Modulational instability of surface gravity waves on water of finite depth with constant vorticity

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WIND $\nearrow$ Pressure at the sea surface $\rightarrow$ wave amplification

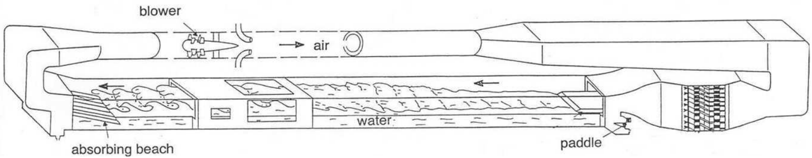
WIND $\searrow$ Shear flow $\rightarrow$ Vorticity in water

CURRENT + BOTTOM FRICTION $\rightarrow$ Shear flows $\rightarrow$ Vorticity

A very brief summary of our previous results on wind effect on steep water wave dynamics

◇ Experiments (LASIF)

- Air sea fluxes are strongly enhanced in the presence of steep wave events
- Wind increases the duration and amplitude of SWE



Sketch of the Large Air-Sea Interaction Facility (Marseille).

- **Water tank dimensions**: 40m long, 2.6m wide, 1m deep
- **Wind tunnel dimensions**: 40m long, 3.2m wide, 1.6m high
- **Paddle**: 0.5Hz - 2Hz (regular or random waves)

$\diamondsuit$ Numerical simulations (Sheltering mechanism $\rightarrow$ Pressure)

- Wind increases the lifetime and amplitude of SWE
- Wind driven current may play a significant role in their persistence

Calculation of surface water waves propagating steadily on a rotational current

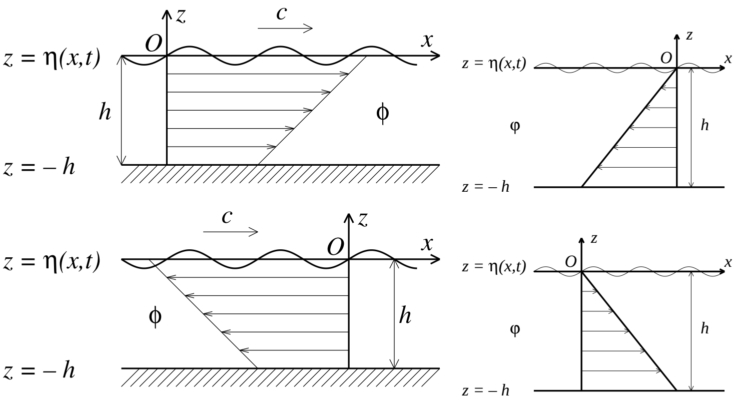
- Tsao(1959), Dalrymple (1974), Brevik (1979), Simmen & Saffman (1985), Teles da Silva & Peregrine (1988), Kishida & Sobey (1988), Pak & Chow (2009), etc.
- Constantin & Strauss (2004), Constantin & Escher (2004), Hur (2008), Wahlen (2009), Constantin (2011), Kozlov & Kuznetsov (2013)

For a general description of the problem of waves on current refer to the following reviews

- Peregrine (1976)
- Jonsson (1990)
- Thomas & Klopman (1997)

Few studies on the modulational instability (Benjamin-Feir instability) of progressive waves in the presence of vorticity

- Johnson (1976)
- Oikawa, Chow & Benney (1987)
- Li, Hui & Donelan (1987)
- Okamura & Oikawa (1989)
- Baumstein (1998)
- Choi (2009)
- Nwogu (2009)
- Thomas, Kharif & Manna (2012)



Shear profiles in the fixed and moving reference frames
Waves propagating downstream ($\Omega > 0$)
Waves propagating upstream ($\Omega < 0$)

- In the fixed reference frame the shear current is

$$\mathbf{U} = (\Omega z + U_0, 0)^T$$

- In the moving reference frame with velocity U_0 the shear current is

$$\mathbf{U} = (\Omega z, 0)^T$$

We assume that the wave-induced motion in water is potential

The Kelvin-Lagrange theorem states that the vorticity is conserved
$\Rightarrow \Omega$ remains constant

- In the moving frame the water velocity is described by

$$\mathbf{u} = (\Omega z + \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial z})^T$$

The Euler equation writes

$$\nabla(\varphi_t + \frac{1}{2}\mathbf{u}^2 + \frac{\mathrm{p}}{\rho} + gz) = \mathbf{u} \wedge (\nabla \wedge \mathbf{u})$$

Introducing the stream function that satisfies the Cauchy-Riemann conditions $\psi_z = \varphi_x$ and $\psi_x = -\varphi_z$, then

$$\mathbf{u} \wedge (\nabla \wedge \mathbf{u}) = \nabla(\frac{1}{2}\Omega^2 z^2 + \Omega\psi)$$

