Scattering theory and non-commutative geometry

Fields 2013

Alan Carey, ANU and Muenster

This is an overview of a topic that is somewhat speculative.

Recently with Gayral, Rennie and Sukochev I established the local index formula of noncommutative geometry in a form applicable to non-compact spaces and non-unital operator algebras.

In the course of proving this result, we organised in collaboration with Fritz Gesztesy (Missouri) who had been working on a question in mathematical scattering theory a workshop in ESI in February 2011 and this talk outlines some ideas that have come from that meeting.

The physical models occur in two and three dimensional Euclidean space with electrons moving in the presence of a magnetic field and colliding with an obstacle.

In the 1950s I. Lifshitz [8] introduced the spectral shift function. He was motivated by studying the S-matrix in quantum scattering problems. This spectral shift function was made mathematically precise (in the framework of trace class perturbations) by M. Krein in his famous 1953 paper.

It says the following (roughly).

For a pair of self-adjoint bounded operators A_0 and A_1 such that their difference (being the perturbation) is trace class, there exists a unique function $\xi \in L^1(\mathbf{R})$ satisfying the trace formula:

$$Tr(\phi(A_1) - \phi(A_0)) = \int \xi(\mu)\phi'(\mu)d\mu$$

whenever ϕ belongs to a class of admissible functions. This theorem has been substantially generalised over the intervening years.

Spectral flow

At first sight this is an unrelated concept.

Recall that spectral flow was introduced in the index theory papers of Atiyah-Patodi- Singer in the 1970's. They give Lusztig the credit for the idea.

Spectral flow is usually defined by considering the net number of eigenvalues of a one parameter family of Fredholm operators that change sign as one moves along the path.

In the spectral shift case we would consider a path $A(t); t \in \mathbf{R}$ joining A_1 and A_0. One may introduce the 'spectral flow function' sf on $\mathbf{R}$, $\mu \to sf(\mu)$ by counting the net number of eigenvalues that cross the point $\mu \in \mathbf{R}$ as one moves along the path $A(t)$.

Modern History

To my knowledge the first person to understand that the spectral shift function and the spectral flow function are the same in some examples was Werner Mueller (Bonn) in 1998.

In the last 60 years a huge industry has grown up around generalisations and applications of the spectral shift function. Similarly there is a large literature on spectral flow. These are beginning to come together.

In 2005-2007 Sukochev and his student Azamov began a systematic study of the relationship between the two.

Note that one usually considers the spectral shift function for operators with continuous spectrum and spectral flow for operators with a discrete spectrum.

However in 2007, in collaboration with Azamov and Sukochev, we showed that under very general conditions that guarantee that both are defined, the spectral shift and spectral flow functions are the same.

The proof was motivated by ideas from non-commutative geometry or more particularly non-commutative analysis. One of the key tools was the creation of a very general theory of ‘non-commutative calculus’ (known as the theory of double operator integrals).

Azamov has continued this work on his own. He has argued that the spectral shift function should be viewed as a generalisation of the spectral flow function that applies to non-Fredholm operators, that is operators with continuous spectrum as occur in quantum mechanical scattering theory. See http://arxiv.org/pdf/0810.2072.pdf.

Note that in the scattering theory literature one usually considers second order operators (e.g. Schrödinger operators) and in the study of spectral flow one is primarily concerned with first order operators. The theory of double operator integrals applies to both and is used to relate the first and second order theory.

The difficulty with Azamov's idea is that the spectral shift function is a purely analytic notion that does not have, as yet, a topological counterpart when one deals with non-Fredholm operators as occur in real scattering problems.

Index theory in the modern sense is about expressing the analytic index in topological terms. So if the spectral shift function is giving some generalised notion of spectral flow what is the corresponding topology?

It seems likely that it is related to geometry.

The Witten index

Witten was studying supersymmetric quantum field theory and as a toy model introduced supersymmetric quantum mechanics in the 1980s. He speculated that there might be a notion of index for non-Fredholm operators and proposed a formula for this.

This idea was taken up by mathematical physicists, Bollé, Gesztesy, Grosse and Simon in 1987. First of all they found simple examples of Dirac operators coupled to connections that had continuous spectrum (in two dimensions) for which the Witten formula gave a finite answer.

They also observed that if, on the other hand, the connection was chosen so that these operators were Fredholm then the Witten index is the Fredholm index. In non-Fredholm situations it can be any real number.

Today I want to describe a way to connect NCG to a paper of Gesztesy-Simon (1987) where they proved that the Witten index is stable under trace class perturbations.

We are given a Hilbert space $\mathcal{H}$ equipped with a $\mathbf{Z}^2$ grading γ and a self adjoint unbounded operator $\mathcal{D}$ such that $\mathcal{D}\gamma + \gamma\mathcal{D} = 0$. Then we may write:

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}.$$

The operator $\mathcal{D}^2$ is the Hamiltonian. In supersymmetric language $\mathcal{D}$ is the supercharge generator.

