Lie group actions, spectral triples and generalized crossed products

Olivier Gabriel

Georg-August-Universität Göttingen

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Part I: spectral triples from ergodic actions

Theorem (G. & Grensing – 2013)

If

- a compact Lie group G acts ergodically
- on a (unital) C^*-algebra A,

then

- a n^+-summable spectral triple $(A, \mathscr{H}, D)$ is defined.

Remarks:

- Links algebraic and analytic properties.
- Recovers spectral triples on NC tori.

Plan of the talk

Joint project with M. Grensing.

Aims of the talk:

1. Construct a spectral triples from ergodic actions.
2. Introduce Generalized Crossed Products (GCPs).
3. Sketch construction of spectral triple on these GCPs.

Disclaimers:

- all algebras are unital.
- Part on GCPs is still work in progress!

Ergodic actions of compact Lie groups

Definition

$\alpha \colon G \curvearrowright A$ is *ergodic* if $(\forall g \in G, \alpha_g(a) = a) \implies a \in \mathbb{C}1$.

Theorem (Høegh-Krohn, Landstad & Størmer – 1981)

If $\alpha \colon G \curvearrowright A$ is ergodic, then

the unique G-invariant state of A is a trace τ.

Corollary

The Hilbert space $\mathscr{H}_0 := GNS(A, \tau)$ is endowed with a covariant representation of A and G.

Covariance relation: $\forall a \in A, \forall x \in \mathscr{H}_0$,

$$\alpha_g(a)x = U_g a U_g^* x. \qquad \text{(Covariance)}$$

Dense G-smooth $\mathscr{A} \subseteq A$ and $\mathscr{H}_0^\infty \subseteq \mathscr{H}_0$. Basis (∂_j) of $\mathfrak{g}$,

$$\partial_j^{\mathscr{A}}(a)\xi = \partial_j(ax) - a\partial_j(x) = [\partial_j, a]x \qquad \text{(Comm)}$$

Relation $[\partial_j, a] = \partial_j^{\mathscr{A}}(a)$: yields bounded commutators.

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Definition and properties of the Dirac operator

$\mathbb{C}l(n)$, (complexified) Clifford algebra gen. by n elements F_j s.t.

$$F_j^* = -F_j \qquad F_jF_k + F_kF_j = -2\delta_{jk}. \tag{Def-F}$$

Let S be a (fin. dim.) Clifford module, identify F_j with $\pi(F_j)$,

$$D := \sum \partial_j \otimes F_j \tag{Dirac}$$

is a *symmetric* unbounded operator on $\mathscr{H}_0^\infty \otimes S$.

Properties:

(i) D has bounded commutators – clear from (Comm).

(ii) D is essentially selfadjoint: $\mathrm{ran}(D \pm i)$ dense, *via* Peter-Weyl decomposition of $\mathscr{H}_0$. Details

(iii) Grading, first order condition, Real structure... Details

(iv) D is n^+-summable.

Summability condition

If T *compact* op., then $|T| := (T^*T)^{1/2}$ compact positive.

- $|T|$ admits a basis of eigenvectors,
- with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$ (with multiplicities).

The ideal $\mathcal{L}^{n^+} \subseteq B(\mathscr{H})$ is defined by

$$\mathcal{L}^{n^+} := \left\{ T \in B(\mathscr{H}) \,\middle|\, \sup_k \frac{\lambda_1 + \cdots + \lambda_k}{k^{(n-1)/n}} < \infty \right\}.$$

Definition

A spectral triple is n^+-*summable* if $(1 + D^2)^{-1/2} \in \mathcal{L}^{n^+}$.

Such summable spectral triple defines a cyclic cocycle.

Example:

The spectral triple $(C^\infty(M), \mathscr{H}, \not{D})$ on a dimension n spin manifold is n^+-summable.

Spectral subspaces for ergodic actions

Given E_ℓ, irrep. of G of dim. d_ℓ, with norm. char. $\chi_\ell(g) = d_\ell \operatorname{Tr}(\pi_\ell(g^{-1}))$, the associated *spectral subspace* is:

$$A_\ell := \overline{\left\{ \int_G \chi_\ell(g)\alpha_g(a)dg \middle| a \in A \right\}} \subseteq A.$$

It decomposes into m_ℓ copies of E_ℓ.

Theorem (Høegh-Krohn, Landstad & Størmer – 1981)

If $\alpha\colon G \curvearrowright A$ is ergodic, then

the multiplicity m_ℓ as above is bounded: $m_\ell \leqslant d_\ell$.

Theorem (G. & Grensing – 2013)

Given an ergodic action on A, with $\mathscr{H}_0$ as above,

D has compact resolvent and is n^+-summable.

Part I: GNS for G compact

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Idea of proof: comparison with spectral triple on $\mathscr{A} = C^\infty(G)$.