Hence

$$\nabla(\varphi_t + \frac{1}{2}\mathbf{u}^2 + \frac{\mathrm{p}}{\rho} + gz - \frac{1}{2}\Omega^2 z^2 - \Omega\psi) = 0$$

$$\varphi_t + \frac{1}{2}\mathbf{u}^2 + \frac{\mathrm{p}}{\rho} + gz - \frac{1}{2}\Omega^2 z^2 - \Omega\psi = \mathrm{C}(t)$$

$$\varphi_t + \frac{1}{2}(\varphi_x^2 + \varphi_z^2) + \Omega z \varphi_x + \frac{\mathrm{p}}{\rho} + gz - \Omega\psi = \mathrm{C}(t)$$

The governing equations are

$$\nabla^2\varphi = 0, \qquad -h < z < \eta(x,t) \qquad (1)$$

$$\varphi_z = 0, \qquad y = -h \qquad (2)$$

$$\eta_t + (\varphi_x + \Omega\eta)\eta_x - \varphi_z = 0, \qquad z = \eta(x,t) \qquad (3)$$

$$\varphi_t + \frac{1}{2}(\varphi_x^2 + \varphi_z^2) + \Omega\eta\varphi_x + g\eta - \Omega\psi = 0, \qquad z = \eta(x,t) \qquad (4)$$

The atmospheric pressure at the free surface is assumed constant.

- Equations (1)-(4) are invariant under the following transformations : $\varphi \to -\varphi$, $t \to -t$, $\Omega \to -\Omega$ and $\psi \to -\psi$. Hence, there is no loss of generality if the study is restricted to waves with positive phase velocities so long as both positive and negative values of Ω are considered.

- To reduce the number of dependent variables, we derive the dynamic boundary condition with respect to x and we use the Cauchy-Riemann conditions to eliminate the stream function. The dynamic boundary condition at the free surface becomes

$$\begin{aligned} \Phi_{tx} + \Phi_{tz}\eta_x + \Phi_x(\Phi_{xx} + \Phi_{xz}\eta_x) + \Phi_z(\Phi_{xz} + \Phi_{zz}\eta_x) \\ +\Omega\eta_x\Phi_x + \Omega\eta(\Phi_{xx} + \Phi_{xz}\eta_x) + g\eta_x + \Omega(\Phi_z - \Phi_x\eta_x) = 0 \end{aligned} \tag{5}$$

The notation Φ means that φ is calculated at the free surface.

Approximate approach

Multiple scale method

- A weakly nonlinear wave train propagating on finite depth at the free surface of a uniform shear current of constant intensity Ω

$$\eta(x,t) = \frac{1}{2}(\epsilon a(\xi,\tau)\exp[i(kx-\omega t)] + c.c) + \mathcal{O}(\epsilon^2)$$

$$\text{where } \xi = \epsilon(x - c_g t) \text{ and } \tau = \epsilon^2 t$$

- The evolution of the complex envelope is governed by the vor-NLS equation

$$ia_\tau + La_{\xi\xi} + N \mid a \mid^2 a = 0$$

- Explicit formulas for L and N are derived

Let $\mu = kh$, $\sigma = \tanh(\mu)$, $r = c_g/c_p$, $\overline{\Omega} = \Omega/\omega$ and $X = \sigma\overline{\Omega}$

$$L=\frac{\omega}{k^2\sigma(2+X)}\{\mu(1-\sigma^2)[1-\mu\sigma+(1-r)X]-\sigma r^2\}$$

$$N=-\frac{\omega k^2(U+VW)}{2(1+X)(2+X)\sigma^4}$$

$$U=9-12\sigma^2+13\sigma^4-2\sigma^6+(27-18\sigma^2+15\sigma^4)X$$
$$+(33-3\sigma^2+4\sigma^4)X^2+(21+5\sigma^2)X^3+(7+2\sigma^2)X^4+X^5$$

$$V=(1+X)^2(1+r+\mu\overline{\Omega})+1+X-r\sigma^2-\mu\sigma X$$

$$W=2\sigma^3\frac{(1+X)(2+X)+r(1-\sigma^2)}{\sigma r(r+\mu\overline{\Omega})-\mu(1+X)}$$

The coupling between the mean flow and vorticity

At third-order, before deriving the vor-NLS equation, presented previously, the following equations emphasize the coupling between the mean flow $\phi_{01\xi}$, due to modulation, and vorticity, $-\overline{\Omega}$

$$\frac{k^3 c_p^2}{g\sigma}[(1+\sigma\overline{\Omega})^2(c_p+c_g+c_p kh\overline{\Omega})+c_p(1+\sigma\overline{\Omega})-(c_g+c_p kh\overline{\Omega})\sigma^2]\phi_{01\xi}A$$

$$-i\omega(2+\sigma\overline{\Omega})A_\tau$$

$$+\{c_g^2-gh+gh\sigma[\sigma+kh(1-\sigma^2)]+c_p kh(1-\sigma^2)(c_g-c_p kh\sigma)\overline{\Omega}\}A_{\xi\xi}$$

$$+\frac{k^5 c_p^2}{2g\sigma^3}[9-12\sigma^2+13\sigma^4-2\sigma^6+(27-18\sigma^2+15\sigma^4)\sigma\overline{\Omega}+$$