Simon-Gesztesy et al consider a special case. They consider for z in the intersection of the resolvent sets the situation where

$$(z + \mathcal{D}_-\mathcal{D}_+)^{-1} - (z + D_+D_-)^{-1}$$

is in the trace class,

Using these operators, define the nonnegative self-adjoint operators

$$H_1 = \mathcal{D}_- \mathcal{D}_+; \qquad H_2 = \mathcal{D}_+ \mathcal{D}_-$$

and introduce the spectral shift function $\xi(\mu; H_2; H_1)$ associated with a pair $(H_2; H_1)$ such that

(i) $\xi(\mu; H_2; H_1) = 0$, $\mu < 0$

(ii)

$$\mathrm{Tr}(z + H_1)^{-1} - (z + H_2)^{-1} = -\int_0^\infty \xi(\mu; H_2; H_1)(\mu - z)^{-2} d\mu$$

In the 80s papers of Bollé et al they introduce the ‘Witten index’ of $\mathcal{D}_+$ as

$$\lim_{z \to 0} z \mathrm{Tr}[(z + H_1)^{-1} - (z + H_2)^{-1}]$$

whenever the limit exists.

What is known about this index requires the use of the spectral shift function.

In examples it can be shown that the spectral shift function is a constant or more generally that the function:

$$\Delta(z) = z\mathrm{Tr}[(z + H_1)^{-1} - (z + H_2)^{-1}]$$

is independent of z. Gesztesy has conjectured that more generally this function has a simple dependence on z that follows from properties of the spectral shift function.

Reminder: for some non-Fredholm operators $\mathcal{D}_+$ there are examples where the Witten index can be any real number.

The example

Let $\sigma_j; j = 1, 2, 3$ be the Pauli matrices. Consider the following operator densely defined on $L^2(R^2, C^2)$:

$$D = \sigma_1(\frac{\partial}{i\partial x} - a_x) + \sigma_2(\frac{\partial}{i\partial y} - a_y)$$

where $a = (a_x, a_y)$ is a connection. Physically we are looking at a fermion in a magnetic field.

This gives an example of a supersymmetric quantum system with grading $\gamma = \sigma_3$. For one particular choice of connection the Witten index can be shown to be given by the total magnetic flux calculated from the curvature da which need not be integral. By perturbing this case one gets other examples with the same index.

There is a connection between the previous discussion and the 2009 thesis of Jens Kaad. Kaad's thesis built on older work of Helton and Howe, R. Carey and J. Pincus, Connes and Connes-Karoubi.

Consider bounded operators T satisfying $[T, T^*]$ is trace class. R. Carey-Pincus wrote $T = X + iY$ where X is the real part and Y is the imaginary part of T. Then we have $[T, T^*] = 2i[X, Y]$ and thus we are dealing with an 'almost commuting' pair of self adjoint operators. They then introduced their 'principal function', observed that it can be related to the spectral shift function and created a functional calculus for almost commuting pairs. They introduce an index which is the same as the Witten index provided one connects the two formalisms as I will now do.

For non-Fredholm operators T with $[T, T^*]$ trace class, the relevant Carey-Pincus paper was in 1986 and this led in part to Kaad's thesis.

Suppose we are given Banach algebras A and J where J is an ideal in A (not necessarily closed).

Let $b : J \otimes A + A \otimes J \to J$ be an extension of the Hochschild boundary given by

$$b : s \otimes a + a' \otimes t \to sa - as + a't - ta'$$

where $s, t \in J$ and $a, a' \in A$.

Choose for example A to be the C^*-algebra generated by T, T^* and $J = L^1(\mathcal{H}) \cap A$.

There is an exact sequence

$$X : 0 \to J \to A \to B$$

where $i : J \to A$ is the inclusion and $q : A \to B$ is the quotient.

By the zeroth relative continuous cyclic homology group of the pair (J, A) we will understand the quotient space $HC_0(J, A) = J/Im(b)$. We also have the first cyclic homology group of B denoted $HC_1(B)$ (cf Loday).

Kaad in Chapter 1 of his thesis proves the following result:

Theorem. (i) The operator trace determines a well defined map on the zeroth continuous relative cyclic homology group $\mathrm{Tr}^* : HC_0(J; A) \to \mathbf{C}$

(ii) This can be extended to a map on $HC_1(B)$ using the connecting map

$$\partial_X : HC_1(B) \to HC_0(J; A)$$

coming from the above exact sequence.

Notice that B is a commutative algebra. The pair $q(T); q(T^*)$ defines a class $q(T) \otimes q(T^*)$ in $HC_1(B)$. This class maps to the commutator $[T, T^*]$ under ∂_X.

