Which wave system is more turbulent: strongly or weakly nonlinear?

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Outline

Some (accepted?) statements:

- Strong turbulence ⇒ High nonlinearity
- Wave turbulence ⇒ small amplitudes & resonant *N*-wave interactions (plus quasi-resonant)

Evidence against these statements:

- Discovery of a new nonlinear transfer mechanism, stronger at **intermediate values of nonlinearity**
- Mechanism favours transfers towards **non-resonant** triads rather than quasi-resonant / resonant triads
- **Robust result**, backed up with Direct Numerical Simulations of nonlinear wave systems

Interaction Representation

- Equations for generic finite-sized 3-wave system:

$$\frac{\partial b_{\boldsymbol{k}}}{\partial t} = \sum_{\boldsymbol{k_1},\boldsymbol{k_2} \in \mathbb{Z}^2} V^{(1)}_{12,\boldsymbol{k}}\, \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\, b_{\boldsymbol{k_1}}\, b_{\boldsymbol{k_2}}\, e^{i(\omega_{\boldsymbol{k}} - \omega_1 - \omega_2)t}$$

$$+ \sum_{\boldsymbol{k_1},\boldsymbol{k_2} \in \mathbb{Z}^2} V^{(2)}_{12,\boldsymbol{k}}\, \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2)\, \overline{b_{\boldsymbol{k_1}}}\, b_{\boldsymbol{k_2}}\, e^{i(\omega_{\boldsymbol{k}} + \omega_1 - \omega_2)t}$$

$$+ \sum_{\boldsymbol{k_1},\boldsymbol{k_2} \in \mathbb{Z}^2} V^{(3)}_{12,\boldsymbol{k}}\, \delta(\boldsymbol{k} + \boldsymbol{k}_1 + \boldsymbol{k}_2)\, \overline{b_{\boldsymbol{k_1}}}\, \overline{b_{\boldsymbol{k_2}}}\, e^{i(\omega_{\boldsymbol{k}} + \omega_1 + \omega_2)t}$$

Robust Instability Mechanism (1/3)

$$\frac{\partial b_{\boldsymbol{k}}}{\partial t} = \sum_{\boldsymbol{k_1},\boldsymbol{k_2} \in \mathbb{Z}^2} V_{12,\boldsymbol{k}}^{(1)}\, \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\, b_{\boldsymbol{k_1}}\, b_{\boldsymbol{k_2}}\, e^{i(\omega_{\boldsymbol{k}} - \omega_1 - \omega_2)t}$$

$$+\ \mathbf{etc} \ldots$$

- Limit of small amplitudes ($|b_{\boldsymbol{k_j}}| \ll 1, \quad \text{for all } j$): phases $e^{i(\omega_{\boldsymbol{k}} - \omega_1 - \omega_2)t}$ rotate faster than amplitudes $b_{\boldsymbol{k_j}}$ $\Rightarrow$ each term in the RHS averages to zero at intermediate time scales

(Exception: exact resonances $\omega_{\boldsymbol{k}} - \omega_1 - \omega_2 = 0$, but these are **irrelevant** for the mechanism of this talk)

Robust Instability Mechanism (2/3)

$$\frac{\partial b_{\boldsymbol{k}}}{\partial t} = \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2 \in \mathbb{Z}^2} V_{12,\boldsymbol{k}}^{(1)}\,\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\, b_{\boldsymbol{k}_1}\, b_{\boldsymbol{k}_2}\, e^{i(\omega_{\boldsymbol{k}} - \omega_1 - \omega_2)t}$$

$$+\ \textbf{etc}\ \ldots$$

- Increase initial amplitudes $b_{\boldsymbol{k}_j}$ from infinitesimally small to finite values:

⇒ $b_{\boldsymbol{k}_j}$'s nonlinear oscillation frequency, Γ, grows proportional to the amplitudes, until **resonance** occurs:

nonlinear $\Gamma \sim \omega_{\boldsymbol{k}} - \omega_1 - \omega_2$ linear

⇒ some terms in the RHS will contain **zero modes**

Robust Instability Mechanism (3/3)

$$\frac{\partial b_{\boldsymbol{k}}}{\partial t} = \sum_{\boldsymbol{k_1},\boldsymbol{k_2} \in \mathbb{Z}^2} V_{12,\boldsymbol{k}}^{(1)} \, \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) \, b_{\boldsymbol{k_1}} \, b_{\boldsymbol{k_2}} \, e^{i(\omega_{\boldsymbol{k}} - \omega_1 - \omega_2)t}$$

$$+ \textbf{etc} \ldots$$

- **Zero modes** in RHS:

$\Rightarrow$ Some amplitudes $b_{\boldsymbol{k}}$ grow linearly in time, without bound: $b_{\boldsymbol{k}}(t) \sim c\, t$, $c = \text{const.}$, **even from zero i.c.**

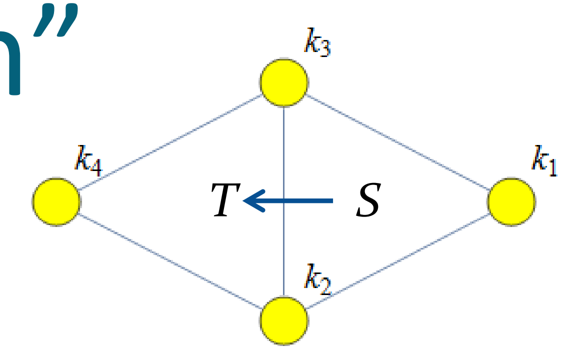
$\Rightarrow$ Far more robust than modulational instability!!!

