Locally compact quantum groups:
The von Neumann algebra versus the C*-algebra approach.

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Abstract

The operator algebra approach to quantum groups resulted in a nice theory of locally compact quantum groups (see [K-V1] and related papers). It seems natural to formulate such a theory in the context of C*-algebras because the theory of C*-algebras can be viewed as the non-commutative version of the theory of locally compact spaces. However, for many other, more practical reasons, it turns out to be more appropriate to develop locally compact quantum groups in a von Neumann algebraic framework. In this note, we will explain why this is so. We will also show that it is rather easy to pass from one setting to the other and in particular, we will see how the two approaches eventually yield the same objects and the same results.

The results have been known for some time (see references in this paper), but our approach is somewhat new. Furthermore, this note is of an expository nature and details will be found in other publications.

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0. Introduction

Let G be a *locally compact group.* Consider the C*-algebra $C_0(G)$ of complex continuous functions on G, tending to 0 at infinity. We will identify the C*-tensor product $C_0(G) \otimes C_0(G)$ with $C_0(G\times G)$ and its multiplier algebra $M(C_0(G)\otimes C_0(G))$ with $C_b(G\times G)$, the C*-algebra of all bounded complex continuous functions on the cartesian product $G \times G$. The product in G yields a non-degenerate *-homomorphism $\Delta : C_0(G) \to M(C_0(G) \otimes C_0(G))$, called a *coproduct* and defined by $\Delta(f)(p, q) = f(pq)$ where $p, q \in G$ and $f \in C_0(G)$. The associativity of the product in G gives the coassociativity of the coproduct $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ (where ι is the identity map).

In *Section 1*, we will recall the notion of a *coproduct on any C*-algebra* (see Definition 1.1). It is an obvious generalization of the above, motivating example.

It would be most natural to give a set of axioms for a pair (A, Δ) of a C*-algebra A and a coproduct Δ to be called a *locally compact quantum group* in terms of the existence of a counit and an antipode (the analogues of the unit and the inverse in a group). Unfortunately, this seems to be rather difficult. At this moment, there is no such theory where it is possible to prove the existence of the Haar weights (the analogues of the Haar measures on a locally compact group). On the other hand, there is a nice theory of locally compact quantum groups where the starting point is a pair (A, Δ) of a C*-algebra with a coproduct and where the existence of the Haar weights is assumed (see Definition 1.3 in Section 1). This theory was developed by Kustermans and Vaes, see [K-V1], [K-V2], [K-V3] and [K-V4].

The development of the theory of locally compact quantum groups has a *long history* and it would take us too far to describe this in all details. Such a survey can e.g. be found in [E-S2], [K-V2], [M-N-W] and other papers. Because of the scope of this paper, let us however look at the following steps.

The main result that people had in mind when developing this theory, was the *duality theorem of Pontryagin* for locally compact abelian groups, see e.g. [P]. It says, roughly speaking, that the dual of a locally compact abelian group is again a locally compact abelian group and that taking the dual again, gives the original group. This result is the basis of abstract harmonic analysis (generalizing Fourier analysis).

Of course, the result is no longer valid when the group is non-abelian. In the 60's and 70's, many results have been obtained by various researchers giving suitable *generalizations.* It was Kac and Vainerman [V-K] on the one hand, and Enock an Schwartz [E-S1] on the other hand, who developed, independently, the theory now known as *Kac algebras.* These objects contain all locally compact groups, allow the construction of a dual object within the same category, and the duality extends the known dualities, in particular, the Pontryagin duality for locally compact abelian groups. It is important to notice that the theory of Kac algebras is formulated completely within the *von Neumann algebraic context.*

Later, mainly because of the work of Drinfel'd [D] and Jimbo [Ji] on quantum groups and the work of Woronowicz on the quantum $SU_q(2)$ [W1], people realized that the axioms for a Kac algebra where too restricive. The condition that the square of the antipode is the identity (assumed for Kac algebras), is not fulfilled in these examples. This motivated researchers to look for generalizations of the theory without assuming this condition.

The first succesful structure was the one of a *compact quantum group*, due to Woronowicz, see [W2] and [W3]). According to the philosophy at that time, the theory was now developed in the framework of C*-algebras. Later, *discrete quantum groups* where introduced, first as duals of compact quantum groups, see [P-W], and later as independent objects, see [E-R] and [VD2]. In any case, the dual of a compact quantum group is a discrete quantum group, and the dual of a discrete quantum group is a compact quantum group. Later, the so-called *algebraic quantum groups* were introduced, see [VD3] and [VD4]. Compact and discrete quantum groups fall into this category and again, the dual could be defined within the same category, thus extending Pontryagin duality between compact abelian groups and discrete abelian groups to compact quantum groups and discrete quantum groups. This theory is formulated in a purely algebraic context, but it is also possible to develop an operator algebra version of it, see [K] and [K-VD].

Later, Masuda and Nakagami extended the notion of a Kac algebra, replacing the condition that the square of the antipode is the identity by some kind of (non-trivial) polar decomposition of the antipode, see [M-N]. They work in the von Neumann algebra context (as in the case of the original Kac algebras). A few years later, Woronowicz joined this project and the result is a theory, along the same lines as the one developed in [M-N], but now in the C*-algebraic setting, see [M-N-W]. Observe that it took many years between the moment the work was announced (in 1995) and the paper actually appeared (in 2003).