- Set $\mathscr{H}_{\text{ref}} := L^2(G) \otimes S$ and D_{ref} defined by (Dirac).
- Peter-Weyl's decomposition for $\mathscr{H}_{\text{ref}}$:

$$\mathscr{H}_{\text{ref}} = \bigoplus E_\ell \otimes \mathbb{C}^{d_\ell} \otimes S.$$

- Considering the trivial spin structure on G, D_{ref} is a Dirac operator on $\mathscr{A} = C^\infty(G)$.
- Hence D_{ref} has compact resolvent and is n^+-summable.

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- Thus $\mathscr{H}_0 \otimes S \hookrightarrow \mathscr{H}_{\mathrm{ref}}$; D_{ref} and D coincide on $E_{\ell,k}$.
- Writing λ_k (resp. μ_k) for eigenvalues of D_{ref} (resp. D).
- Get $\lambda_k \leqslant \mu_k$: suppressing terms in increasing sequence yields a faster increasing sequence.
- Consider $f(x) = (1+x^2)^{-1/2}$.

  Setting $\lambda'_k := f(\lambda_k)$ and $\mu'_k := f(\mu_k)$, we get $\mu'_k \leqslant \lambda'_k$.
- D_{ref} is n^+-summable means

$$\left\| \left(1 + D_{\mathrm{ref}}^2\right)^{-1/2} \right\|_{n^+} = \sup_k \frac{\sum_{p=0}^{k-1} \lambda'_p}{k^{(n-1)/n}} < \infty.$$

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Remarks on Part I and further research

About the degree of summability:

- We only get an upper bound on summability.
- It is not saturated in general!
- However, *orientability condition* – Hochschild cycycle $c = \sum c_{0,j} \otimes c_{1,j} \otimes \cdots \otimes c_{n,j} \in Z_p(\mathscr{A}, \mathscr{A})$ s.t.

$$\sum c_{0,j}[D, c_{1,j}] \cdots [D, c_{n,j}] = \gamma.$$

Perspectives:

- Is the trace $\varphi(a) = \mathrm{Tr}_\omega\, a|D|^{-n}$ G-invariant?

Consequences: if φ is G-inv. then

- $\exists \lambda \geqslant 0$ s.t. $\varphi = \lambda\tau$ (unicity of G-inv. τ),
- thus we should get:

$$\mathrm{Tr}_\omega\left(\gamma a_0[D, a_1] \cdots [D, a_n][D|^{-n}\right) = \sum \varepsilon(\sigma)\tau\Big(a_0\partial_{\sigma(1)}(a_1) \cdots \partial_{\sigma(n)}(a_n)\Big)$$

It works for NC 2-tori and Quantum Heisenberg Manifolds!

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About the degree of summability:

- We only get an upper bound on summability.
- It is not saturated in general!
- However, *orientability condition* – Hochschild cocycle $c = \sum c_{0,j} \otimes c_{1,j} \otimes \cdots \otimes c_{n,j} \in Z_p(\mathscr{A}, \mathscr{A})$ s.t.

$$\sum c_{0,j}[D, c_{1,j}] \cdots [D, c_{n,j}] = \gamma.$$

Perspectives:

- Is the trace $\varphi(a) = \mathrm{Tr}_\omega\, a|D|^{-n}$ G-invariant?

Consequences: if φ is G-inv. then

- $\exists \lambda \geqslant 0$ s.t. $\varphi = \lambda\tau$ (unicity of G-inv. τ),
- thus we should get:

$$\mathrm{Tr}_\omega\left(\gamma a_0[D, a_1]\cdots[D, a_n][D|^{-n}\right) = \sum \varepsilon(\sigma)\tau\Big(a_0\partial_{\sigma(1)}(a_1)\cdots\partial_{\sigma(n)}(a_n)\Big)$$

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Generalized Crossed Products (GCP)

A with σ, pointwise continuous *gauge action* of $S^1 = \mathbb{R}/\mathbb{Z}$.

- $\forall a \in A$, $t \mapsto \sigma_t(a)$ 1-periodic, Banach-valued cont. funct.
- Fourier series: introduce subspaces A_n, $n \in \mathbb{Z}$

$$A_n = \left\{ a \in A \middle| \forall t \in \mathbb{R}, \sigma_t(a) = e^{i2\pi nt} a \right\}.$$

- "$\cdots \oplus A_{-2} \oplus A_{-1} \oplus A_0 \oplus A_1 \oplus A_2 \oplus \cdots$" is dense in A.

Properties: $A_{-1} = (A_1)^*$; A_1, Hilbert bimodule over A_0 ▸ Def..

Definition (Generalized Crossed Product)

The C^*-algebra A is a *generalized crossed product* iff it is generated (as C^*-algebra) by A_0 and A_1.

Inversely, given $B = A_0$ (*basis algebra*) and $E = A_1$,
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Examples of GCP:

Crossed products by $\mathbb{Z}$: take $A_0 = B$ and $E = B\mathbb{U}$;

Commutative case: if moreover $A_1^* A_1 = A_0$, then
continuous functions on a S^1-principal bundle $P \to X$:

$$B = C(X) \qquad A = C(P).$$

The gauge action corresponds to the principal action.