$$(33-3\sigma^2+4\sigma^4)\sigma^2\overline{\Omega}^2+(21+5\sigma^2)\sigma^3\overline{\Omega}^3+(7+2\sigma^2)\sigma^4\overline{\Omega}^4+\sigma^5\overline{\Omega}^5]\mid A\mid^2 A=0$$

where $A=c_p a/i\sigma$ is the envelope of the velocity potential

The mean flow

$$\phi_{01\xi} = \frac{gk\sigma(2+\sigma\overline{\Omega}) + k^2 c_p c_g (1-\sigma^2)}{c_p[c_g(c_g+\Omega h) - gh]} \mid A \mid^2$$

satisfies

$$\phi_{01\xi} \to \frac{gk(2+\overline{\Omega})}{hc_p(\Omega c_g - g)} \mid A \mid^2 \quad \text{as} \quad h \to +\infty$$

The coefficient in brackets of $\phi_{01\xi}A$ is of $\mathcal{O}(h)$ whereas $\phi_{01\xi}$ is of $\mathcal{O}(h^{-1})$ and consequently the product has a finite limit when $h \to +\infty$. Finally

$$iA_\tau - \frac{\omega(1+\overline{\Omega})^2}{k^2(2+\overline{\Omega})^3} A_{\xi\xi} + \frac{\omega k^2}{8c_p^2} \frac{\overline{\Omega}^2(2+\overline{\Omega})^2}{1+\overline{\Omega}} \mid A \mid^2 A$$

$$-\frac{\omega k^2}{8c_p^2}(4 + 6\overline{\Omega} + 6\overline{\Omega}^2 + \overline{\Omega}^3) \mid A \mid^2 A = 0$$

The NLS equation for the envelope of the elevation in deep water and constant vorticity is

$$ia_\tau - \frac{\omega(1+\overline{\Omega})^2}{k^2(2+\overline{\Omega})^3} a_{\xi\xi} + \frac{\omega k^2}{8} \frac{\overline{\Omega}^2(2+\overline{\Omega})^2}{1+\overline{\Omega}} \mid a \mid^2 a$$

$$-\frac{\omega k^2}{8}(4+6\overline{\Omega}+6\overline{\Omega}^2+\overline{\Omega}^3) \mid a \mid^2 a = 0$$

or

$$ia_\tau - \frac{\omega(1+\overline{\Omega})^2}{k^2(2+\overline{\Omega})^3} a_{\xi\xi} - \frac{\omega k^2}{8(1+\overline{\Omega})}(4+10\overline{\Omega}+8\overline{\Omega}^2+3\overline{\Omega}^3) \mid a \mid^2 a = 0$$

or

$$ia_\tau - \frac{\omega(1+\overline{\Omega})^2}{k^2(2+\overline{\Omega})^3} a_{\xi\xi} - \frac{\omega k^2}{8} \frac{(\overline{\Omega}+\ 2/3)(3\overline{\Omega}^2+6\overline{\Omega}+6)}{1+\overline{\Omega}} \mid a \mid^2 a = 0$$

with $\overline{\Omega} > -1$

For $\overline{\Omega} > -2/3 \Rightarrow$ NLS+ (self focusing NLS equation)

For $-1 < \overline{\Omega} < -2/3 \Rightarrow$ NLS- (defocusing NLS equation)

- The vor-NLS equation admits the following Stokes' wave solution

$$a = a_0 \exp(iNa_0^2\tau)$$

- Perturbation

$$a = a_0(1 + \delta a) \exp[i(\delta\omega + Na_0^2\tau)]$$

with

$$\delta a = (\delta a)_0 \exp[i(l\xi - \lambda\tau)]$$

$$\delta\omega = (\delta\omega)_0 \exp[i(l\xi - \lambda\tau)]$$

- Condition of instability with respect to sidebands perturbations

$$LN > 0$$

$$\text{Let } L = L_1\omega/k^2 \text{ and } N = N_1\omega k^2$$

- The growth rate of instability is

$$\gamma = \frac{l\omega}{k^2}\sqrt{2N_1L_1k^4a_0^2 - l^2L_1^2}$$

- Maximal growth rate

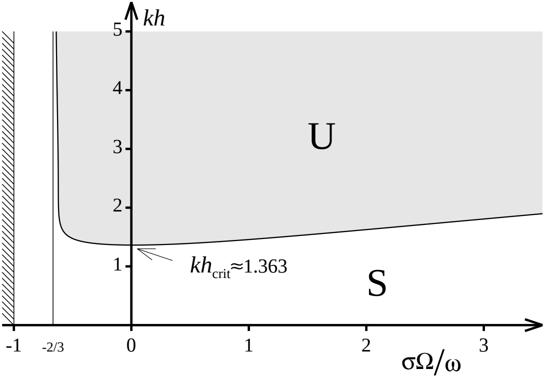
$$\gamma_{\max} = -N_1\omega(a_0k)^2 \quad \text{for} \quad l_{\max} = \sqrt{N_1/L_1}a_0k^2$$