- **Can this mechanism be observed/modelled?**

Quantitative Studies of Strong Transfer Mechanism

- Finite-dimensional ODE model:
  Two triads connected via two common modes
  - Initial conditions: $b_{\boldsymbol{k}}(0) = A\, b_{\boldsymbol{k}}^{\mathrm{ref}}(0)$, A: arbitrary const.
  - Energy flows from **source** triad to **target** triad
  - Physical mechanism and "linear-nonlinear" resonance
- Full direct numerical simulation of a PDE model
  - Initial conditions: $b_{\boldsymbol{k}}(0) = A\, b_{\boldsymbol{k}}^{\mathrm{ref}}(0)$, A: arbitrary const.
  - Study turbulent cascades & transfer efficiency as a function of A

ODE Model: The "Atom"



- Wave-vectors: $\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3, \quad \boldsymbol{k}_2 + \boldsymbol{k}_3 = \boldsymbol{k}_4$
- Frequency mismatches:

$$\delta_S = \omega_1 + \omega_2 - \omega_3 \qquad \delta_T = \omega_2 + \omega_3 - \omega_4$$

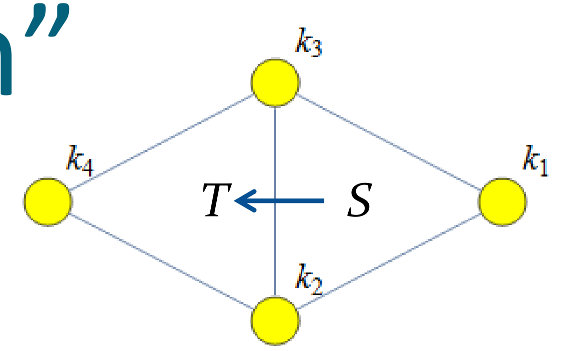
- Take $\delta_S = 0$ <u>for simplicity</u> (not essential)
- Equations of motion:

$$\dot{B}_1 = S_1\, B_2^*\, B_3$$
$$\dot{B}_2 = S_2\, B_1^*\, B_3 + T_1\, B_3^*\, B_4\, e^{i\delta_T\, t}$$
$$\dot{B}_3 = S_3\, B_1\, B_2 + T_2\, B_2^*\, B_4\, e^{i\delta_T\, t}$$
$$\dot{B}_4 = T_3\, B_2\, B_3\, e^{-i\delta_T\, t}, \qquad B_j(t) \in \mathbb{C}$$

- Initial Conditions:

$$B_1(0), B_2(0), B_3(0) \neq 0 \qquad B_4(0) = 0$$

ODE Model: The "Atom"



$$\dot{B}_1 = S_1\, B_2^*\, B_3$$
$$\dot{B}_2 = S_2\, B_1^*\, B_3 + T_1\, B_3^*\, B_4\, e^{i\delta_T\, t}$$
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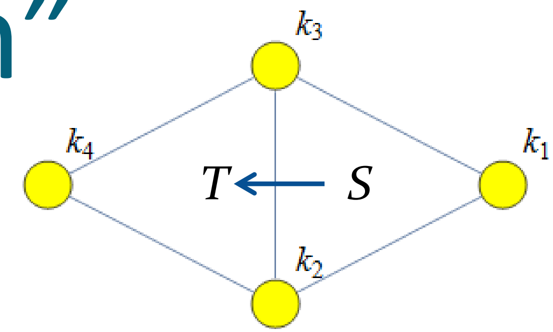
$$\dot{B}_4 = T_3\, B_2\, B_3\, e^{-i\delta_T\, t},$$

$B_j(t) \in \mathbb{C}$

- $B_1(0), B_2(0), B_3(0) \neq 0 \qquad B_4(0) = 0$
- 8-dimensional phase space
- 2 quadratic conservation laws & 2 slave variables
  ⇒ Effectively 4 **degrees of freedom**
- **Boundedness:** ∃ positive-definite conservation law

ODE Model: The "Atom"



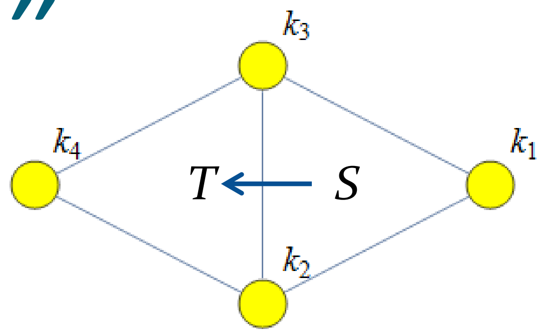
$$\dot{B}_1 = S_1\, B_2^*\, B_3$$
$$\dot{B}_2 = S_2\, B_1^*\, B_3 + T_1\, B_3^*\, B_4\, e^{i\delta_T\, t}$$
$$\dot{B}_3 = S_3\, B_1\, B_2 + T_2\, B_2^*\, B_4\, e^{i\delta_T\, t}$$
$$\dot{B}_4 = T_3\, B_2\, B_3\, e^{-i\delta_T\, t}, \qquad B_j(t) \in \mathbb{C}$$

- Initially $B_4(0) = 0, \quad B_1(0), B_2(0), B_3(0) \neq 0$
- Assume $|B_4(t)|$ remains small for all times $\Rightarrow$ system is further approximated by:

$$\dot{B}_1 = S_1\, B_2^*\, B_3 \qquad \dot{B}_2 = S_2\, B_1^*\, B_3 \qquad \dot{B}_3 = S_3\, B_1\, B_2$$

$$\dot{B}_4 = T_3\, B_2\, B_3\, e^{-i\delta_T\, t}$$

ODE Model: The "Atom"



- Assume $B_4(t)$ is small:

$$\dot{B}_1 = S_1\, B_2^*\, B_3 \qquad \dot{B}_2 = S_2\, B_1^*\, B_3 \qquad \dot{B}_3 = S_3\, B_1\, B_2$$

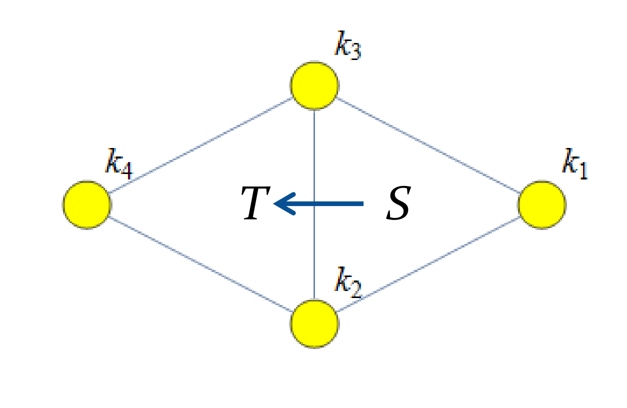
$$\dot{B}_4 = T_3\, B_2\, B_3\, e^{-i\delta_T\, t}$$

- B_1, B_2, B_3 satisfy the usual integrable triad equations
- $B_4(t)$ obtained by quadratures after B_2, B_3 are known
- Triad: Jacobi Elliptic functions
  $\Rightarrow$ **bounded**, quasi-periodic motion