Quantum Heisenberg Manifolds (QHM – Rieffel, 1989):

- Take $B := C(T^2)$, $E := \Gamma(T^2; \mathcal{L})$, line bundle $\mathcal{L} \to T^2$.
- Natural right action B on E and Hermitian structure.
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Vertical class in K-homology

Def.

All GCP come with a natural class in $KK_1(A, B)$.

- The action σ yields a conditional expectation $\mathbb{E}\colon A \to B$.

It induces a A-B-C^*-correspondance X:

- X is a right B-Hilbert module, completion of A for

$$\langle a_1, a_2 \rangle_B = \mathbb{E}(a_1^* a_2).$$

  A acts naturally on the left of X.

- Gauge action $\sigma_t(b) = b$, $\sigma_t(\xi) = e^{i2\pi t}\xi$ extends naturally to X. Denote ∂_t its derivative.

$(X, \partial_t) = [\partial]$ is an unbounded Kasparov module in $KK_1(A, B)$ (see *e.g.* Wahl '10 or Carey, Neshveyev, Nest & Rennie '11).

Definition (Vertical class)

We call $[\partial]$ the *vertical class* of the GCP A.

Summary of the construction

Idea: investigate "permanence properties" of spectral triples, just like in Adam Skalski's talk.

Assume that:

1. $S^1 \curvearrowright A$ is a GCP with $B := A^{S^1}$,
2. $(B, \mathscr{H}, D)$ is a spectral triple on B with D described by (Dirac) as in Part I,
3. we have a *two-sided Hermitian connexion* ∇ on $E = A_1$ which is associated to D

then

Conjecture

1. we construct a spectral triple $(A, \underline{\mathscr{H}}, \underline{D})$ on $A = B \rtimes_E \mathbb{Z}$,
2. in KK-theory, $[\underline{D}]$ represents the (inner) Kasparov product:

$$[\underline{D}] = [\partial] \otimes_B [D].$$

Connexions

For $D = \sum \partial_j \otimes F_j$, then B-bimodule of differential forms is

$$\Omega^1_D := \overline{\left\{ \sum b_{0,j}[D, b_{1,j}] \middle| b_{0,j}, b_{1,j} \in \mathscr{B} \right\}} \subseteq B \otimes \langle F_1, \ldots, F_n \rangle.$$

Definition (connexion)

A *connexion* is densely defined map $\nabla : \mathcal{E} \to E \otimes_B \Omega^1_D$ s.t.

$$\nabla(\xi b) = (\nabla \xi) b + \xi \otimes [D, b] \qquad \text{(R-Connexion)}$$

Proposition

∇ satisfies (R-Connexion) iff there are maps $\nabla_j : \mathcal{E} \to E$ s.t.

$$\nabla(\xi) = \sum \nabla_j(\xi) \otimes F_j \qquad \nabla_j(\xi b) = \nabla_j(\xi) b + \xi \partial_j(b)$$

Proof: identify $E \otimes_B \Omega^1_D$ with $E \otimes \langle F_1, \ldots, F_n \rangle$ and expand...

Two-sided Hermitian connexions: definition

For our purposes, we will need more properties:

Definition

A *two-sided Hermitian connexion* on E is $\nabla = \sum \nabla_j \otimes F_j$ s.t.

$$\nabla_j(\xi b) = \nabla_j(\xi)b + \xi\partial_j(b) \quad \partial_j(\xi^*\eta) = \nabla_j(\xi)^*\eta + \xi^*\nabla_j(\eta)$$

(Hermitian right-connexion) and:

$$\nabla_j(b\xi) = \partial_j(b)\xi + b\nabla_j(\xi) \quad \partial_j(\xi\eta^*) = \nabla_j(\xi)\eta^* + \xi\nabla_j(\eta)^*.$$

Define $\mathscr{A}$ as $*$-algebraic span of $\mathscr{B}$ and $\mathcal{E}$ inside $A = B \rtimes_E \mathbb{Z}$.

Hypotheses on ∇_j and ∂_j suffice to obtain:

$\underline{\nabla}_j$ unique $*$-derivation on $\mathscr{A}$ extending ∇_j and ∂_j.