On the other hand, motivated by the work done in [VD3], [K] and [K-VD], Kustermans started (in 1998) to develop a *general theory of locally compact quantum groups*, independently of Masuda, Nakagami and Woronowicz. Soon, Vaes joined the project and this resulted in their joined works [K-V*] (written independently of [M-N-W]). Their axioms are simpler than the ones in [M-N-W]. The main difference is that Kustermans and Vaes assume the existence of a left and of a right Haar weight and thus are able to prove the existence of the antipode, whereas Masuda, Nakagami and Woronowicz only assume the existence of a right Haar weight but also the existence of the antipode (with its polar decomposition). However, it does not take too much effort to obtain the right Haar weight from the left Haar weight and the polar decomposition of the antipode. So, I believe it is fair to say that the axioms of [K-V*] are simpler and weaker and that therefore, their theory is stronger than the one of [M-N-W]. Moreover, under the infuence of [K-V*], the axioms in [M-N-W] have been slightly weakened so as to allow a theory with a non-trivial scaling constant. This possibility was foreseen by Kustermans and Vaes and it was shown in [VD5] that some of Woronowicz' examples have this property. Nevertheless, it is right to say that also the work of Masuda, Nakagami and Woronowicz is an important contribution to the theory and that the paper [M-N-W] contains interesting results and uses fine techniques.

Let us recall that in all of these more general theories, one has to assume the existence of the Haar weights. A theory with reasonable axioms from which the Haar weights can be obtained is (still) not available. Only in special cases (like the compact and the discrete quantum groups), there are existence theorems. Fortunately however, in all the known examples, the Haar weights exist. In fact, there is even a more or less standard method to obtain the Haar weights for concrete examples (see e.g. [VD5], [V-VD] and [Ja]).

The theory of *multiplicative unitaries* also has a long history and in some sense, runs parallel with the other attempts to generalize Pontryagin duality. Vanheeswijck in [VH] was the first to develop an independent theory, but the objects where considered long before (and carried different names: the Kac-Takesaki operator, the fundamental operator, ...). However, it was Baaj and Skandalis in [B-S] who studied multiplicative unitaries in most detail and they obtained many strong results. Later also Woronowicz published a fundamental paper on the subject, introducing the notion of manageability, see [W4] (and also [S-W]). The theory of multiplicative unitaries however is of a different kind than the other theories, previously discussed. Some people consider it as one form of the theory of locally compact quantum groups, but I would mainly consider it as an *important (and indispensible) tool* for this theory. Indeed, the theory of multiplicative unitaries is a theory without the presence of the Haar weights. Also here, still it is not possible to give axioms so that the Haar weights can be found. It must be mentioned however that in this case, there are some indications that this might be possible in the near future (see again [VD5] and [Ja]).

The *present note* is part of a set of papers that continues along the lines of [K-V*]. Recall that the earlier theories (Kac algebras and the work of Masuda and Nakagami) are formulated in the von Neumann algebraic setting. Later, mainly under the influence of the leading works of Woronowicz on compact quantum groups, there was a strong tendency to move to the C^*-algebra framework. There are certainly good reasons to do this. After all, we are studying locally compact quantum groups and C^*-algebras can be viewed as locally compact quantum spaces.

On the other hand, working with C^*-algebras also has some (more technical) disadvantages and the von Neumann algebra setting is easier. First of all, the axioms in the C^*-setting are somewhat more involved (see Section 1 of this paper). Secondly, the theory of weights on von Neumann algebras is better developed and more widespread. And there are other arguments for which I refer to the other papers [VD7] and [VD8] where the theory is being developed within the von Neumann algebras.

In this note, we will focus on the *intimate relation* between the C^*-algebra approach and the von Neumann algebra approach. In *Section* 1 we simply give the two main definitions of locally compact quantum groups in each of these two frameworks. We illustrate these with the two basic examples. In *Section* 2 we show that it is possible to pass from a locally compact quantum group in the C^*-algebraic sense to one in the von Neumann algebraic sense. This is done in a rather quick way and very little of the theory of locally compact quantum groups (C^*-algebraic version) is needed for this step. This fact is very important because it makes it possible to develop the theory of locally compact quantum groups within the von Neumann algebra setting and deduce results about the C^*-theory. This has technical advantages as we mentioned already. Also in this section, we look at our examples. In *Section* 3 we discuss the other procedure. In this case, more of the theory is needed, but this is no problem as we want to develop the theory in the framework of von Neumann algebras anyway. We also see what happens when the two steps are performed one after the other. Finally, in *Section* 4 we draw some conclusions.

The paper is of an *expository nature.* Some of the basic proofs are briefly scetched but for details, we refer to other papers. In [VD7] we develop the theory of locally compact quantum groups completely within the von Neumann algebra setting. This paper is still quite concisely written and it is aimed at people already familiar with the subject. On the other hand, we also plan to write a set of lecture notes ([VD8]), especially for those who are not familiar with the theory of locally compact quantum groups as it is known now, but who like to learn more about this subject. In this note, we try to use as little as possible of these other two papers because we focuss here on the relation between the two approaches. Especially, as we already mentioned earlier, because the theory in these two other papers is developed within the von Neumann algebra context, it is important that we do not rely on the C*-algebraic development of the theory. This point of view also makes the difference between this note on the two approaches and the work of Kustermans and Vaes about this subject (as found in [K-V3]).

