K-theory and the Lefschetz fixed-point formula

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Poincaré duality for C*-algebras

Definition

Two C*-algebras A and B are *Poincaré dual* if there exist classes

$$\Delta \in \mathrm{KK}(A \otimes B, \mathbb{C}) \text{ (the `unit')}, \quad \widehat{\Delta} \in \mathrm{KK}(\mathbb{C}, A \otimes B) \text{ (`co-unit')}$$

such that

$$\widehat{\Delta} \otimes_A \Delta = 1_B, \quad \widehat{\Delta} \otimes_A \Delta = 1_A.$$

In this case, one can check that the map

$$\Delta \cap \cdot : \mathrm{K}_*(A) \to \mathrm{K}^*(B), \quad \Delta \cap a := (a \otimes_{\mathbb{C}} 1_B) \otimes_{A \otimes B} \Delta$$

is an isomorphism interchanging the K-theory of A and the K-homology of B.

Self-duality for K-oriented manifolds

Example

If X is a compact, K-oriented manifold, there is a distinguished elliptic operator on X called the *Dirac operator*. It determines a class $[D] \in \mathrm{KK}(C(X), \mathbb{C})$. Let $\delta \colon X \to X \times X$ be the diagonal map. Set $\Delta := \delta_*([D]) \in \mathrm{KK}(C(X) \otimes C(X), \mathbb{C})$. For $\widehat{\Delta}$, let

- ν be the normal bundle to the embedding $\delta \colon X \to X \times X$
- ξ_ν be the Thom class in $\mathrm{KK}(\mathbb{C}, C_0(\nu))$ of the vector bundle ν over X
- $\widehat{\Delta} \in \mathrm{KK}(\mathbb{C}, C(X \times X))$ be the image of ξ_ν under the map $\mathrm{KK}(\mathbb{C}, C_0(\nu)) \to \mathrm{KK}(\mathbb{C}, C(X \times X))$ induced from tubular neighbourhood embedding of ν in $X \times X$.

Then Δ and $\widehat{\Delta}$ induce a Poincaré duality between $C(X)$ and itself.

Remark

$\Delta \cap [E]$ is the class of the Dirac operator 'twisted' by E.

Other examples of Poincaré dual C*-algebras

- $C(X)$ for any compact smooth manifold X ($\mathbb{K}$-oriented or not) is dual to $C_0(TX)$ where TX is the tangent bundle. (Kasparov, Connes, Skandalis)
- The irrational rotation algebra A_θ is self-dual (Connes).
- The Cuntz-Krieger algebras O_A and O_{A^T} are Poincaré dual (Kaminker and Putnam)
- If G is a Gromov hyperbolic group and ∂G its Gromov boundary then $C(\partial G) \rtimes G$ is self-dual. (Emerson)
- If G is a discrete group acting properly, co-compactly and smoothly on a smooth manifold X then the orbifold C*-algebra $C_0(X) \rtimes G$ is Poincaré dual to $C_0(TX) \rtimes G$. (Emerson, Echterhoff, Kim)

What Poincaré duality is good for

To describe the K-homology of a C*-algebra in some geometric fashion.

Example

Poincaré self-duality for K-oriented manifolds implies that every K-homology class for $C(X)$ is represented by a $d := \dim(X)$-dimensional spectral triple over $C^\infty(X)$ (principal values grow like $\lambda_n \sim n^{\frac{1}{d}}$) – important for noncommutative geometry.

Another consequence of Poincaré duality:

Proposition

If A is separable with a separable dual in KK, then the K-theory of A has finite rank.

Poincaré duality categorically

Poincaré duality means that the functor

$$T_A\colon \mathrm{KK} \to \mathrm{KK},\ D \mapsto A \otimes D,$$
$$f \in \mathrm{KK}(D_1, D_2) \mapsto 1_A \otimes f \in \mathrm{KK}(A \otimes D_1, A \otimes D_2)$$

is left adjoint to the functor T_B similarly defined, *i.e.* there is a natural system of isomorphisms

$$\mathrm{KK}(A \otimes D_1, D_2) = \mathrm{Hom}_{\mathrm{KK}}(T_A(D_1), D_2) \cong$$
$$\mathrm{Hom}_{\mathrm{KK}}(D_1, T_B(D_2)) = \mathrm{KK}(D_1, B \otimes D_2).$$

one for each pair D_1, D_2.