ODE Model: The “Atom”

$$\dot{B}_1 = S_1\, B_2^*\, B_3$$
$$\dot{B}_2 = S_2\, B_1^*\, B_3$$
$$\dot{B}_3 = S_3\, B_1\, B_2$$

$$\dot{B}_4 = T_3\, B_2\, B_3\, e^{-i\delta_T\, t}$$



- Amplitude-Phase representation:

$$B_j(t) = \left|B_j(t)\right| e^{i\,\varphi_j(t)}$$

- $|B_1(t)|, |B_2(t)|, |B_3(t)|,\ \ \varphi(t) = \varphi_1(t) + \varphi_2(t) - \varphi_3(t)$ are periodic functions with nonlinear frequency

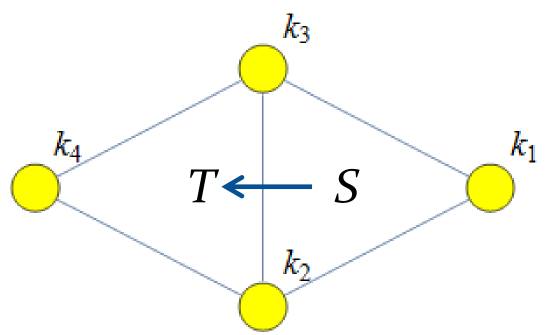
$$\Gamma = \ \Gamma(|B_1(0)|, |B_2(0)|, |B_3(0)|;\ \ \varphi(0))$$

- Homogeneity: $\Gamma(A\,x, A\,y, A\,z; \alpha) = A\,\Gamma(x, y, z; \alpha)$

ODE Model: The "Atom"

$$\dot{B}_1 = S_1\, B_2^*\, B_3$$
$$\dot{B}_2 = S_2\, B_1^*\, B_3$$
$$\dot{B}_3 = S_3\, B_1\, B_2$$

$$\dot{B}_4 = T_3\, B_2\, B_3\, e^{-i\delta_T\, t}$$

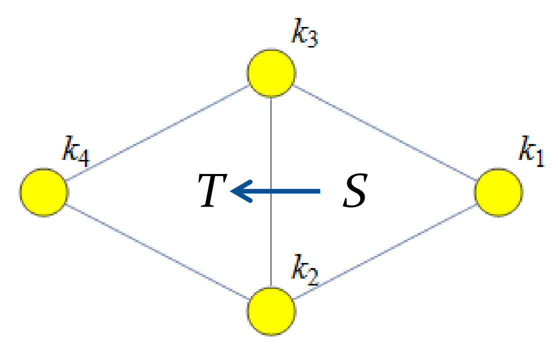


- $B_2(t) = |B_2(t)| e^{i\,(\varphi_2^{\mathrm{per}}(t) + \Omega_2\, t)}$
  $B_3(t) = |B_3(t)| e^{i\,(\varphi_3^{\mathrm{per}}(t) + \Omega_3\, t)}$ (exact triad solutions)
- $|B_2(t)|, |B_3(t)|, \varphi_2^{\mathrm{per}}(t), \varphi_3^{\mathrm{per}}(t)$ are periodic: nonlinear frequency $\Gamma \sim$ amplitudes (homogeneity)
- Ω_2, Ω_3: precession frequencies, also $\sim$ amplitudes

ODE Model: The "Atom"

$$\dot{B}_1 = S_1\, B_2^*\, B_3$$
$$\dot{B}_2 = S_2\, B_1^*\, B_3$$
$$\dot{B}_3 = S_3\, B_1\, B_2$$

$$\boldsymbol{\dot{B}_4 = T_3\, B_2\, B_3\, e^{-i\delta_T\, t}}$$



- Solving by quadratures:
- $B_4(t) = T_3 \int_0^t f^{\text{per}}(\tau)\; e^{i\,(\Omega_2+\Omega_3-\delta_T)\tau}\; d\tau$
- $f^{\text{per}}(t) \in \mathbb{C}$: periodic, nonlinear frequency Γ

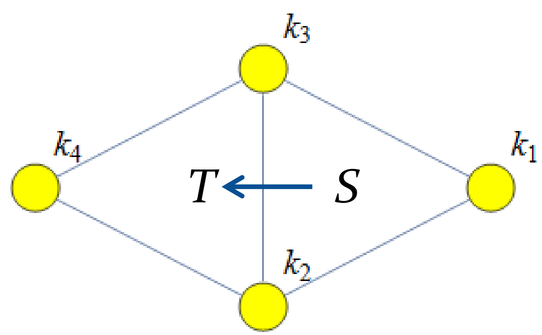
$\Rightarrow$ Unbounded growth if resonance occurs:

$$n\,\Gamma + \Omega_2 + \Omega_3 - \delta_T = 0,\ \text{for some}\ \ n \in \mathbb{Z}$$

ODE Model: The “Atom”

$$\dot{B}_1 = S_1\, B_2^*\, B_3$$
$$\dot{B}_2 = S_2\, B_1^*\, B_3$$
$$\dot{B}_3 = S_3\, B_1\, B_2$$

$$\dot{\boldsymbol{B}}_4 = \boldsymbol{T}_3\, \boldsymbol{B}_2\, \boldsymbol{B}_3\, \boldsymbol{e}^{-i\delta_T\, t}$$



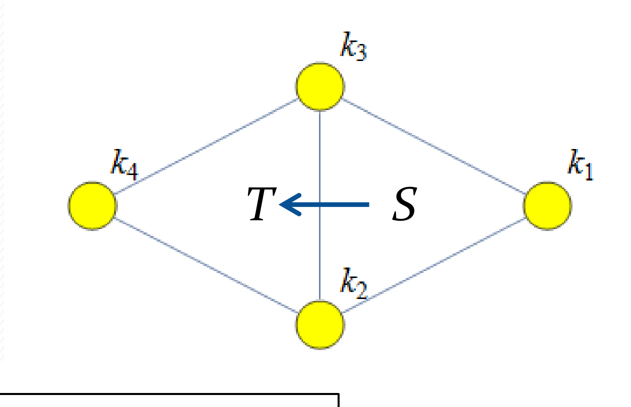
- $B_4(t) = T_3 \int_0^t f^{\text{per}}(\tau)\; e^{i\,(\Omega_2+\Omega_3-\delta_T)\tau}\, d\tau$
- Fine-tuning initial conditions via simple re-scaling:

$$|B_j(0)| \to A\, |B_j(0)|$$

$\Rightarrow$ Instability if $A = A_n \left(\equiv \dfrac{\delta_T}{n\,\Gamma^{\text{ref}} + \Omega_2^{\text{ref}} + \Omega_3^{\text{ref}}} \right)$, for some $n \in \mathbb{Z}$