Let us finish this introduction with some conventions about the *notations* and by giving some *basic references.*

For a C*-algebra A, we use A^* for the dual space of all continuous linear functionals on A. When M is a von Neumann algebra, we use M_* to denote the predual, i.e. the space of all σ-weakly continuous linear functionals on M. The tensor product of C*-algebras will always be the (completed) minimal tensor product. The tensor product of von Neumann algebras will be the von Neumann tensor product and the tensor product of Hilbert spaces will be the Hilbert space tensor product. We will use the same symbol $\otimes$ for all these different tensor products but it will be clear from the context which one is considered. We will use ι to denote the identity map and so, when e.g. ω is a bounded linear functional on a C*-algebra A the slice map $\iota \otimes \omega$ will be a linear map from the minimal C*-tensor product $A \otimes A$ to A.

A good reference for the theory of operator algebras is [T] where also the weights on C*-algebras and von Neumann algebras, as well as the necessary elements of the Tomita-Takesaki theory, needed for this note, are found. The standard works on Hopf algebras are [A] and [S] and for multiplier Hopf algebras, it is [VD1]. Moreover, [K-S] can be consulted for aspects of the standard theory of quantum groups. A survey on the theory of compact quantum groups is found in [M-VD] and on locally compact quantum groups in [K-*]. In [VD6], a short introduction to algebraic quantum groups is found and a discussion about its role in the general theory.

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2005) for inviting me to this meeting and to give a talk on the same subject. The present paper is based on these two talks.

I would also like to express my appreciation for my former Ph.D. students and coworkers J. Kustermans and S. Vaes for their development of the theory of locally compact quantum groups on which this note is based.

1. Two definitions

In this section, we will focus on the *precise definitions.* First we will recall the definition of a locally compact quantum group in the C*-algebraic formulation (due to Kustermans and Vaes, see [K-V1] and [K-V2]). Secondly, we will give the definition in the von Neumann algebra framework (also given by Kustermans and Vaes, see [K-V3]). In both cases, we use the motivating example, coming from a locally compact group, to illustrate the definition.

First we recall what is meant by a coproduct on a C*-algebra.

1.1 Definition Let A be a C*-algebra and consider the minimal C*- tensor product $A\otimes A$ of A with itself. Let $M(A\otimes A)$ be the multiplier algebra of $A\otimes A$. A *comultiplication* (or a coproduct) on A is a non-degenerate *-homomorphism $\Delta : A \to M(A\otimes A)$ satisfying coassociativity $(\Delta\otimes\iota)\Delta = (\iota\otimes\Delta)\Delta$.

This definition requires some remarks. Non-degeneracy of Δ means that $\Delta(A)(A\otimes A)$ is dense in $A\otimes A$. It is easily seen that the maps $\Delta\otimes\iota$ and $\iota\otimes\Delta$, extended to $A\otimes A$ by continuity, are still non-degenerate. It is also known that non-degenerate *-homomorphisms have unique extensions to unital *-homomorphisms on the multiplier algebra. Therefore, the formula for coassociativity makes sense.

Sometimes, for a comultiplication, it is required that $\Delta(A)(A\otimes 1)$ and $\Delta(A)(1\otimes A)$ are subsets of $A\otimes A$. In this case, one can express coassociativity without the need to extend Δ, see [VD1]. The comultiplications considered in this theory, satisfy these conditions. However, usually, a weaker condition is required:

1.2 Assumption The slices of the form $(\omega\otimes\iota)\Delta(a)$ and $(\iota\otimes\omega)\Delta(a)$ are well-defined in $M(A)$ for all $a\in A$ and $\omega\in A^*$. It will be assumed that these elements actually belong to A and that, in each of the two cases, they span a dense subspace of A.

Also here, we like to add some remarks. First, notice that any $\omega\in A^*$ has the form $\rho(\,\cdot\, b)$ for some $\rho\in A^*$ and $b\in A$. This implies that slices as above are well-defined in the multiplier algebra. Moreover, if e.g. $\Delta(A)(1\otimes A)$ is a subset of $A\otimes A$, then also $(\iota\otimes\omega)\Delta(a)\in A$ for all $a\in A$ and $\omega\in A^*$. Similarly for the other slices. Furthermore, if the spaces $\Delta(A)(1\otimes A)$ and $\Delta(A)(A\otimes 1)$ are assumed to be dense, then also the above assumption is fulfilled. In some other approaches to quantum groups, these stronger conditions are part of the axioms. In the theory, developed by Kustermans and Vaes, the weaker conditons are assumed and the stonger conditions follow. These two forms of the density conditions are related with the cancellation law in a group.

Now, we are ready for the first of the two *main definitions*.

1.3 Definition A pair (A, Δ) of a C*-algebra A and a coproduct Δ on A is called a *locally compact quantum group* if there exist a left and a right Haar weight on A.