Euler characteristics

A and B Poincaré dual with unit $\Delta \in \mathrm{KK}(A \otimes B, \mathbb{C})$, co-unit $\widehat{\Delta} \in \mathrm{KK}(\mathbb{C}, A \otimes B)$ we can pair them to get

$$\widehat{\Delta} \otimes_{A \otimes B} \Delta \in \mathrm{KK}(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}.$$

Proposition

If A and B are Poincaré dual and satisfy the Künneth and UCT theorems then

$$\widehat{\Delta} \otimes_{A \otimes B} \Delta = \mathrm{rank}(\mathrm{K}_0(A)) - \mathrm{rank}(\mathrm{K}_1(A)).$$

Proof.

Use the Künneth and Universal coefficient theorems to write $\widehat{\Delta} = \sum_i x_i \otimes y_i$ where (x_i) is a basis for $\mathrm{K}_*(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, y_i the dual basis for $\mathrm{K}_*(B) \otimes_{\mathbb{Z}} \mathbb{Q}$, do the same for Δ, and compute. □

Why is this interesting?

The left-hand side of the 'Gauss-Bonnet' theorem

$$\widehat{\Delta} \otimes_{A \otimes B} \Delta = \mathrm{rank}(\mathrm{K}_0(A)) - \mathrm{rank}(\mathrm{K}_1(A))$$

of the previous slide is a straight Kasparov product which can be computed geometrically if Δ and $\widehat{\Delta}$ have nice geometric descriptions. The right-hand side – by contrast – is a global homological invariant of A, you need to compute the K-theory of A to decide what it is.

Example

If X is a K-oriented manifold then it is a simple exercise to check that $\widehat{\Delta} \otimes_{C(X \times X)} \Delta$ is the Fredholm index of the *de Rham operator* on X.

The Lefschetz Theorem in KK

Theorem

If A and B are Poincaré dual with unit and co-unit $\Delta, \widehat{\Delta}$, and if $f \in \mathrm{KK}(A, A)$, then

$$(f \otimes 1_B)_*(\widehat{\Delta}) \otimes_{A \otimes B} \Delta = \mathrm{Tr}_s(f_*)$$

where Tr_s is the graded trace of f acting on $\mathrm{K}_*(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

- We call the invariant on the left-hand side the *geometric trace of f*. The geometric trace of any $f \in \mathrm{KK}(A, A)$ is defined for any dualizable A (pick a dual; the trace is independent of the choice).
- The invariant on the right-hand side is the *homological trace of $f \in \mathrm{KK}(A, A)$*. It is defined for any A satisfying the UCT and Künneth theorems and for which the K-theory has finite rank.

Why the Lefschetz theorem is useful

As with the 'Gauss-Bonnet theorem', the left-hand side of the Lefschetz theorem

$$(f \otimes 1_B)_*(\widehat{\Delta}) \otimes_{A \otimes B} \Delta = \mathrm{Tr}_s(f_*)$$

can be computed geometrically if one has a geometrically interesting dual $B, \Delta, \widehat{\Delta}$, and a geometrically interesting f to compute with.

Example – the classical Lefschetz theorem

X a K-oriented manifold, $[f^*] \in \mathrm{KK}(C(X), C(X))$ the class of a smooth map $f\colon X \to X$ such that the graph $x \mapsto (x, f(x))$ is transverse to the diagonal embedding $\delta\colon X \to X \times X$.

Exercise. The geometric trace $([f^*] \otimes 1_{C(X)})_*(\widehat{\Delta}) \otimes_{C(X\times X)} \Delta$ is the algebraic fixed-point set

$$\sum_{x\in\mathrm{Fix}(f)} \det(1 - D_x f) \in \mathbb{Z}.$$

We deduce the traditional Lefschetz fixed-point theorem.

The idea of the proof

From the definitions of Δ and $\widehat{\Delta}$

$$([f^*] \otimes 1_{C(X)})_*(\widehat{\Delta}) \otimes_{C(X \times X)} \Delta = ([f^*] \otimes 1_{C(X)})_*(\xi_\nu) \otimes_{C(X \times X)} \delta_*([D]) = \xi_\nu \otimes_{C(X \times X)} \Gamma(f)_*([D])$$

where $\Gamma(f) \colon X \to X \times X$ is the graph of f. Now roughly ξ_f is a cohomology class supported near the diagonal in $X \times X$ and $\Gamma(f)_*([D])$ is a homology class supported on the graph of f in $X \times X$. The pairing only depends on what happens on the intersection of these two supports, which is a neighbourhood of the fixed-point set of f, a finite set of points in X. The result follows from a local, linear index computation at each fixed-point.