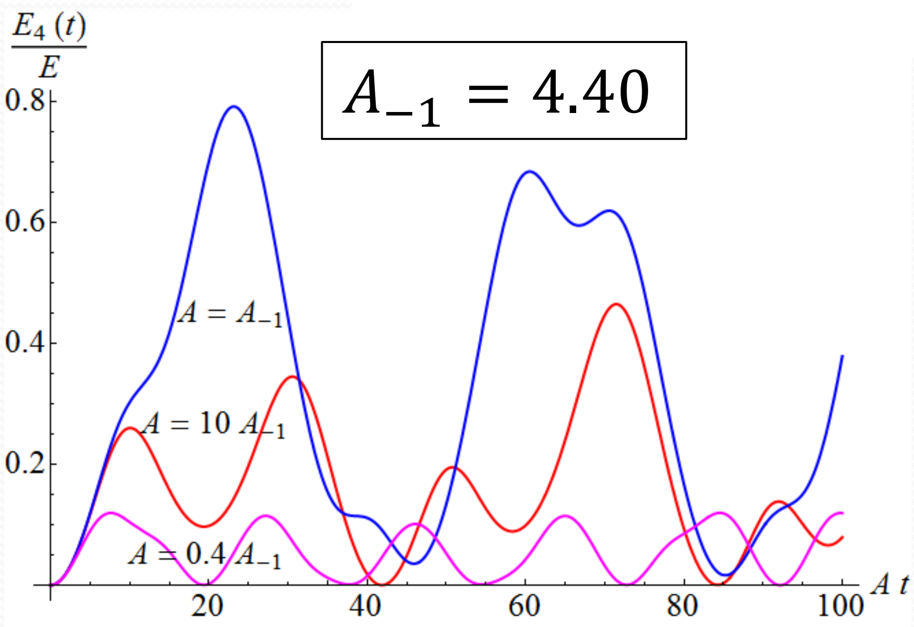
ODE Model: Numerical Study

$$\dot{B}_1 = S_1\, B_2^*\, B_3$$
$$\dot{B}_2 = S_2\, B_1^*\, B_3 + T_1\, B_3^*\, B_4\, e^{i\delta_T t}$$
$$\dot{B}_3 = S_3\, B_1\, B_2 + T_2\, B_2^*\, B_4\, e^{i\delta_T t}$$
$$\dot{B}_4 = T_3\, B_2\, B_3\, e^{-i\delta_T t}$$



$$\delta_T = -\frac{8}{9},$$
$$S_1 = 1, S_2 = 9, S_3 = -8$$
$$T_1 = -1, T_2 = \frac{8}{3}, T_2 = -\frac{9}{5}$$

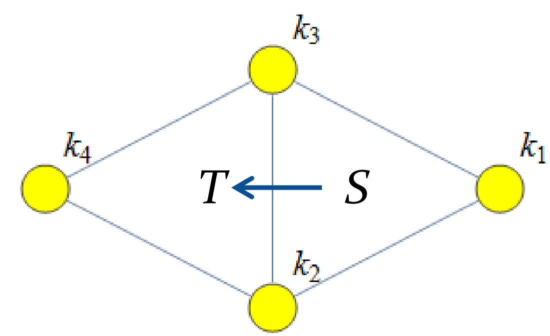
$$B_1(0) = 0.007772\ A$$
$$B_2(0) = 0.038582\ A$$
$$B_3(0) = -\,0.0358876\, i\, A$$
$$B_4(0) = 0$$



$$A_{-1} = 4.40$$

ODE Model: Numerical Study

Introduce an ε-family of systems:



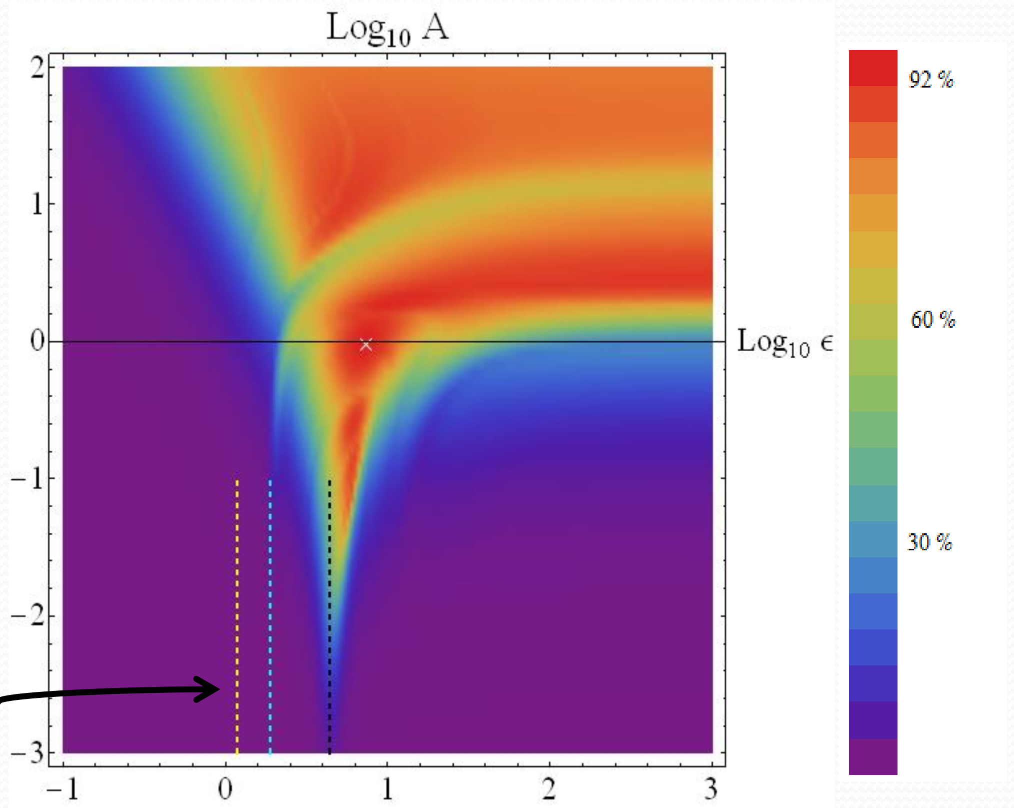
$$\dot{B}_1 = S_1\, B_2^*\, B_3$$
$$\dot{B}_2 = S_2\, B_1^*\, B_3 + \varepsilon\, T_1\, B_3^*\, B_4\, e^{i\delta_T\, t}$$
$$\dot{B}_3 = S_3\, B_1\, B_2 + \varepsilon\, T_2\, B_2^*\, B_4\, e^{i\delta_T\, t}$$
$$\dot{B}_4 = T_3\, B_2\, B_3\, e^{-i\delta_T\, t}$$

where $0 \le \varepsilon \le 1$

- $\varepsilon = 0$: Integrable system, with new resonant instability
- $\varepsilon = 1$: Full original system
- Study transfer efficiency as function of A & ε