What is a *left Haar weight*? It is a faithful, lower semi-continuous and densely defined weight φ on A satisfying left invariance, i.e. for all $\omega \in A^*$ with $\omega \geq 0$ and all $a \in A$ with $a \geq 0$ and $\varphi(a) < \infty$, we require that $\varphi((\omega \otimes \iota)\Delta(a)) = \|\omega\|\varphi(a)$. Some extra condition on the weight is necessary: it is assumed to be *central* as will be explained in Section 2 (see Defition 2.3). Similarly, a right Haar weight is defined. It is an important result of the theory that such Haar weights are unique (up to a scalar) if they both exist.

Let us now consider the first of our two *motivating examples*.

1.4 Example Let G be a locally compact group and let A be the C*-algebra $C_0(G)$ of continuous complex functions on G tending to 0 at infinity. Identifying, as explained in the introduction, $A \otimes A$ with $C_0(G \times G)$ and $M(A \otimes A)$ with $C_b(G \times G)$, one can define a comultiplication Δ on A by $\Delta(f)(p, q) = f(pq)$ where $f \in C_0(G)$ and $p, q \in G$. In this case, we see that $(\Delta(f)(1 \otimes g))(p, q) = f(pq)g(q)$ and it follows easily that the map $f \otimes g \to \Delta(f)(1 \otimes g)$ extends to an isomorphism of $A \otimes A$ with itself. In particular, the stronger density conditions are fulfilled and also Δ is non-degenerate. Coassociativity comes from the associativity of the product in G. Integrating w.r.t. the left Haar measure, gives rise to the left Haar weight and the right Haar measure will give the right Haar weight. Hence, $(C_0(G), \Delta)$ is a locally compact quantum group in the sense of Definition 1.3.

Conversely, it can be shown that any locally compact quantum group (A, Δ) (in the sense of Definition 1.3) with an underlying *abelian* C*-algebra A must be of the above form.

Later in this section, we will consider a second example (dual to the previous one), but it will be easier to give this example after we have described the notion of a locally compact quantum group in the von Neumann algebraic setting.

1.5 Definition Let M be a von Neumann algebra and $M \otimes M$ the von Neumann algebraic tensor product. A *comultiplication* on M is a unital, normal *-homomorphism $\Delta : M \to M \otimes M$, satisfying coassociativity $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.

Observe that the definition here is less complicated than in the case of a C*-algebra. There is no need to consider multiplier algebras and non-degeneracy is simply replaced by $\Delta(1) = 1 \otimes 1$. The maps $\Delta \otimes \iota$ and $\iota \otimes \Delta$ extend by continuity and so coassociativity makes sense without further considerations.

Also, there is no need for any form of density conditions. These will turn out to be automatic. We will discuss more about this in Section 3 (where we describe the passage from von Neumann algebras to C*-algebras).

So, we are ready immediately for the second of our two main definitions:

1.6 Definition A pair (M, Δ) of a von Neumann algebra M and a comultiplication Δ on M is called a locally compact quantum group (in the von Neumann algebraic sense) if there exist a left and a right Haar weight.

In this context, a left Haar weight is a faithful, normal, semi-finite weight φ on M satisfying left invariance, i.e. for all $\omega \in M_*$ with $\omega \geq 0$ and $x \in M$ with $x \geq 0$ and $\varphi(x) < \infty$, we have $\varphi((\omega \otimes \iota)\Delta(x)) = \omega(1)\varphi(x)$. Here, unlike the case in the C*-algebraic context, no extra condition is needed. Similarly, a right Haar weight is defined. Also here, it can be shown that these weights are unique (up to a scalar) if they both exist.

The example that we considered in Example 1.4 can also be formulated in this context:

1.7 Example Let G be a locally compact group as before. Now let $M = L^\infty(G)$ where G is considered with the left (or equivalently, with the right) Haar measure. Define again $\Delta(f)(p, q) = f(pq)$ where now $f \in L^\infty(G)$ and $p, q \in G$. We get obviously a comultiplication on the von Neumann algebra M. Left and right Haar weights are obtained again by integrating respectively over the left and the right Haar measures on G.

Also in this case, one can show that any locally compact quantum group (M, Δ) where the underlying von Neumann algebra M is abelian, has to be of the above form.

Let us now also consider the dual example.

1.8 Example Let G be a locally compact group. Consider the Hilbert space $L^2(G)$ (for the left Haar measure) and the left regular representation λ of G on $L^2(G)$, defined as usual by $(\lambda_p \xi)(q) = \xi(p^{-1}q)$ where $p, q \in G$ and $\xi \in L^2(G)$. Consider the von Neumann algebra M, generated by the operators λ_p with $p \in G$. Define a unitary operator W on $L^2(G \times G)$ by $(W\xi)(p, q) = \xi(p, p^{-1}q)$. It is straighforward to verify that $W(\lambda_p \otimes 1)W^* = \lambda_p \otimes \lambda_p$ for all $p \in G$. Then $x \mapsto W(x \otimes 1)W^*$ is a normal and unital *-homomophism from M to $M \otimes M$. It is a comultiplication on M. Again, the pair (M, Δ) will be a locally compact quantum group in the sense of Definition 1.6. In this case, the left and right Haar weights coincide (because Δ is 'co-abelian') and they are given by the formula $\varphi(\lambda(f)) = f(e)$ when $\lambda(f) = \int f(p)\lambda_p dp$ for a continuous function f on G with compact support (and where of course e is the identity in the group). In order to define φ in a correct way and to prove the left invariance properly, one can use standard techniques from the Tomita-Takesaki theory.