Necessary properties:

$$\underline{\nabla}_j(\eta \cdot \xi) := \underline{\nabla}_j(\eta) \cdot \xi + \eta \cdot \nabla_j(\xi) \quad \underline{\nabla}_j(\xi^*) := \left(\underline{\nabla}_j(\xi)\right)^*.$$

Two-sided Hermitian connexions: examples

Example 1: action β of Lie group G on E over B:

Definition

A *Hilbert bimodule action* β associated to α satisfies:

$$\beta(\xi b) = \beta(\xi)\alpha(b) \qquad \alpha\left(\langle \xi, \eta \rangle_B\right) = \langle \beta(\xi), \beta(\eta) \rangle_B$$
$$\beta(b\xi) = \alpha(b)\beta(\xi) \qquad \alpha\left({}_B\langle \xi, \eta \rangle\right) = {}_B\langle \beta(\xi), \beta(\eta) \rangle$$

Infinitesimal generators of $\beta \rightsquigarrow$ two-sided Hermitian connexion.
Link part 1: $\beta \rightsquigarrow$ action $G \curvearrowright A$, combine gauge action,
$\rightsquigarrow$ obtain action of $G \times S^1$ and apply previous theory!

Example 2: quantum Heisenberg manifolds. Reminder:
$B = C(T^2)$ and $E = C(T^2; \mathcal{L})$ with $\mathcal{L} \to T^2$, line bundle.

- Action α of $G := T^2$ on $B \rightsquigarrow$ canonical Dirac on B.
- Connexion ∇ on E assoc. to α, two-sided and Hermitian.
- Not of the previous type: curvature $\nabla^2 \neq 0$!

Spectral triple: algebraic expression

Reminder: X, C^*-correspondence constructed from A to B
obtained out of $\mathbb{E} : A \to B$, conditional expectation.

If $(B, \mathscr{H}, D)$, spectral triple on basis B of A,
define a spectral triple $(A, \underline{\mathscr{H}}, \underline{D})$ by

- $\underline{\mathscr{H}} := X \otimes_B \mathscr{H}$ (well-defined Hilbert space),
- A represented on $\underline{\mathscr{H}}$ by $a \cdot ([a'] \otimes x) = [aa'] \otimes x$,
- If $(B, \mathscr{H}, D)$ is even with grading γ (acting on S), set

$$\underline{D} := \sum (\underline{\nabla}_j \otimes 1 + 1 \otimes \partial_j) \otimes F_j + \partial_t \otimes 1 \otimes \gamma$$

with domain $\mathrm{Dom}(\underline{D}) = \mathscr{A} \odot_{\mathscr{B}} \mathscr{H}_0^\infty \otimes S$.

For odd spectral triple, double S and more involved expression.

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Unbounded symmetric operator

$$\underline{D} = \sum(\underline{\nabla}_j \otimes 1 + 1 \otimes \partial_j) \otimes F_j + \partial_t \otimes 1 \otimes \gamma$$

on $\mathrm{Dom}(\underline{D}) = \mathscr{A} \odot_{\mathscr{B}} \mathscr{H}_0^\infty \otimes S$ defines a symmetric operator:

1. Is $\underline{D}$ well-defined?
   Problem of the tensor product over $\mathscr{B}$:
   $$\underline{\nabla}_j(ab) \otimes x + ab \otimes \partial_j(x) = \underline{\nabla}_j(a) \otimes bx + a \otimes \partial_j(bx).$$
   Requires:
   - right-connection property: $\underline{\nabla}_j(ab) = \underline{\nabla}_j(a)b + a\partial_j^{\mathscr{B}}(b)$,
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2. Is $\underline{D}$ symmetric?
   - $\underline{\nabla}_j$ and ∂_t commute with the gauge action...
   - ... hence consider $\eta \otimes x$ and $\eta' \otimes x'$ for $\eta, \eta' \in X_n$.
   - Check property for all j separately. Clear for ∂_t.
3. Does $\underline{D}$ have bounded commutators?
   Yes! Action of $\mathscr{A}$ on $\mathrm{Dom}(\underline{D})$ and $\underline{\nabla}_j$ derivations.

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Selfadjoint operator

The difficulties that remain:

- show that $\underline{D}$ is selfadjoint;
- prove that it has compact resolvent.

To prove this rely on:

Spectral flow and the unbounded Kasparov product
by J. Kaad and M. Lesch (to appear)

Given two unbounded Kasparov modules, they show how to:

1. construct another unbounded Kasparov module,
2. prove this is the Kasparov product of the original modules

- Similar to B. Mesland '09...
- ...major technical improvements: "C^1-version" of Hilbert module (*operator $*$-module*) instead of "smooth version".

Operator $*$-algebra

Definition (Operator $*$-algebra, Mesland '09 & Ivankov '11)

A Banach algebra $\mathscr{B}$ is an operator $*$-algebra if

1. $\mathscr{B}$ is an operator space, ▸ Def.
2. the multiplication m on $\mathscr{B}$ is completely bounded,
3. the involution $*$ on $\mathscr{B}$ is also completely bounded.