ODE Model: Analysis of Results

- **Transfer Efficiency as function of A & ε**



$$A_n = \frac{\delta_T}{n\,\Gamma^{\text{ref}} + \Omega_2^{\text{ref}} + \Omega_3^{\text{ref}}}$$

Predicted Resonances

A_{-3}, A_{-2}, A_{-1}

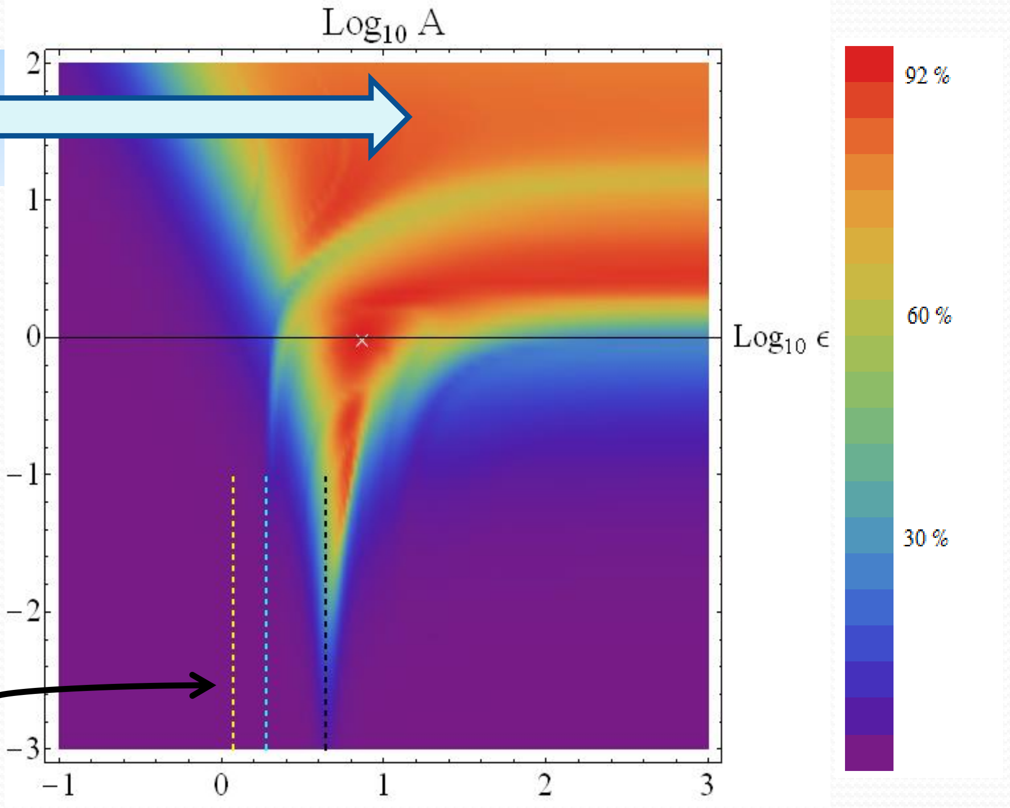
ODE Model: Analysis of Results

- **Transfer Efficiency as function of A & ε**

Efficiency Plateau:
$|\omega_2 + \omega_3 - \omega_4| \leq \Gamma, \quad \varepsilon \gg 1$

$$A_n = \frac{\delta_T}{n\,\Gamma^{\mathrm{ref}} + \Omega_2^{\mathrm{ref}} + \Omega_3^{\mathrm{ref}}}$$

Predicted Resonances
A_{-3}, A_{-2}, A_{-1}

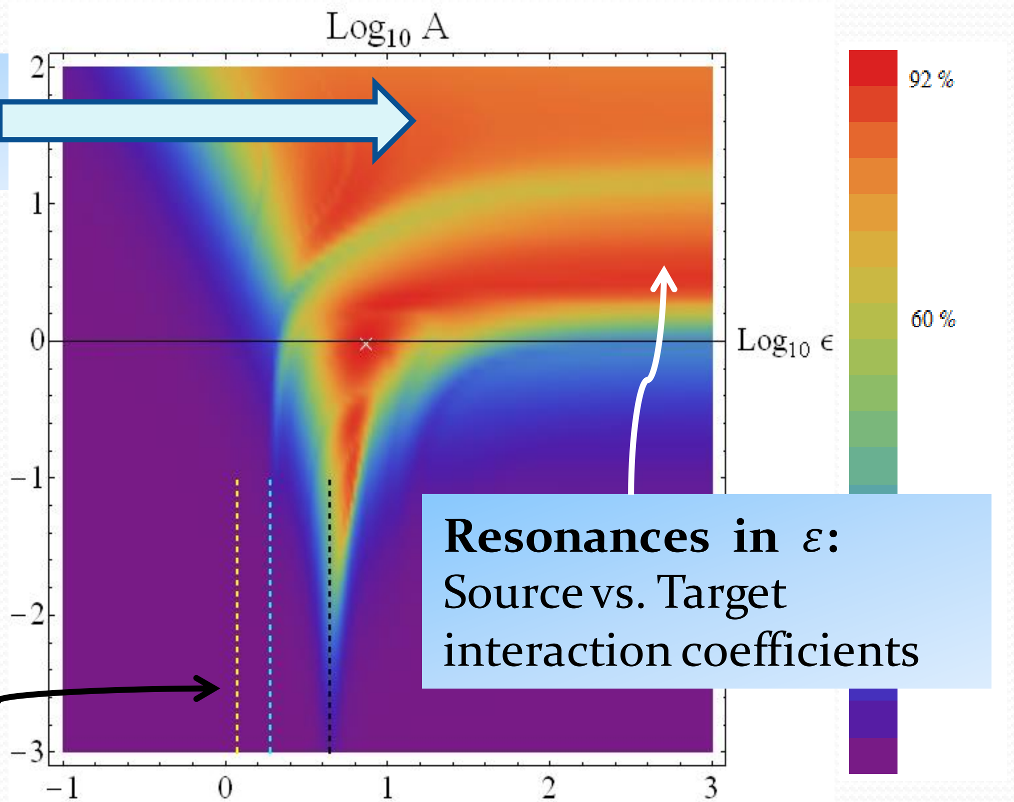


ODE Model: Analysis of Results

- **Transfer Efficiency as function of A & ε**

Efficiency Plateau:

$$|\omega_2 + \omega_3 - \omega_4| \leq \Gamma, \quad \varepsilon \gg 1$$



$$A_n = \frac{\delta_T}{n\,\Gamma^{\text{ref}} + \Omega_2^{\text{ref}} + \Omega_3^{\text{ref}}}$$