From this example, we see that we can easily deduce the dual example in the C*-algebraic context. Simply consider the reduced group C*-algebra $C_r^*(G)$ which is the C*-subalgebra of M generated by the convolution operators $\lambda(f)$ with f continuous and of compact support. It can easily be shown that the restriction of Δ to this C*-algebra is a comultiplication and that the restricion of the Haar weight is a Haar weight on the C*-algebra. Also compare with Example 3.6.ii).

All these examples will be used throughout the note to illustrate several of the main ideas and the different procedures.

Let us finish this section with a few remarks already about the two different approaches. First of all, it is clear that the definition in the C*-algebraic setting is already more involved (as we mentioned already) than in the von Neumann algebraic setting. But there is more. Also when we look at the second example in the von Neumann algebra case (Example 1.8), we see that it is easier to first consider it in the von Neumann algebraic framework and then obtain the C*-algebraic version by 'restriction'. This procedure is a special case of the general one as will be described in Section 3. And even though the first example (Example 1.4) is somewhat easier, and certainly more natural within the C*-frame, it should be mentioned that this is rather an exception. In general, examples are more easily constructed in the von Neumann algebra setting first (as is already the case with the dual example here).

2. From C*-algebras to von Neumann algebras

In this section, we will describe a procedure to pass from a locally compact quantum group in the C*-algebra formulation (cf. Definition 1.3) to the one in the von Neumann algebra setting (cf. Definition 1.6). The way we will do this here is different from (and more direct than) the method used in the original papers by Kustermans and Vaes (cf. [K-V3]). As we already mentioned, the fact that this procedure is simplified, is one of the reasons why the theory of locally compact quantum groups is more easily developed wihtin the von Neumann algebra framework.

So, in what follows, let (A, Δ) be a pair of a C*-algebra A and a comultiplication Δ on A (cf. Definition 1.1), satisfying the Assumption 1.2 and such that there exist a left and a right Haar weight (cf. Definition 1.3).

Consider the double dual A^{**} of A, i.e. the *enveloping von Neumann algebra* of A. We will denote it by $\widetilde{A}$. We have the following result.

2.1 Proposition There is a unique coproduct $\widetilde{\Delta}$ on the von Neumann algebra $\widetilde{A}$ (in the sense of Definition 1.5) that extends the original coproduct Δ from A to $\widetilde{A}$.

This result is not very deep. We know that $M(A \otimes A)$ sits in the von Neumann algebra tensor product $\widetilde{A} \otimes \widetilde{A}$ and that A is σ-weakly dense in $\widetilde{A}$. Then, essentially by definition of $\widetilde{A}$, the *-homomorphism Δ extends uniquely to a normal *-homomorphism $\widetilde{\Delta} : \widetilde{A} \to \widetilde{A} \otimes \widetilde{A}$. It is unital because Δ is assumed to be non-degenerate so that the extension to $M(A)$ is already unital. The coassociativity of $\widetilde{\Delta}$ on $\widetilde{A}$ follows from the coassociativity of Δ on A.

Next, let φ be a left Haar weight on A. Then the following is true.

2.2 Proposition There is a unique normal semi-finite weight $\widetilde{\varphi}$ on $\widetilde{A}$, extending φ on A. Moreover, $\widetilde{\varphi}$ is still left invariant.

The first property is standard. Indeed, by lower semi-continuity, we have $\varphi(a) = \sup \omega(a)$ for all $a \in A$ with $a \geq 0$ where the supremum is taken over all $\omega \in A^*$ with $\omega \geq 0$ and $\omega \leq \varphi$. By definition, any ω extends to an element $\widetilde{\omega}$ in $\widetilde{A}_*$ and $\widetilde{\varphi}$ is defined as the supremum of these extensions (see e.g. [B]). The second property is certainly expected but the proof is not so simple. It first requires some left Hilbert algebra techniques to show that the G.N.S.-representation of $\widetilde{\varphi}$ is realized, in a natural way, in the same Hilbert space as the G.N.S.-representation of φ on A. Then, it is relatively straightforward to show that $\widetilde{\varphi}$ is still invariant. For details, we refer to [VD7] and [VD8].

Now, we will formulate the *extra assumption* on the Haar weights on a locally compact quantum group (A, Δ) that we announced earlier (cf. Definition 1.3).

2.3 Definition Let φ be a lower semi-continuous weight on a C*-algebra A. Consider the normal extension $\widetilde{\varphi}$ on $\widetilde{A}$ as in Proposition 2.2. We call φ *central* if the support of $\widetilde{\varphi}$ is a central projection in the von Neumann algebra $\widetilde{A}$.

There are various other ways to formulate this condition on the weights. In [K-V2], it is assumed that the 'extension' of the weight φ to the von Neumann algebra $\pi_\varphi(a)''$ (where π_φ is the G.N.S. representation) is faithful and then the weight is called *approximately K.M.S.* It is more or less clear that this condition is equivalent with the one formulated in Definition 2.3 above. In [M-N-W], still another condition is assumed and the weights satisfying this condition are called *strictly* faithful. Using left Hilbert algebra thechniques, one can relatively easily show that all these notions are the same. We have chosen to work with the above characterization because of the scope of this work. After all, we focus on the von Neumann algebra approach. Moreover, since in most cases, examples are first constructed in the von Neumann algebra setting, this condition is essentially automatic because the constructed weights are faithful on the von Neumann algebra.