Example: $\pi : \mathscr{B} \to \mathcal{L}(F_C)$ faithful rep. and $\delta : \mathscr{B} \to \mathcal{L}(F)$ s.t.

$$\delta(bb') = \delta(b)\pi(b') + \pi(b)\delta(b') \qquad \delta(b^*) = U\delta(b)^* U$$

for some unitary $U \in \mathcal{L}(F)$ which commutes with $b \in \mathscr{B}$, we obtain an *operator $*$-algebra* B_1 as completion of $\mathscr{B}$ for:

$$\rho_B(b) = \begin{pmatrix} \pi(b) & 0 \\ \delta(b) & \pi(b) \end{pmatrix} \in \mathcal{L}(F \oplus F).$$

Properties:

- B_1 is a subalgebra of B iff δ is closable.
- In this case, B_1 is stable under holom. funct. calculus.

Operator $*$-module

Similar "C^1-version" for Hilbert module: *operator $*$-module*.

Definition (operator $*$-module, Kaad & Lesch '11)

Y_1 is an operator $*$-module over the operator $*$-algebra A_1 if:

- Y_1 is an operator space,
- the product $Y_1 \times A_1 \to Y_1$ is completely bounded,
- there is a completely bounded pairing $Y_1 \times Y_1 \to A_1$ with the usual properties of Hilbert modules,
- Y_1 is a direct summand of the standard module over A_1.

Example: given

- (π, δ) for $\mathscr{B} \subseteq B$ as before and
- $\mathcal{E}$ dense in E, f.g proj. Hilbert module with $\langle \mathcal{E}, \mathcal{E} \rangle \subseteq B_1$,
- with a Hermitian closable connection ∇ assoc. to δ,

then we get an operator $*$-module E_1 as completion of $\mathcal{E}$ for

$$\rho_E(\xi) = \begin{pmatrix} \pi(\xi) & 0 \\ \nabla(\xi) & \pi(\xi) \end{pmatrix} \in \mathcal{L}(F \oplus F).$$

Frames of operator $*$-module

Proposition (G. & Grensing - '13)

Given a Hermitian closable connexion ∇ and a finitely generated proj. E with $\langle E_1, E_1 \rangle \subseteq B_1$ then

there is a frame of E inside E_1.

Conversely, a frame of E inside E_1 imposes that ∇ is closable.

In particular, E_1 is a direct summand of B_1^N.

Proof: (first implication only)

- Consider $C_1 := \{T \in \mathrm{End}_B(E) | T(E_1) \subseteq E_1\}$.
- $\partial(T)(\xi) := \nabla(T(\xi)) - (T \otimes 1)(\nabla(\xi))$ is a densely defined and closed derivation on $\mathrm{End}(E)$.
- $C_1 \subseteq \mathrm{End}(E)$, dense and stable under holom. calculus.
- Frame for E, perturb and rectify $\rightsquigarrow$ frame for E_1.

Use this to construct a closable connexion $\underline{\nabla}$ on X,

assuming E is left and right f.g. projective.

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Kasparov product (Kaad–Lesch)

Theorem (Kaad-Lesch, to appear)

If

- (X, D_1) *and* (Y, D_2) *are two unbounded Kasparov modules for* (A, B) *and* (B, C) *resp.*
- *there is a correspondence* (X_1, ∇) *from* (X, D_1) *to* (Y, D_2)*,*
- $\nabla_{D_2} : X_1 \to X \hat{\otimes}_B \mathcal{L}(Y)$ *be any Hermitian* D_2*-connexion,*

then

- $(D_1 \times_\nabla D_2, (X \hat{\otimes}_B Y)^2)$*, even Kasparov A-C module...*
- *...which is the Kasparov product of* (X, D_1) *and* (Y, D_2)*.*

Definition

A D_2-connexion ∇ is a completely bounded linear map $\nabla : X_1 \to X \hat{\otimes} \mathcal{L}(Y)$ which is a (R-Connexion).

We can now apply this theorem to (X, D_1) the vertical class $[\partial]$ and (Y, D_2) the spectral triple on B.

Outline

Conclusion

Summary:

- Construction of spectral triples from ergodic actions.
- Introduction of Generalized Crossed Products (GCP).
- Extension of spectral triples from basis to GCP.

Perspectives:

- Link between $⨍ a|D|^{-n}$ and τ?
- Do the same "extension construction" and Kasparov product for $SU(2)$-principal bundles?

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References

- For ergodic actions:

O. G. and M. Grensing
Ergodic actions and spectral triples
http://arxiv.org/abs/1302.0426

- For generalized crossed products:

O. G. and M. Grensing
Generalized crossed products and spectral triples
Coming soon!

Thank you for your attention!

. . .

Additional properties

Spectral triple of dimension $n = \dim G$.

Parity For even n, *grading operator* γ s.t. $\gamma^2 = 1$, $\gamma^* = \gamma$

$$a\gamma = \gamma a \qquad D\gamma = -\gamma D$$

Motivations: K-homology.

Real structure and order one

Norm-preserving antilinear operator $J \colon \mathscr{H} \to \mathscr{H}$ s.t.

$$[a, Jb^*J] = 0, \qquad [[D, a], Jb^*J] = 0, \qquad J^2 = \varepsilon_J$$

and

$$J(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D) \qquad JD = \varepsilon_D DJ \qquad J\gamma = \varepsilon_\gamma \gamma J,$$

with $\varepsilon_J, \varepsilon_D$ and (possibly) ε_γ in ± 1, depending on n Table.

