frac{1}{2k})$ times the measure of $b$. Choose a $y \in b$ so that the measure of the fiber $K_y = K \cap (\{y\} \times 2^\omega)$ is greater than $1-\frac{1}{k}$. Finish with a compactness argument to find a positive measure set $c \subset b$ with $y \in c$ and an open set $U \supset K_y$ (a finite union of intervals) such that $c \Vdash U \subset \tau$. $\square$

**Question 4.** *Assume CH in $V$; let $P = Fn(\omega_2, 2)$ and $Q = \mathcal{M}_{\omega_2}$. Let $G \times H$ be generic for $P \times Q$. In $V[G \times H]$, every set of $\aleph_1$ many reals has measure 0, but is there is a non-meager set of size $\aleph_1$?*