- The wavenumber l of unstable modulational instabilities satisfies

$$-\sqrt{2\frac{N_1}{L_1}}k^2 a_0 < l < \sqrt{2\frac{N_1}{L_1}}k^2 a_0$$

- For $\Omega < -2\sqrt{\frac{gk}{3}} \Rightarrow$ no BF instability

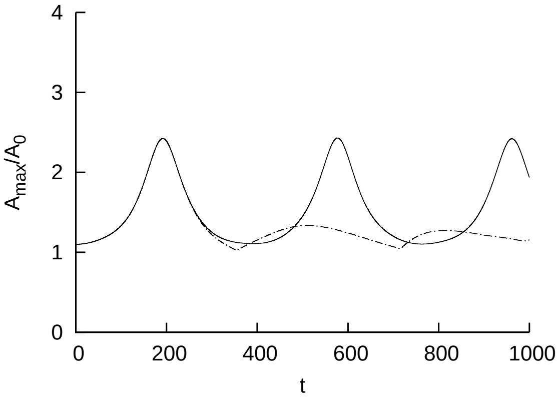
Thomas, Kharif & Manna (Phys. Of Fluids, 2012)



Stability diagram: **S** : stable, **U** : unstable.

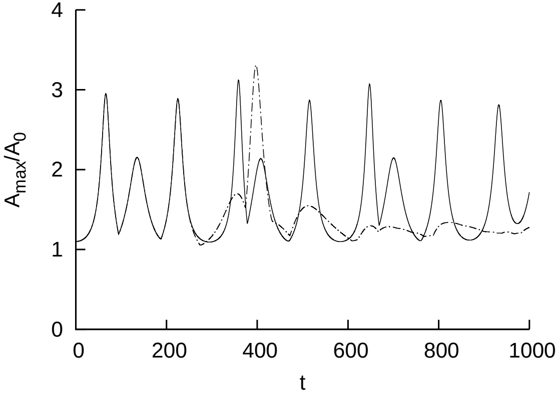
$$-1 < \frac{\sigma\Omega}{\omega} \leq -\frac{2}{3}\sigma \Leftrightarrow -\infty < \Omega \leq -2\sqrt{\frac{gk}{3}}$$

Thomas, Kharif & Manna (POF, 2012)



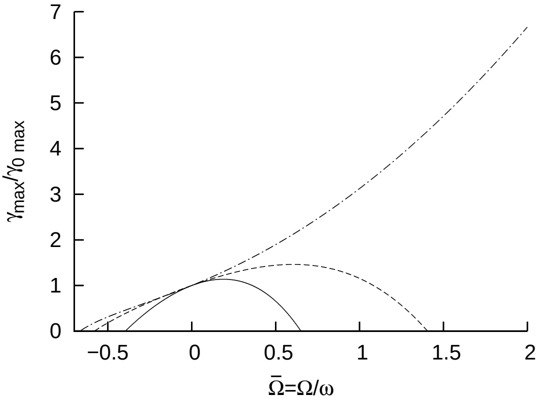
Dimensionless maximum amplitude of the envelope of a group of 8 waves for a simple recurrence with $kh = \infty$, $a_0 k_0 = 1/16$: $\Omega/\omega = 0$ (solid line) and $\Omega/\omega = -0.83$ (dash-dotted line)

Thomas, Kharif & Manna (POF, 2012)

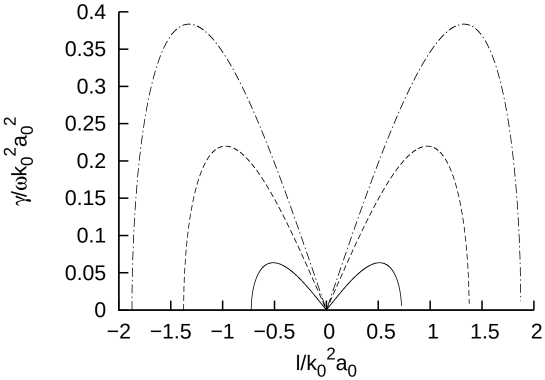


Dimensionless maximum amplitude of the envelope of a group of 8 waves for a double recurrence with $kh = \infty$, $a_0 k = \sqrt{3}/16$: $\Omega/\omega = 0$ (solid line) and $\Omega/\omega = -0.83$ (dash-dotted line)

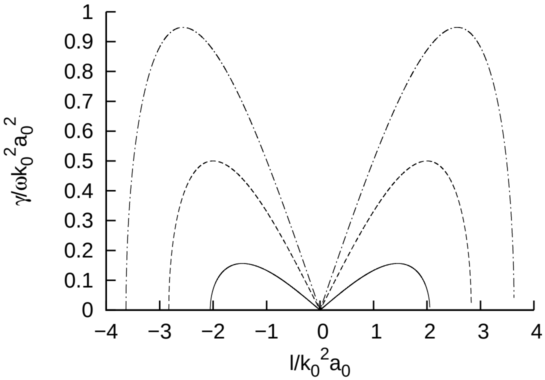
Thomas, Kharif & Manna (POF, 2012)