Now, we prove a crucial lemma.

2.4 Lemma Let φ be a left Haar weight on (A, Δ) and let ψ be a right Haar weight. Denote by e the support of $\widetilde{\varphi}$ and by f the support of $\widetilde{\psi}$ (the extension of ψ). Then $e = f$.

Proof: First observe that e and f are central projections in $\widetilde{A}$ because Haar weights are supposed to be central (see the remark after Definition 1.3).

We have $\widetilde{\varphi}(1-e) = 0$ because e is the support of $\widetilde{\varphi}$. By left invariance of φ, we also get $\varphi((\rho\otimes\iota)\widetilde{\Delta}(1-e)) = 0$ whenever $\rho \in \widetilde{A}_*$ and $\rho \geq 0$. It follows that $((\rho\otimes\iota)\widetilde{\Delta}(1-e))e = 0$ for all such ρ and therefore also $\widetilde{\Delta}(1-e)(1\otimes e) = 0$. Now, multiply from the left with $\widetilde{\Delta}(x)$ where x is any element in $\widetilde{A}$ satisfying $x \geq 0$ and $\widetilde{\psi}(x) < \infty$. Because e is central, we still have $x(1-e) \geq 0$ and $\widetilde{\psi}(x(1-e)) < \infty$. Then, from right invariance of $\widetilde{\psi}$, we find $\widetilde{\psi}(x(1-e))e = 0$ and because $e \neq 0$, also $\widetilde{\psi}(x(1-e)) = 0$. Because this is true for all such x, we must have $f(1-e) = 0$ and so $f = fe$.

Similarly, $e = ef$ and so $e = f$.

The above result is not new. Essentially, it is already present in the work of Kustermans and Vaes (see [K-V2]). Certainly, the above argument is original, relatively simple and straightforward. But more important is the following observation. It is known that Haar weights are unique, but in the original papers, this result is only proven in an advanced stage of the theory. From the lemma however, we see immediately that two left invariant weights must have the same central support in the von Neumann algebra $\widetilde{A}$ (because these supports are equal to the support of any right invariant weight). But this means that there is a distinguished von Neumann algebra, given by this support. It is an important result and it gives us quicky the following theorem.

2.5 Theorem Let (A, Δ) be a locally compact quantum group (in the sense of Definition 1.3). Consider the associated pair $(\widetilde{A}, \widetilde{\Delta})$ as in Proposition 2.1. Define the von Neumann algebra $M = \widetilde{A}e$ where as before, e is the support of the extension $\widetilde{\varphi}$ of a left Haar weight φ on A. If we define Δ_1 on M by $\Delta_1(x) = \widetilde{\Delta}(x)(e \otimes e)$, we get a comultiplication on M and (M, Δ_1) is a locally compact quantum group (in the sense of Definition 1.6). Moreover, A sits inside M as a dense C*-subalgebra and Δ_1 coincides with Δ on A.

Proof: First, it is clear that Δ_1 is a normal *-homomorphism from M to $M \otimes M$. In the proof of Lemma 2.4 we have seen that $\widetilde{\Delta}(e)(1 \otimes e) = 1 \otimes e$ and it follows that $\Delta_1(e) = e \otimes e$. Therefore, Δ_1 is a unital *-homomorphism on M (because e is the identity in M). The formula $\widetilde{\Delta}(e)(e \otimes e) = e \otimes e$ is also used to show that Δ_1 is still coassociative.

Of course, the Haar weights on M are obtained by restricting the weights $\widetilde{\varphi}$ and $\widetilde{\psi}$ to M (where, as before, φ and ψ are left and right Haar weights on A). Cutting down with e makes these restrictions faithful and it has also no effect on the invariance.

That A is a subalgebra of M is a simple consequence of the fact that the Haar weights are assumed to be faithful on A and from $\Delta(A) \subseteq M(A \otimes A)$ it then also follows that Δ_1 coincides with the original comultiplication Δ on A.

Because of this last statement, it makes sense to denote the coproduct Δ_1 on M again simply with Δ.

We will finish this section by looking at our main examples. But first, we would like to make an important remark.

2.6 Remark We see from the results above that little is needed to pass from a locally compact quantum group in the C*-algebraic sense to one in the von Neumann algebraic sense. This makes it appropriate to develop the theory of locally compact quantum groups mainly within the setting of von Neumann algebras. This is also easier. Uniqueness of Haar weights is e.g. proven using Connes' cocycle Radon Nikodym theorem in an elegant way (see [VD7] and [VD8]). But there are also other reasons for working with von Neumann algebras. The theory of weights on von Neumann algebras is better known than the one on C*-algebras e.g.

In the next section, we will see how one goes back from von Neumann algebras to C*-algebras and how results obtained in the von Neumann algebra context give rise to similar results for the C*-algebras (without much effort).

Now, let us discuss briefly the above procedure in the case of our main examples.