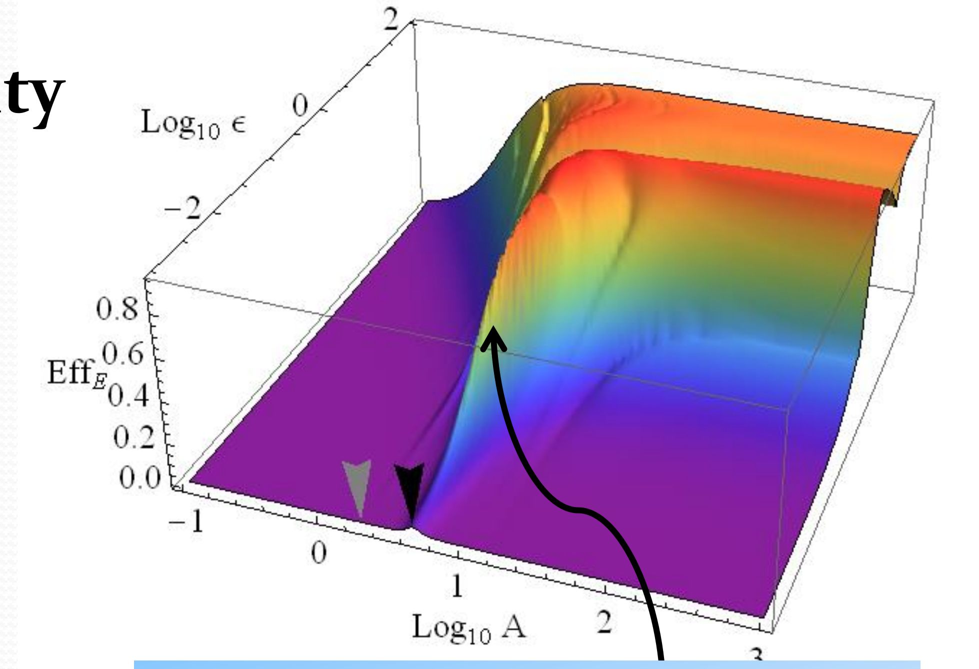
Predicted Resonances

A_{-3}, A_{-2}, A_{-1}

Resonances in ε:
Source vs. Target interaction coefficients

ODE Model: Analysis of Results

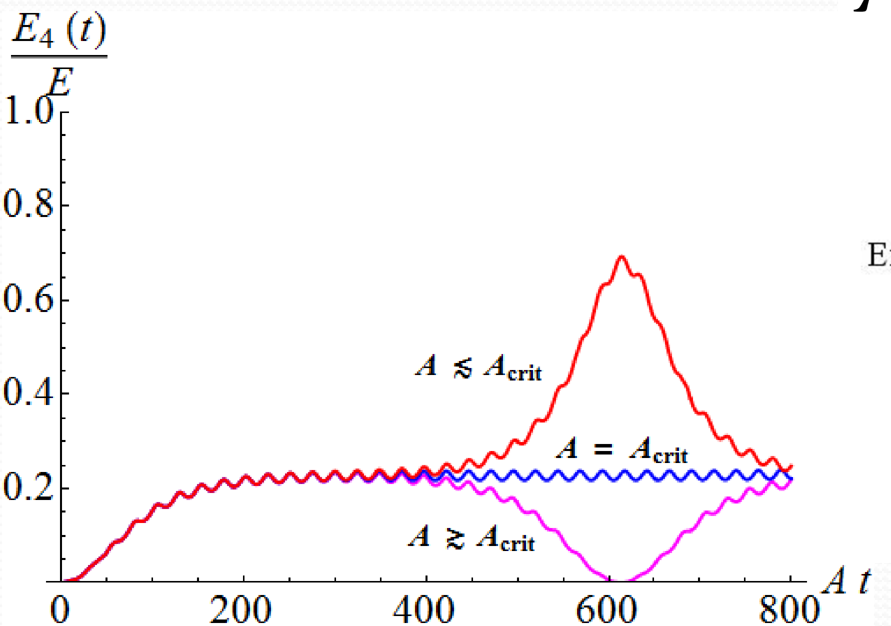
- **Nature of the Instability**

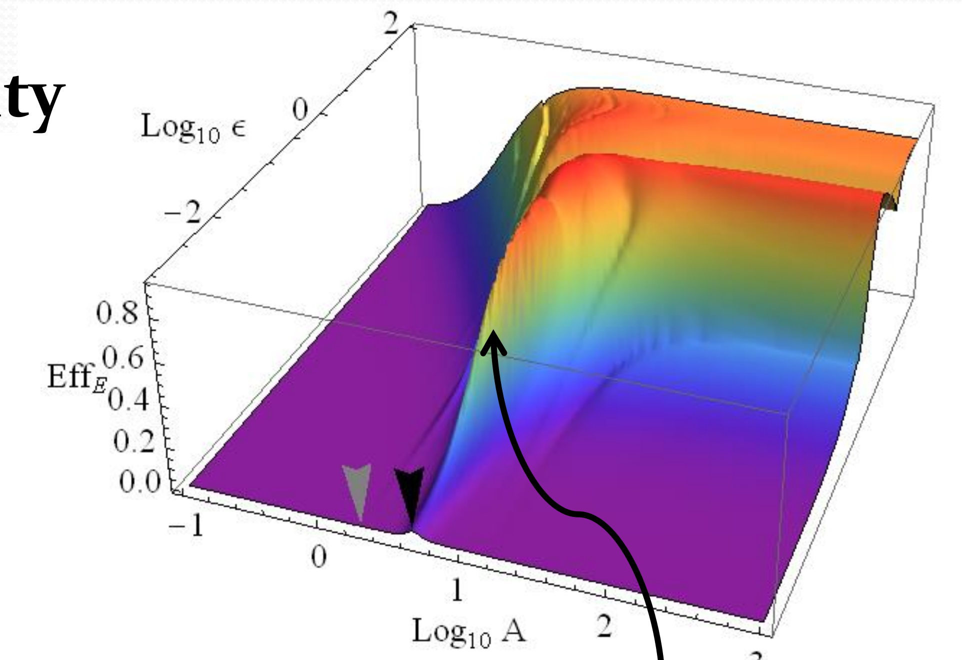


Stiff Ridge:
- Persistence of invariant manifold
- Unstable periodic orbit

ODE Model: Analysis of Results