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- KR-homology for $\mathscr{A} \otimes \mathscr{A}^0$ with $\Sigma(a \otimes b^0) = b^* \otimes (a^*)^0$,
- $\mathscr{H}$ as A-bimodule, Poincaré duality in KK-theory. More

For Poincaré duality: K-theory class in $K(A \otimes A^0)$? Back

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For Poincaré duality: K-theory class in $K(A \otimes A^0)$? Back

Operator spaces

A Banach space $(X, \|\cdot\|)$ is an *operator space* if there exists a norm $\|\cdot\|_X \colon M(X) \to [0,\infty)$ on the finite matrices over X s.t.

- for all finite matrices over $\mathbb{C}$ $v, w \in M(\mathbb{C})$, and any matrix $x \in M(X)$, we have:

$$\|v \cdot x \cdot w\|_X \leqslant \|v\|_{\mathbb{C}} \|x\|_X \|w\|_{\mathbb{C}}$$

- for any projections $p, q \in M(\mathbb{C})$ with $pq = 0$ and $x, y \in M(X)$, we have:

$$\|pxp + qyq\|_X = \max\{\|pxp\|_X, \|qyq\|_X\}$$

- for any projection $p \in M(\mathbb{C})$ of rank 1 and $x \in X$, we have $\|p \otimes x\|_X = \|x\|$.

Last condition: original $\|\cdot\|$ is “compatible” with $\|\cdot\|_X$.

Back

Unbounded Kasparov modules

Definition

An *unbounded Kasparov module A-B* module is (X, D) where

- X, B-Hilbert module with action $\varphi \colon A \to \mathcal{L}(X_B)$,
- D is an unbounded regular selfadjoint operator on X,

such that

- there is a dense subalgebra $\mathscr{A} \subseteq A$ with
    - $a(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D)$,
    - and $[D, a]$ extends to a bounded operator on X,
- the resolvent $(D - i)^{-1} \in \mathcal{K}(X)$ is B-compact.

In particular, D has to be selfadjoint.

Regular operator
Back

Covariant representation and compact Lie groups

Proposition

If G is *compact*, then

$$D \text{ defined in (Dirac) is essentially selfadjoint.}$$

▸ Def.

Proof:

Criterion: both $\operatorname{ran}(D \pm i)$ are dense in $\mathscr{H} = \mathscr{H}_0 \otimes S$. ▸ Reminder

- By Peter-Weyl's decomposition theorem:

$$\mathscr{H}_0 = \bigoplus E_\ell \otimes \mathbb{C}^{m_\ell}$$

- For each E_ℓ, choose spaces $E_{\ell,k}$. Projections $P_{\ell,k}$ on $\mathscr{H}_0$.
- $Q_{\ell,k} := P_{\ell,k} \otimes 1_S$ commutes with D.
- $Q_{\ell,k} D$ selfadjoint on finite dimensional space,
- hence it has real eigenvalues and...
- ... $Q_{\ell,k} D \pm i$ is surjective!

Corollary of proof: D admits a basis of eigenvectors.

◂ Back

Regular operator

Let E and F be two Hilbert modules over A.

Definition

A *regular* (unbounded) operator from E to F is a densely defined closed A-linear map $T\colon \mathrm{Dom}(T) \to F$ s.t.

- T^* is densely defined,
- and $1 + T^*T$ has dense range.

Lemma

If $T\colon E \to E$ is densely defined and selfadjoint, then

T is regular if and only if the operators $T \pm i$ are surjective.

Back

Clifford algebra $\mathbb{C}l(n)$ – II

Proposition (Dabrowski & Dossena – 2011)

For any $n \in \mathbb{N}$, consider S with its matrices as in (Def-F).

- For even n, grading operator γ_S with $\gamma_S^* = \gamma_S$, $\gamma_S^2 = 1$ and $\gamma_S F_j = -F_j \gamma_S$.
- Antilinear map J_S s.t. $\langle J_S s, J_S s' \rangle = \langle s', s \rangle$ and

$$J_S^2 = \varepsilon_J \qquad J_S F_j = \varepsilon_D F_j J_S \qquad J_S \gamma_S = \varepsilon_\gamma \gamma_S J_S,$$

where $\varepsilon_J, \varepsilon_D$ and ε_γ: either -1 or 1, as in Table ▸ Real structure.

If $\mathscr{H}_0 = \mathrm{GNS}(A, \tau)$ for a G-invariant trace τ on A,

- $\mathscr{H}_0$ is naturally endowed with a covariant rep. of (A, G),
- we use the above to get better properties for D.