2.7 Examples i) First, consider Example 1.4 where $A = C_0(G)$ for a locally compact group G. Because left and right Haar measures are absolutely continuous with respect to each other, it follows that they will yield the same von Neumann algebra $L^\infty(G)$. Obviously, the above procedure will yield Example 1.7.

ii) Secondly, consider Example 1.8 where M is the von Neumann algebra generated by the left regular representation of the group G. Because here, the C*-algebra A is $C_r^*(G)$, as sitting in M, and because the pair (A, Δ) is obtained from the pair (M, Δ) by 'restriction', again it is quite obvious that the procedure, described in this section, when applied to the pair (A, Δ) will give back the original pair (M, Δ).

3. From von Neumann algebras to C*-algebras

In this section, we start with a locally compact quantum group (M, Δ) in the von Neumann algebraic sense. We do not assume from the start that it is coming from a pair (A, Δ) as in the previous section. Only later in this section, we will consider this case. First, we will briefly describe the (more or less standard) procedure to obtain a distinguished C*-subalgebra A of M, left invariant by the coproduct.

So, let (M, Δ) be a pair of a von Neumann algebra M with a comultiplication Δ as in Definition 1.5 and assume the existence of left and right Haar weights (as in Definition 1.6).

We first need to define the *left regular representation*.

Let φ be a left Haar weight on M. Consider the G.N.S.-representation of M associated with φ. As usual, we denote by $\mathcal{H}_\varphi$ the Hilbert space. The left ideal $\mathcal{N}_\varphi$ is defined as the set of elements $x \in M$ satisfying $\varphi(x^*x) < \infty$ and the canonical map $\Lambda_\varphi : \mathcal{N}_\varphi \to \mathcal{H}_\varphi$ satisfies $\langle \Lambda_\varphi(x), \Lambda_\varphi(x) \rangle = \varphi(x^*x)$ for all $x \in \mathcal{N}_\varphi$. We let M act directly on $\mathcal{H}_\varphi$ so that $y\Lambda_\varphi(x) = \Lambda_\varphi(yx)$ whenever $y \in M$ and $x \in \mathcal{N}_\varphi$.

3.1 Proposition There exists a unitary operator $W \in M \otimes \mathcal{B}(\mathcal{H}_\varphi)$ defined by

$$\Lambda_\varphi((\omega \otimes \iota)\Delta(x)) = ((\omega \otimes \iota)W^*)\Lambda_\varphi(x)$$

for all $x \in \mathcal{N}_\varphi$ and $\omega \in M_*$. Moreover, we have

i) $(\Delta \otimes \iota)W = W_{13}W_{23}$ (where we use the standard leg numbering notation),

ii) $W^*(1 \otimes x)W = \Delta(x)$ for all $x \in M$.

Most of the proof of this result is standard. First one must argue that $(\omega \otimes \iota)\Delta(x) \in \mathcal{N}_\varphi$ when $x \in \mathcal{N}_\varphi$ and $\omega \in M_*$. This is a straightforward consequence of the left invariance of φ. Then the formula in the definition makes sense. The next step is to show that this formula defines a bounded operator W and that $WW^* = 1$. Also this part follows directly from the left invariance of φ. Essentially from the definition, we get $W \in M \otimes \mathcal{B}(\mathcal{H}_\varphi)$. The two formulas i) and ii) follow after some careful calculations. To complete the proof of ii) however, it is already needed that also $W^*W = 1$. Only this last part is non-trivial. Usually, the right Haar weight is used to show this result.

The details of this proof can be found in the original paper [K-V2]. In [VD7] and [VD8] we present another argument for the more difficult aspect (the unitarity of W).

Once this is done, it is relatively easy to get the associated C*-algebra.

3.2 Theorem Let A be the norm closure of the set $\{(\iota \otimes \rho)W \mid \rho \in \mathcal{B}(\mathcal{H}_\varphi)_*\}$. Then A is a C*-algebra and a subalgebra of M. The restriction of Δ to A is a comultiplication on A and the restrictions of the Haar weights give Haar weights on A. So, (A, Δ) is a locally compact quantum group in the C*-algebraic sense (cf. Definition 1.3).

Also the proof of this result is more or less standard and it can be found in [K-V3]. Again, in [VD7] and [VD8] we present a slightly different approach. We do not use the manageability of W to show that A is invariant under the involution (the only non-easy part to show that it is a C*-algebra).

The procedure to go from the von Neumann algebra to the C*-algebra, although more standard, is also more involved than the other one (as we saw in the previous section). Here, it is necessary to develop the theory up to a certain stage before it can be shown that the pair (A, Δ), as defined in 3.2, indeed satisfies the axioms. Fortunately, from our point of view, this is not a problem. Indeed, we propose in [VD7] the development of the theory of locally compact quantum groups in the von Neumann algebra setting. Therefore, as we have said already, it is more important that the other stap does not require too much (as we saw is the case).

Also the following remark is important.

3.3 Remark When developing the theory in the von Neumann algebra setting, several objects are constructed. One of them is the antipode S with its polar decompostion. This polar decomposition has the form $S = R\tau_{-\frac{i}{2}}$ where R is an involutive *-anti-automorphism (the *unitary antipode*) and where $\tau_{-\frac{i}{2}}$ is the analytic extension to the point $-\frac{i}{2}$ of a one-parameter group $(\tau_t)_{t\in\mathbb{R}}$ of *-automorphisms (the *scaling group*). It can be shown that these objects also leave the C*-algebra, defined in Theorem 3.2, invariant. Similar results hold for other objects constructed in the process.