- **Nature of the Instability**





Stiff Ridge:
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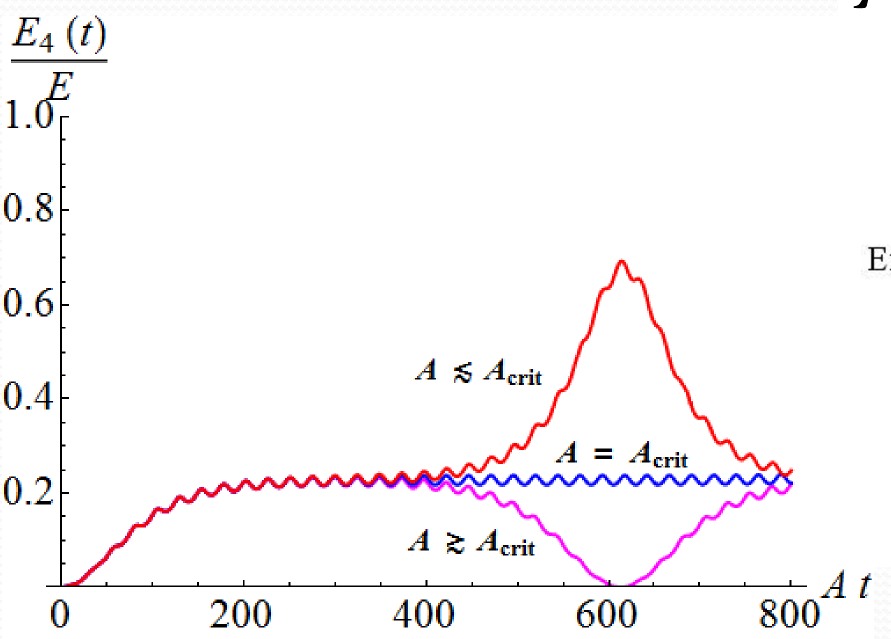
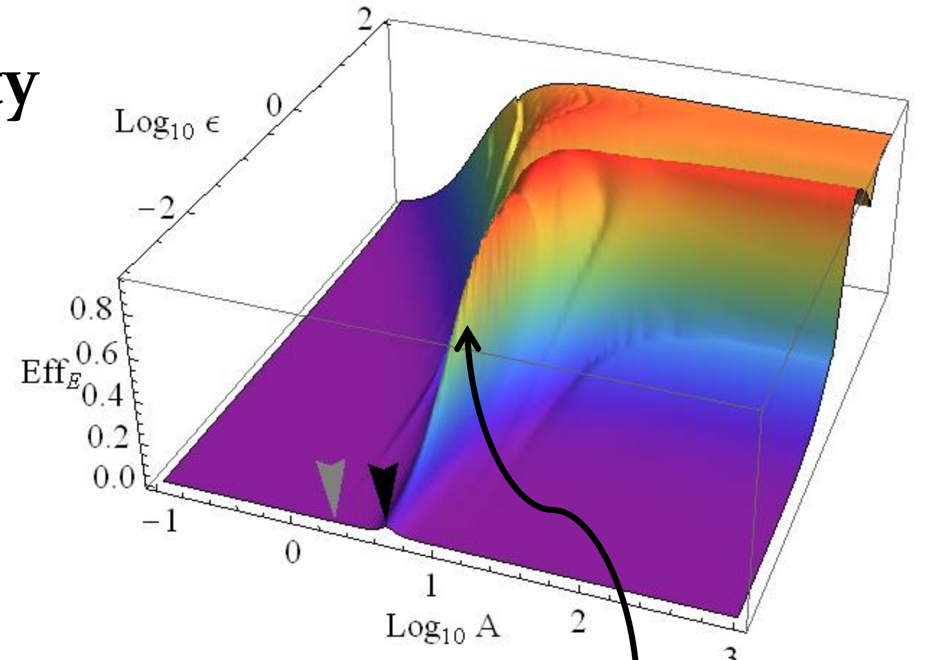
ODE Model: Analysis of Results

- **Nature of the Instability**





Stiff Ridge:
- Persistence of invariant manifold
- Unstable periodic orbit

- Three Lyapunov exponents
- Ratios (-1):(-2):(3)