Unbounded symmetric operator – part II

If $\mathscr{H}_0 = \mathrm{GNS}(A, \tau)$, consider $\mathscr{H} := \mathscr{H}_0 \otimes S$ and still $D = \sum \partial_j \otimes F_j$ defined on $\mathrm{Dom}(D) = \mathscr{H}_0^\infty \otimes_{\mathbb{C}} S \subseteq \mathscr{H}$.

Proposition

The operator D on $\mathscr{H}$ has further properties:

(iii) For even n, grading operator $\gamma = 1 \otimes \gamma_S$ s.t. $\gamma^2 = 1$ and for all $a \in A$,

$$\gamma a = a\gamma \qquad \gamma(\mathrm{Dom}\, D) \subseteq \mathrm{Dom}(D) \qquad \gamma D = -D\gamma;$$

(iv) D has a real structure, *i.e.* antilinear $J = J_0 \otimes J_S$ on $\mathscr{H}$ with commutation relations of ◂ Real structure.

(v) D and J satisfy the *first order condition*, *i.e.* for all $a, a' \in \mathscr{A}$,

$$[[D, a'], Ja^* J^{-1}] = 0;$$

(vi) D admits a selfadjoint extension $\widetilde{D}$.

Sketch of proof

General idea: use properties of tensor product.

(iii) Grading operator: $\gamma = 1 \otimes \gamma_S$ and γ_S satisfies all required properties...

(iv) Real structure: $J = J_0 \otimes J_S$. Since $\mathscr{H}_0 := \mathrm{GNS}(A, \tau)$, the set $[a] \in \mathscr{H}_0$ is dense. Set $J_0([a]) = [a^*]$ then

$$U_g J_0([a]) = [\alpha_g(a^*)] = [\alpha_g(a)^*] = J_0 U_g([a])$$

and all properties follow.

(v) First order condition: notice that $J_0 b J_0^{-1}([a]) = [ab^*]$ so $[D, a']$ and JaJ^{-1} act on "different sides" of $\mathscr{H}$.

(vi) Selfadjoint extension: very different idea. Requires a theorem by von Neumann.

Existence of selfadjoint extension: why is it interesting?

Example of conditions: real structure

- Real structure antilinear operator $J\colon \mathscr{H} \to \mathscr{H}$ s.t. $\langle J\xi, J\eta\rangle = \langle \eta, \xi\rangle$, $J^2 = \varepsilon_J$, $[a, Jb^*J] = 0$ and

$$J(\mathrm{Dom}(D)) \subseteq \mathrm{Dom}(D) \quad JD = \varepsilon_D DJ \quad J\gamma = \varepsilon_\gamma \gamma J,$$

where $\varepsilon_J, \varepsilon_D$ and (possibly) ε_γ are all ± 1, depending on n:

n	0	2	4	6	1	3	5	7
ε_J	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
ε_D	$+$	$+$	$+$	$+$	$-$	$+$	$-$	$+$
ε_γ	$+$	$-$	$+$	$-$				

Motivations:

- Real K-homology (KR-homology). Spin.
- Turns $\mathscr{H}$ into $\mathscr{A} \otimes \mathscr{A}^{\mathrm{op}}$ module. Natural involution $a \otimes b^{\mathrm{op}} \mapsto b^* \otimes (a^*)^{\mathrm{op}}$. Poincaré duality.
- Tomita operator (traceless case).

Clifford algebra

Additional properties

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◂ Clifford algebra

◂ Additional properties

Hilbert bimodule

Hilbert bimodule: a Hilbert module on *both* left and right.

Definition (Hilbert bimodule)

A-B-bimodule E such that

- E is a left A-Hilbert module,
  with an A-valued scalar product $_A\langle\, ,\, \rangle$.
- E is a right A-Hilbert module,
  with an A-valued scalar product $\langle\, ,\, \rangle_A$.
- condition de compatibilité : pour tous ξ, ζ, η dans E,

$$\xi\langle\zeta, \eta\rangle_B = {}_A\langle\xi, \zeta\rangle\eta.$$

- Closely related notion: Morita equivalence bimodule.

Example:

$E = A$ with the standard action on both sides and

$$_A\langle\xi, \eta\rangle = \xi\eta^* \qquad \langle\xi, \eta\rangle_A = \xi^*\eta.$$

Back

Modules hilbertiens : idée générale

Idée : généralisation des espaces hilbertiens pour C^*-algèbres autres que $\mathbb{C}$.

Exemple dans le cas commutatif :

- M, variété riemannienn lisse et $A = C(M)$.
- TM, fibré tangent de M.

E, sections continues de TM: module sur A.
Formule $\langle \xi, \eta \rangle (x) = \langle \xi(x), \eta(x) \rangle$: définit un produit scalaire à valeur dans A !

Definition (: module hilbertien (à droite))

E, A-module (à droite) et produit scalaire $\langle \cdot, \cdot \rangle$ à valeur dans A.

- Définition similaire pour les modules hilbertiens *à gauche*.