What has this to do with the Witten index?

The Witten index is formulated in terms of unbounded operators. It can be reformulated as a bounded operator problem using the map on self adjoint unbounded operators $\mathcal{D}$ on the space $\mathcal{H}^{(2)} = \mathcal{H} \oplus \mathcal{H}$ that takes us to the operator $F_{\mathcal{D}} = \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}$.

It is important to note that in the context of the Witten index we do not have ‘full supersymmetry’. Recall that we have the grading γ on $\mathcal{H}^{(2)}$ and the relation $\gamma\mathcal{D} + \mathcal{D}\gamma = 0$ but we only consider the case where

$$z[(z + \mathcal{D}_-\mathcal{D}_+)^{-1} - (z + D_+D_-)^{-1}]$$

is trace class and do not ask for the individual resolvents to be trace class as we would in a Fredholm problem.

Set $z = \lambda^2$ and rewrite the previous formula as

$$\mathrm{Tr}((1 + \lambda^{-2}\mathcal{D}_-\mathcal{D}_+)^{-1} - (1 + \lambda^{-2}\mathcal{D}_+\mathcal{D}_-)^{-1})$$

so we are scaling $\mathcal{D}$ by λ^{-1} and taking $\lambda \to 0$. This is the same limit as in McKean-Singer situation although now for operators that are not necessarily Fredholm.

So let us define

$$F_{\mathcal{D}}^{\lambda} = \lambda^{-1}\mathcal{D}(1 + \lambda^{-1}\mathcal{D}^2)^{-1/2}$$

Then

$$1 - (F_{\mathcal{D}}^{\lambda})^2 = (1 + \lambda^{-2}\mathcal{D}^2)^{-1}$$

Introduce the notation:

$$F_{\mathcal{D}}^{\lambda} = \begin{pmatrix} 0 & T_{\lambda}^{*} \\ T_{\lambda} & 0 \end{pmatrix}.$$

then putting these definitions together:

$$T^{\lambda} = \lambda^{-1}\mathcal{D}_{+}(1 + \lambda^{-2}\mathcal{D}_{-}\mathcal{D}_{+})^{-1/2}$$

and

$$1 - (F_{\mathcal{D}}^{\lambda})^{2} = \begin{pmatrix} 1 - T_{\lambda}^{*}T_{\lambda} & 0 \\ 0 & 1 - T_{\lambda}T_{\lambda}^{*} \end{pmatrix}.$$

Consequently our assumption that

$$(1 + \lambda^{-2}\mathcal{D}_{-}\mathcal{D}_{+})^{-1} - (1 + \lambda^{-2}\mathcal{D}_{+}\mathcal{D}_{-})^{-1}$$

is trace class translates in the bounded picture to the assumption that the commutator $[T_{\lambda}, T_{\lambda}^{*}]$ is trace class. The Witten index is thus calculating $\lim_{\lambda \to 0} \mathrm{Tr}([T_{\lambda}, T_{\lambda}^{*}]$ when this exists.

Conclusion. The theorem quoted above from Kaad's thesis tells us that the Witten index is calculated using the values of a function defined on $HC_1(B)$.

Next steps. Joachim Cuntz asked the question of whether one can use the S-operator in cyclic homology to understand the homotopy invariance of this number $\mathrm{Tr}([T_\lambda, T_\lambda^*]$. With Jens Kaad I was able to answer this question in the framework of general Schatten ideals, not just the trace class.

We found a bicomplex that defines an homology theory which replaces the cyclic homology discussion of Kaad's thesis in the case of higher Schatten ideals. These ideals and the framework we have are needed to discuss examples in dimensions greater than two (all examples to date have been one and two dimensional).

What we have proved

The natural question is to ask about operators with $[T, T^*]$ lying in a higher Schatten class.

The homology theory suggests however that we consider the stronger condition that

$$\mathsf{T}r[(1 - T^*T)^{-n} - (1 - TT^*]^n] \qquad (*)$$

is finite.

This number we call the homological index. It can be understood as the pairing of a cycle in our bicomplex with a cocycle in a dual cohomology theory. This number is homotopy invariant in cyclic theory as suggested by Joachim.

What is interesting about the construction is that it gives an homology theory into which the old work of R. Carey and Pincus fits as the lowest degree example and for which there are natural higher degree examples.

The proofs at the moment are complicated.

There are examples of unbounded supersymmetric operators $\mathcal{D}$ which map to bounded operators T as before with (*) finite.

There is a corresponding map $\lambda^{-1}\mathcal{D}_+ \to T_\lambda$ for these scaled operators and we can show that the homological index is non-trivial by calculating the $\lambda \to \infty$ limit.

There are no general results as yet on the $\lambda \to 0$ limit.