Dimensionless maximum growth rate as a function of the vorticity for $kh = 1.40$ (solid line), $kh = 1.70$ (dashed line) and $kh = \infty$ (dash-dotted line). $\gamma_{0_{\max}}$ is the maximum growth rate in the absence of shear current



Dimensionless growth rate as a function of the perturbation wavenumber l in finite depth ($kh = 2$) for $\Omega/\omega = -0.50$ (solid line), $\Omega/\omega = 0$ (dashed line), $\Omega/\omega = 0.50$ (dash-dotted line)



Dimensionless growth rate as a function of the perturbation wavenumber l in infinite depth ($kh = \infty$) for $\Omega/\omega = -0.50$ (solid line), $\Omega/\omega = 0$ (dashed line), $\Omega/\omega = 0.50$ (dash-dotted line)

	$F = 0.0$	$F = 0.25$	$F = 0.5$	$F = 1.0$	$F = 1.5$
$kh = 1.5$	$1.6/1.54$	$1.3/1.28$	$1.0/1.00$	$1.2/1.31$	$6.0/5.96$
$kh = 2.0$	$2.8/2.75$	$2.4/2.40$	$2.0/1.97$	$4.8/4.72$	$-/-$

Table : Dimensionless instability bandwidth. Oikawa, Chow & Benney (1987) results from their figures/ Present results (F is the dimensionless vorticity)

Application to Rogue Waves when modulational instability prevails

- The key parameter measuring the importance of the nonlinear four-wave interaction is the Benjamin-Feir Index (BFI) which is the ratio of the wave steepness to the normalized spectral bandwidth.

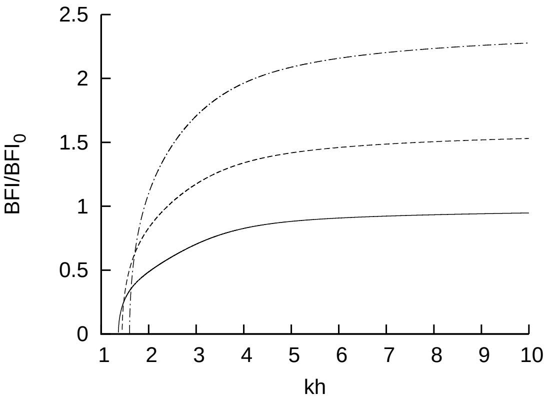
- Within the framework of the NLS equation the BFI writes

$$BFI = \frac{a_0 k}{\Delta K / k} \sqrt{| N_1 / L_1 |}$$

where ΔK is a typical spectral bandwidth (Onorato *et al*, 2006 and Kharif *et al*, 2009)

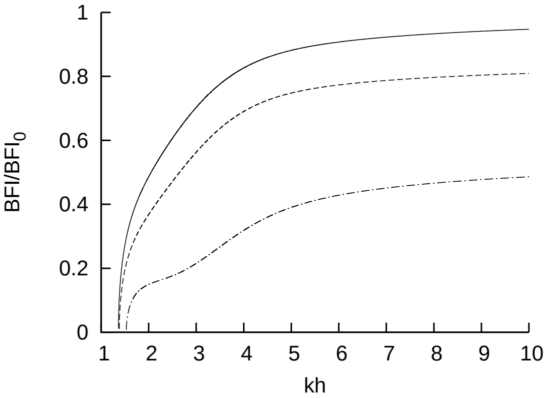
- The BFI is a convenient indicator for prediction of rogue wave occurrence. It is related to the pdf of wave heights. The rogue wave probability occurrence increases with BFI.

Thomas, Kharif & Manna (POF, 2012)



Normalized BFI as a function of kh for several values of $\overline{\Omega}$:
$\overline{\Omega} = 0$ (solid line), $\overline{\Omega} = 1$ (dashed line), $\overline{\Omega} = 2$ (dot-dashed line)

Thomas, Kharif & Manna (POF, 2012)



Normalized BFI as a function of *kh* for several values of $\overline{\Omega}$: $\overline{\Omega} = 0$ (solid line), $\overline{\Omega} = -0.3$ (dashed line), $\overline{\Omega} = -0.6$ (dot-dashed line)

Conclusions on effect of uniform shear currents on modulational instability and steep wave event (within the framework of the vor-NLS equation)

- At $\mathcal{O}(\epsilon^3) \Rightarrow$ importance of the coupling between the mean flow induced by the modulation and the vorticity. At this order and in infinite depth this coupling vanishes in the absence of vorticity as it is expected whereas it is not the case in the presence of vorticity
- The presence of vorticity does not modify the critical value $kh_{\mathrm{crit}} = 1.363$
- Plane wave solutions may be linearly stable to modulational instability for an opposite shear current independently of the dimensionless parameter kh
- The vorticity may increase or decrease the BFI