We will not go further into the details here, but we refer again to [VD7] and [VD8]. However, we must say what happens when the two procedures (the one from this section and the one from Section 2), are applied, one after the other. Do we recover the original object? We will state the results in two separate theorems. We begin with the easier case.

3.4 Theorem Let (M, Δ) be a locally compact quantum group in the von Neumann algebraic sense (see Defintion 1.6). Associate the pair (A, Δ) as in Theorem 3.2. If the procedure of Section 2 is applied to this pair (A, Δ), we recover the original pair (M, Δ).

This result is certainly not very deep. Because the Haar weights on A are obtained by restricting the Haar weights of M and because these are assumed to be faithful, it is quite obvious that M will be the von Neumann algebra when applying Theorem 2.5.

This is the other case:

3.5 Theorem Let (A, Δ) be a locally compact quantum group in the C*-algebraic framework (cf. Definition 1.3). Associate the pair (M, Δ) as in Theorem 2.5. If we apply the procedure, described in this section to (M, Δ), we again recover the original pair (A, Δ).

Why is this result more involved? Well, it is here that, for the first time, the density conditions (Assumption 1.2) are needed. In general, the C*-algebra obtained from (M, Δ), using Theorem 3.2, will yield a smaller one than the one we started with. In order to get all of the original C*-algebra, we need that the spaces, spanned by the slices $(\omega \otimes \iota)\Delta(a)$ or by $(\iota \otimes \omega)\Delta(a)$, where $a \in A$ and $\omega \in A^*$, both are dense in A (cf. 1.2).

In the von Neumann algebra case, such density conditions are not assumed (but still true - they are proven). In the C*-algebra case, they are needed, not in the first place because of Theorem 3.5 however, but in order to be able to apply the results, obtained in the von Neumann algebraic setting, to the C*-algebra pair (A, Δ). In particular, this is needed to show that the antipode, together with its polar decomposition, also exists in the C*-algebraic formulation (cf. Remark 3.3).

Again, we finish this section by illustrating these results in the case of the two main examples.

3.6 Examples i) First, consider Example 1.7 where $M = L^\infty(G)$ for a locally compact group G. In this case, the left regular representation W, as defined in Proposition 3.1, is the unitary operator on $L^2(G \times G)$ given by $(W\xi)(p, q) = \xi(p, p^{-1}q)$ where $p, q \in G$ and $\xi \in L^2(G \times G)$. If now $\rho \in \mathcal{B}(L^2(G))_*$ has the form $\rho = \langle \cdot\, \xi_1, \eta_1 \rangle$ for $\xi_1, \eta_1 \in L^2(G)$, a straightforward calculation gives that the operator $(\iota \otimes \rho)W$ is multiplication on $L^2(G)$ with the function f given by $f(p) = \int \overline{\eta_1(q)}\xi_1(p^{-1}q)\, dq$ (where we integrate over the left Haar measure on G). As a convolution of two functions in $L^2(G)$, we get that $f \in C_0(G)$. So we see that the C*-algebra A, as defined in Theorem 3.2, in this case gives precisely $C_0(G)$.

ii) In the second example, one verifies that W is again a unitary on $L^2(G \times G)$, but now given by $(W\xi)(p, q) = \xi(qp, p)$. If again ρ has the form $\langle \cdot\, \xi_1, \eta_1 \rangle$, we find that $(\iota \otimes \rho)W$ is a convolution operator of the form $\int f(p)\lambda_p\, dp$ with $f(p) = \overline{\eta_1(p)}\xi_1(p)$. Again we see that the C*-algebra from the theorem will be precisely $C^*_r(G)$.

4. Conclusions

The theory of locally compact quantum groups has been developed first in the C*-algebraic framework by Kustermans and Vaes, see [K-V1] and [K-V2]. From a certain point of view, it was quite natural to do so. Shortly later, it was shown in [K-V3] that it is also possible to use von Neumann algebras as the underlying frame and that the two approaches are equivalent. The same objects are studied, but starting from a different set of axioms in a different framework.

In this note, we have given the two definitions of a locally compact quantum group (in each of these settings), see Section 1. We have shown in Section 2 that it is possible to go from a locally compact quantum group in the C*-algebraic formulation to one in the von Neumann algebraic setting without much effort and in particular, without having to prove too many results from the starting definition. Therefore, and for other reasons mentioned in this note, it now seems more appropriate to develop the theory first within the von Neumann algebras. We also saw in Section 3 how one can go back and conclude results in the C*-algebraic setting from the theory in the von Neumann algebraic setting.

In [VD7] we have developed the theory independently in the von Neumann algebraic frame. The approach turns out to be more direct than in [K-V3] and also simpler. Moreover, in combination with the short procedure described in Section 2 of this note, it also gives an easier treatment of the C*-case than in the original papers.

As mentioned already, the paper [VD7] is rather condensed. It is aimed at readers who have at least some knowledge about the theory of locally compact quantum groups already. On the other hand, a paper [VD8] is planned where all details will appear so as to be useful for anyone who wants to learn and get more familiar with the theory of locally compact quantum groups. This note should be considered as preliminary to these two papers.

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