Modules hilbertiens : définition complète

Soit A une C^*-algèbre,

Definition (: module hilbertien (à droite))

E, A-module à droite et $\langle \cdot | \cdot \rangle$, produit scalaire à valeur dans A: for all $\xi, \eta \in E$ and $a \in A$,

1. $0 \leqslant \langle \xi | \xi \rangle$ dans A.
2. $\langle \xi | \xi \rangle = 0 \iff \xi = 0$
3. $\langle \xi | \eta\, a \rangle = \langle \xi | \eta \rangle a$
4. $\langle \xi | \eta \rangle^* = \langle \eta | \xi \rangle$
5. E est complet pour la norme $\|\xi\| = \|\langle \xi | \xi \rangle\|^{\frac{1}{2}}$.

Retour

Poincaré duality in KK-theory

Let A and B be two C^*-algebras, assume we have two elements

$$\alpha \in KK(A \otimes B, \mathbb{C}) \qquad \beta \in KK(\mathbb{C}, A \otimes B)$$

such that

$$\beta \otimes_A \alpha = 1_B \in KK(B, B) \qquad \beta \otimes_B \alpha = 1_A \in KK(A, A)$$

which exchanges K-theory and K-homology for A and B:

$$K_*(A) = KK(\mathbb{C}, A) \simeq KK(B, \mathbb{C}) = K^*(B)$$
$$K_*(B) = KK(\mathbb{C}, B) \simeq KK(A, \mathbb{C}) = K^*(A)$$

Back

Reminder: unbounded operators

Back

Definition

Given densely defined T, $\mathrm{Dom}(T^*)$ set of $x \in \mathscr{H}$ s.t.

$$\exists z \in \mathscr{H}, \forall y \in \mathrm{Dom}(T), \langle x, Ty\rangle = \langle z, y\rangle.$$

- The *adjoint* T^* of T is defined by $T^*x = z$.
- T *selfadjoint* iff $T = T^*$ (in part. $\mathrm{Dom}(T) = \mathrm{Dom}(T^*)$).

Delicate equilibrium: enlarging $\mathrm{Dom}(T)$ puts more constraints, thus restricting $\mathrm{Dom}(T^*)$...

- For *symmetric* T, *i.e.* $\forall x, y \in \mathrm{Dom}(T)$, $\langle Tx, y\rangle = \langle x, Ty\rangle$, we have $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(T^*)$.
- In this case, the *closure* $\overline{T}$ is defined on $\mathrm{Dom}(\overline{T})$, completion of $\mathrm{Dom}(T)$ for $\|x\|_T^2 = \|x\|^2 + \|Tx\|^2$.

T is *essentially selfadjoint* if $\overline{T}$ is selfadjoint.

The spectral theorem only holds for selfadjoint operators!

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- T *selfadjoint* iff $T = T^*$ (in part. $\mathrm{Dom}(T) = \mathrm{Dom}(T^*)$).

Delicate equilibrium: enlarging $\mathrm{Dom}(T)$ puts more constraints, thus restricting $\mathrm{Dom}(T^*)$...

- For *symmetric* T, *i.e.* $\forall x, y \in \mathrm{Dom}(T)$, $\langle Tx, y \rangle = \langle x, Ty \rangle$, we have $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(T^*)$.
- In this case, the *closure* $\overline{T}$ is defined on $\mathrm{Dom}(\overline{T})$, completion of $\mathrm{Dom}(T)$ for $\|x\|_T^2 = \|x\|^2 + \|Tx\|^2$.

T is *essentially selfadjoint* if $\overline{T}$ is selfadjoint.

The spectral theorem only holds for selfadjoint operators!

Reminder: unbounded operators

Back

Definition

Given densely defined T, $\mathrm{Dom}(T^*)$ set of $x \in \mathscr{H}$ s.t.

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Reminder: essentially selfadjoint operators

◂ Back

Proposition

If T is symmetric, TFAE:

1. T is essentially selfadjoint;
2. $\ker(T^* + i) = \{0\}$ and $\ker(T^* - i) = \{0\}$;
3. Both $\mathrm{ran}(T + i)$ and $\mathrm{ran}(T - i)$ are dense in $\mathscr{H}$.

Example: $T = id/ds$ with

$$\mathrm{Dom}(T) := \{f \in H^1([0,1]), f(0) = 0 = f(1)\}$$

- Integration by parts: T is symmetric.
- Adjoint: $T^* = id/ds$ on $\mathrm{Dom}(T^*) = H^1([0,1])$,
  $\rightsquigarrow$ no restriction on $f(0)$ and $f(1)$!
- T is *not* essentially selfadjoint:
  $e^{\pm s} \in \mathrm{Dom}(T^*)$ and $(T^* \pm i)e^{\pm s} = 0$.

Selfadjoint extensions? Yes! T_α for $|\alpha| = 1$ with:

$$\mathrm{Dom}(T_\alpha) := \{f \in AC([0,1]), f(0) = \alpha f(1)\}$$

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