Fully nonlinear approach

In order to use the HOSM, we consider the evolution equations

$$\eta_t + \eta_x(\varphi_x^s + \Omega\eta) - (1 + \eta_x^2)W = 0 \qquad (6)$$

$$\varphi_t^s + \frac{1}{2}(\varphi_x^s)^2 - \frac{1}{2}(1 + \eta_x^2)W^2 + g\eta + \Omega\eta\varphi_x^s - \Omega\psi^s = 0 \qquad (7)$$

where $\varphi^s(x,t) = \varphi(x, \eta(x,t), t)$, $\psi^s(x,t) = \psi(x, \eta(x,t), t)$ and $W(x,t) = \varphi_z(x, \eta(x,t), t)$

- For the computation of ψ^s we use the following relation

$$\psi_x^s = -(1 + \eta_x^2)W + \eta_x\varphi_x^s = -G(\eta)\varphi^s$$

where G is the DNO (Craig & Sulem, 1993)

Computation of the basic wave: Adjustment scheme

- Initialization of nonlinear waves using an adjustment scheme (Dommermuth, 2000)

$$\eta_t - W^{(1)} = -W^{(1)} - \eta_x(\varphi_x^s + \Omega\eta) + (1+\eta_x^2)W = F$$

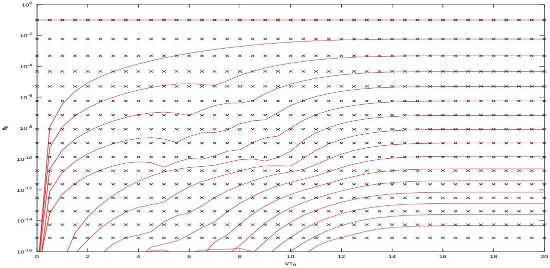
$$\varphi_t^s + g\eta - \Omega\psi^{s(1)} = -\Omega\psi^{s(1)} - \frac{1}{2}(\varphi_x^s)^2 + \frac{1}{2}(1+\eta_x^2)W^2 - \Omega\eta\varphi_x^s + \Omega\psi^s = G$$

$$\eta_t - W^{(1)} = F\{1 - \exp[-(\frac{t}{T_a})]\}$$

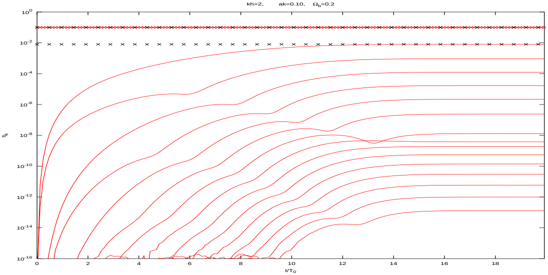
$$\varphi_t^s + g\eta - \Omega\psi^{s(1)} = G\{1 - \exp[-(\frac{t}{T_a})]\}$$

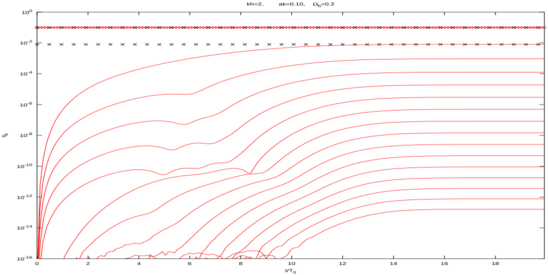
with $T_a = 10T_0$, the period of adjustment and $n = 4$ the rate of adjustment

$\overline{\Omega} = 0,\ ak = 0.10,\ kh = 2,\ M = 6$ (top), $M = 8$ (bottom)

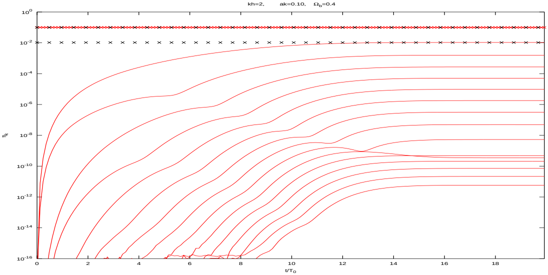


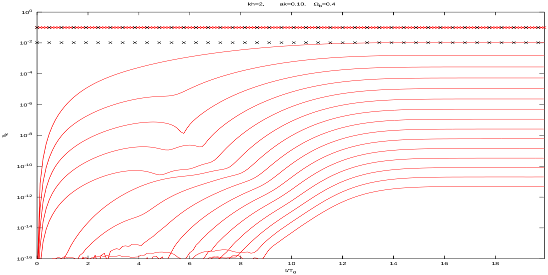
$\overline{\Omega} = 0.20$, $ak = 0.10$, $kh = 2$, $M = 3$ (top), $M = 6$ (bottom)