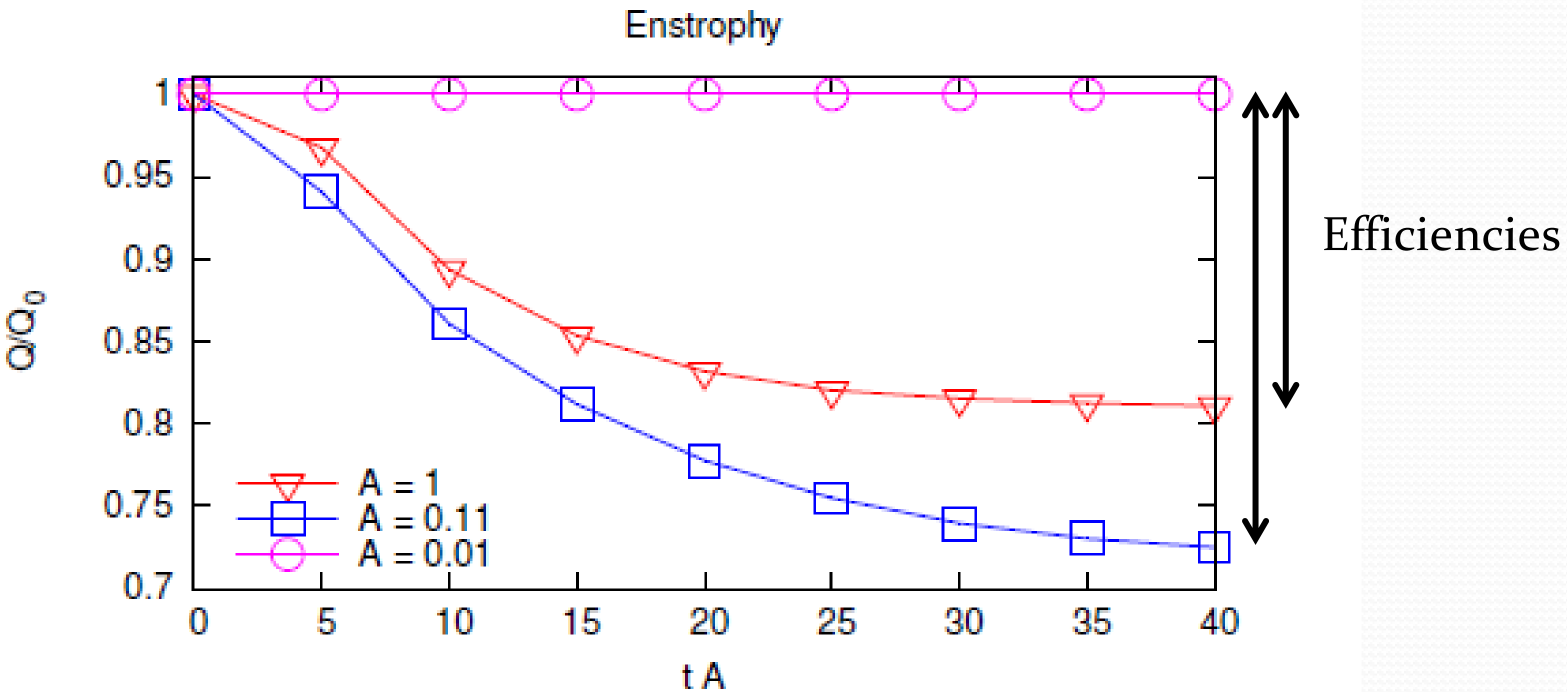
Barotropic Vorticity Equation

$$\frac{\partial}{\partial t}\left(\Delta\psi - \alpha^2\psi\right) - \beta\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} = 0$$

- **Direct Numerical Simulations**
  - Pseudospectral
  - Resolution 128 x 128
  - Initial conditions at large scales, with an overall re-scaling factor *A* in front
  - Dissipation at small scales: ENSTROPHY direct cascade

Barotropic Vorticity Equation

$$\frac{\partial}{\partial t}\left(\Delta\psi - \alpha^2\psi\right) - \beta\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} = 0$$



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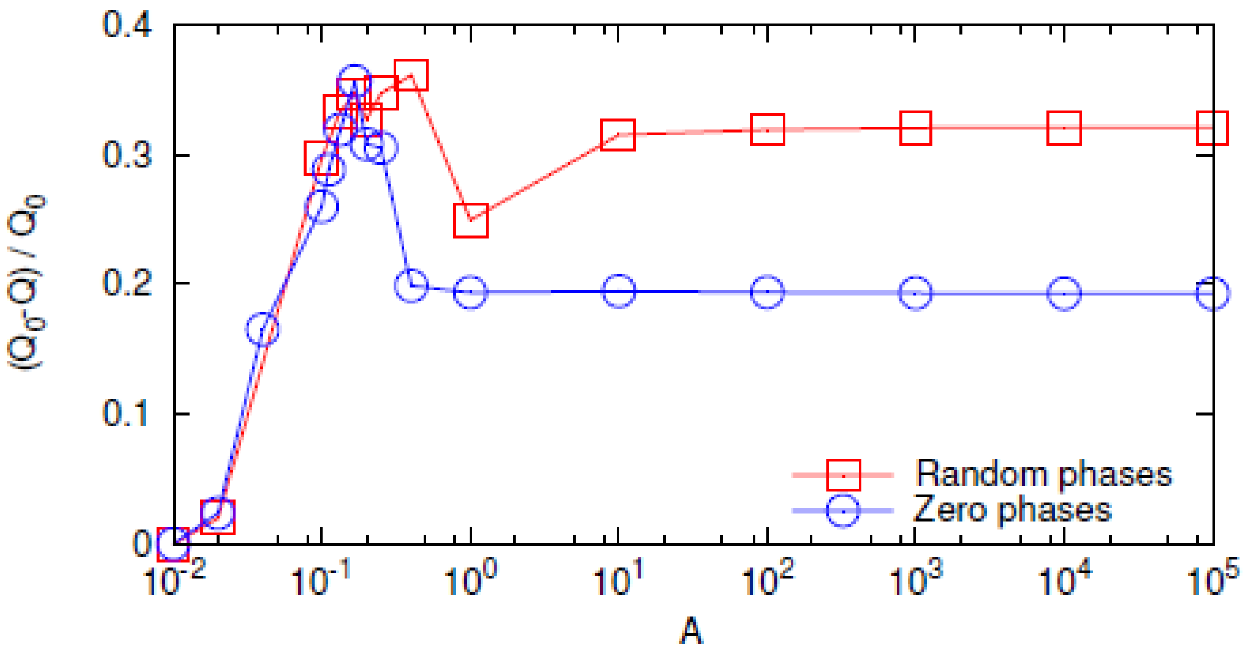
Efficiency as a function of *A*



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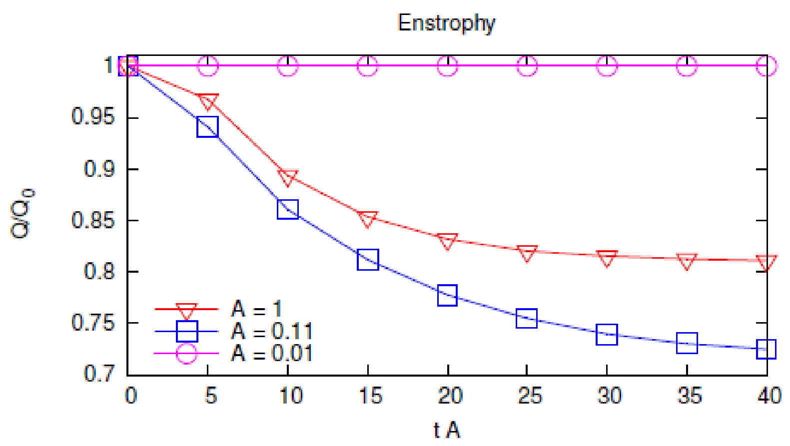

VIDEOS

Conclusions

- Robust energy transfer mechanism towards non-resonant triads
- Analytically derived and verified numerically:
  - ODE “Atom” model
  - Direct numerical simulations of a full PDE model
- Implications of this mechanism:
  - Understanding turbulent cascades as a natural selection mechanism of triads
  - Yet another mechanism of rogue wave generation (triggered either by forcing or dissipation)
  - New paradigm for a complete theory of wave turbulence

Thank You!

- This Paper: ArXiv version http://arxiv.org/abs/1305.5517

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