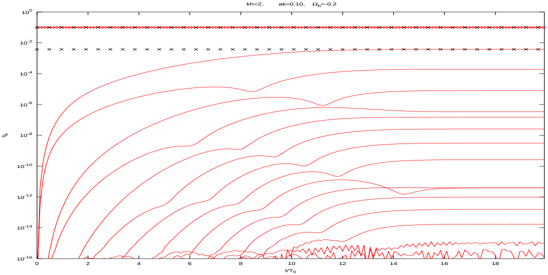


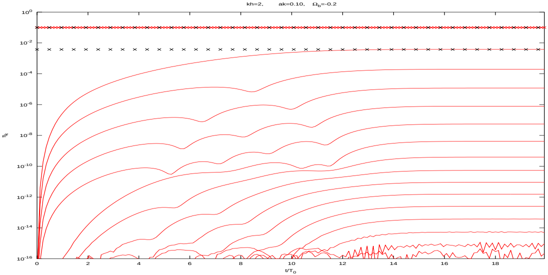
$\overline{\Omega} = 0.40$, $ak = 0.10$, $kh = 2$, $M = 3$ (top), $M = 6$ (bottom)



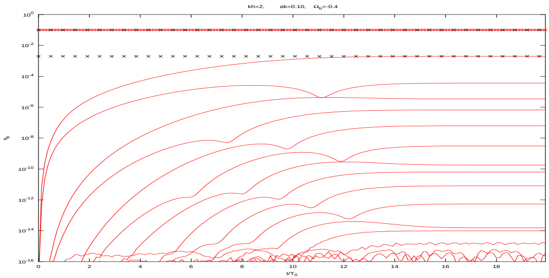


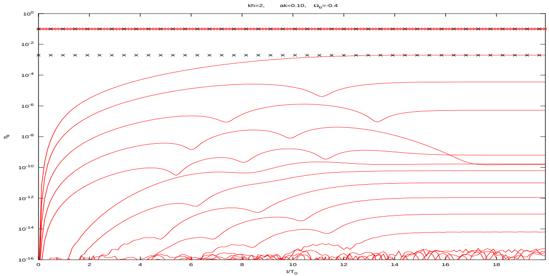
$\overline{\Omega} = -0.20$, $ak = 0.10$, $kh = 2$, $M = 3$ (top), $M = 6$ (bottom)





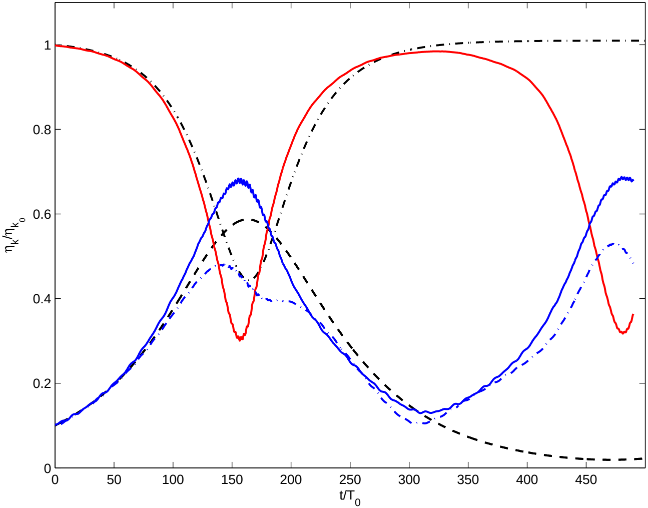
$\overline{\Omega} = -0.40$, $ak = 0.10$, $kh = 2$, $M = 3$ (top), $M = 6$ (bottom)



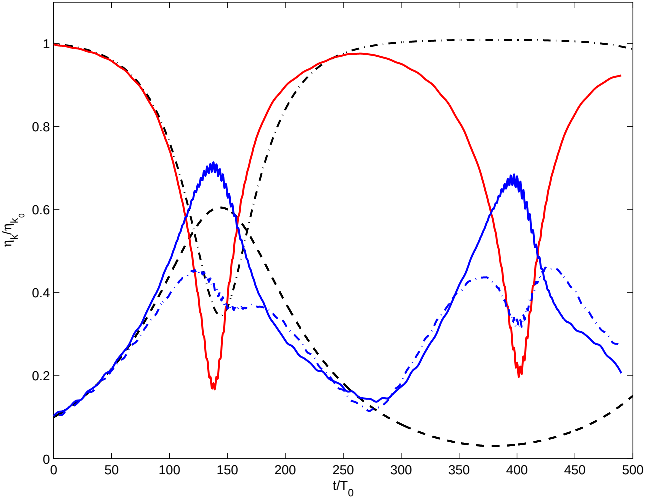


Modulational instability: Comparison (HOSM vs vor-NLS)

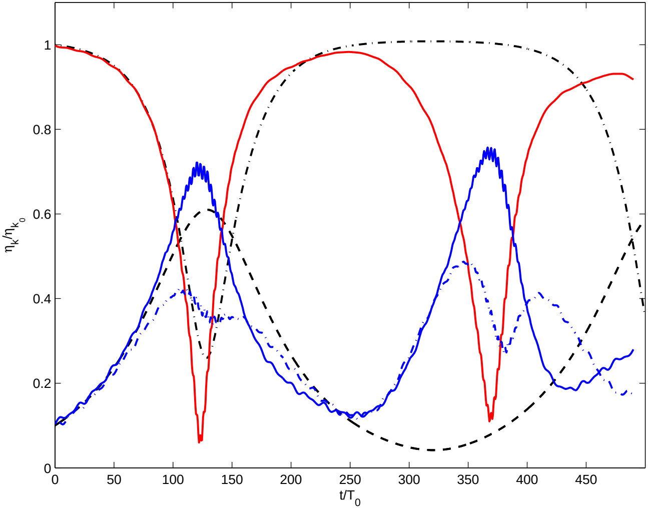
- $ak = 0.10$
- $kh = 2$
- 10 waves in one modulation (k=10 and l=1)
- The amplitude of the sidebands is 10% of the amplitude of the carrier wave.
- $M = 3$



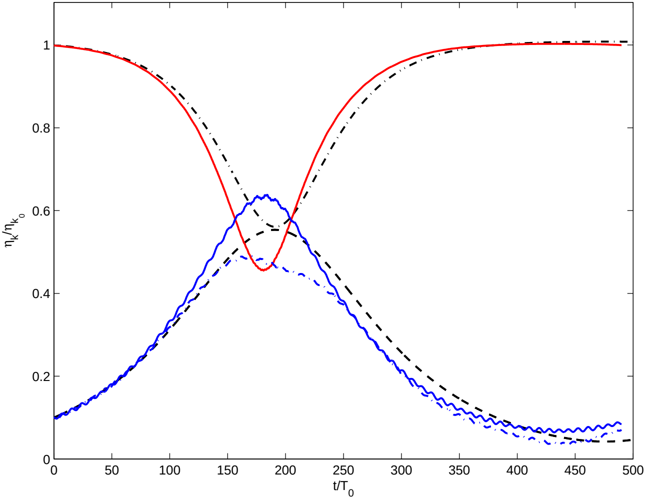
- Carrier wave and sideband evolutions. Black lines refer to NLS and red and blue lines to HOSM. The results of the HOSM are offset by $10\,T_0$ ($\overline{\Omega} = 0$)



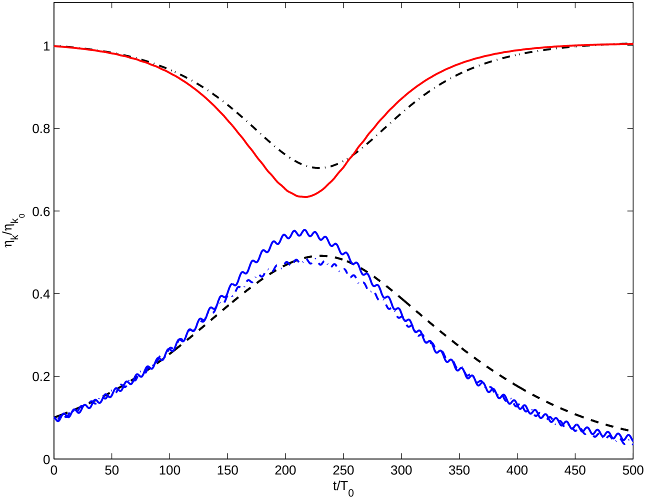
- Carrier wave and sideband evolutions. Black lines refer to NLS and red and blue lines to HOSM. The results of the HOSM are offset by $10T_0$ ($\overline{\Omega} = 0.10$)



- Carrier wave and sideband evolutions. Black lines refer to NLS and red and blue lines to HOSM. The results of the HOSM are offset by $10T_0$ ($\overline{\Omega} = 0.20$)



- Carrier wave and sideband evolutions. Black lines refer to NLS and red and blue lines to HOSM. The results of the HOSM are offset by $10T_0$ ($\overline{\Omega} = -0.10$)



- Carrier wave and sideband evolutions. Black lines refer to NLS and red and blue lines to HOSM. The results of the HOSM are offset by $10\,T_0$ ($\overline{\Omega} = -0.20$)

Perspectives

(within the framework of the fully nonlinear water wave equations with vorticity)

- Linear stability of Stokes waves of arbitrary steepness on a linear shear current
- Extension of the linear stability analysis to 3D rotational perturbations within the framework of the Euler equations
- numerical simulations within the framework of the Euler equations

An asymptotic solution is sought in the following form

$$\varphi = \sum_{n=-\infty}^{+\infty} \varphi_n E^n, \qquad \eta = \sum_{n=-\infty}^{+\infty} \eta_n E^n$$

where $E = \exp[i(kx - \omega t)]$

We assume $\varphi_{-n} = \varphi_n^*$ and $\eta_{-n} = \eta_n^*$

Then φ_n and η_n are written in perturbation series

$$\varphi_n = \sum_{j=n}^{+\infty} \epsilon^j \varphi_{nj}, \qquad \eta_n = \sum_{j=n}^{+\infty} \epsilon^j \eta_{nj}$$

where $\varphi_{nj}(\xi, z, \tau)$ and $\eta_{nj}(\xi, \tau)$

The small parameter ϵ is the wave steepness.

We assume $\varphi_{00} = 0$ and $\eta_{00} = 0$.