Example – a Lefschetz fixed-point theorem for orbifolds

Using the KK-Lefschetz theorem one has the chance to find noncommutative analogues of the classical Lefschetz fixed-point formula. The following is one example.

Let G be a discrete group acting

- Properly
- Isometrically
- Co-compactly

on a smooth Riemannian manifold X.

Example

- The group $\mathbb{Z}/2$ acting on the circle by complex conjugation.
- The infinite dihedral group G, generated by $x \mapsto x + 1$, $x \mapsto -x$, acting on $\mathbb{R}$.

A class of endomorphisms $f \in \mathrm{KK}(C_0(X) \rtimes G, C_0(X) \rtimes G)$

Automorphisms of $C_0(X) \rtimes G$: covariant pairs (ϕ, ζ), $\phi \colon X \to X$ homeomorphism, $\zeta \in \mathrm{Aut}(G)$ a group automorphism, such that $\phi(\zeta(g)x) = g\phi(x)\ \forall x \in X$.

The transversality assumption: *If $x \in X$, $g \in G$ such that $\phi(gx) = x$, then the map*

$$\mathrm{Id} - d(\phi \circ g)(x) \colon T_xX \to T_xX \tag{0.1}$$

is non-singular.
This implies that the fixed-point set of the induced map on the space $G\backslash X$ of orbits is finite.

The set-up

Let G, X as above.

- Let
$$f \in \mathrm{KK}(C_0(X) \rtimes G, C_0(X) \rtimes G)$$
be the class of the *-automorphism from the covariant pair (ϕ, ζ) as above.
- To compute the geometric trace, we use the dual $C_0(TX) \rtimes G, \Delta, \widehat{\Delta}$ of E-E-K.
- By the KK-Lefschetz theorem the answer will equal the graded trace of f acting on $\mathrm{K}_*(C_0(X) \rtimes G)$ ($\cong \mathrm{RK}^*_G(X)$, what topologists call the 'G-equivariant K-theory of X).

Theorem

(Echterhoff-Emerson-Kim) Choose a point p from each fixed orbit of the induced map $\dot{\phi}\colon G\backslash X \to G\backslash X$.
For each p, let $L_p := \{g \in G \mid \phi(gp) = p\}$ *(it is finite); then the isotropy subgroup* $\mathrm{Stab}_G(p)$ *acts on* L_p *by twisted conjugation* $h \cdot g := \zeta(h)gh^{-1}$. *Let the orbits of this action be represented by elements* $g_1, \ldots, g_m$. *For each i, let* $H_{p,i} \subset \mathrm{Stab}_G(p)$ *be the stabilizer of* g_i *under this action.*
Then $H_{p,i}$ *commutes with* $\phi \circ g_i$ *and the geometric trace of the covariant pair* (ϕ, ζ) *is given by*

$$\sum_{\dot{p}\in \mathrm{Fix}(\dot{\phi})} \sum_i \frac{1}{|H_{p,i}|} \sum_{h\in H_{p,i}} \mathrm{sign}\,\det(\mathrm{id} - D_{p_i}(\phi \circ g_i)|_{\mathrm{Fix}(h)})$$

Remark on the proof

The computation of the geometric trace involves a calculation with certain Hilbert modules and a local index calculation.

Lemma

Let H be a finite group acting orthogonally on $\mathbb{R}^n$. Let $A \in \mathrm{GL}(n, \mathbb{R})$ be a matrix commuting with H. Then the H-index of the twisted Schrdinger type operator $D + AX$ has virtual character

$$\chi \colon H \to \{\pm 1\} \subset \mathbb{C}, \ \chi(h) = \mathrm{sign}\, \det(A|_{\mathrm{Fixed}(h)}).$$

- It is not obvious that the right-hand side is a character!
- $D = d + d^*$ is the de Rham operator on $\mathbb{R}^n$, AX denotes Clifford multiplication by the (linear) vector field $V(x) = Ax$ on $\mathbb{R}^n$.

Back to the classical setting

Even in the classical setting of the C*-algebra $A = C(X)$, X a compact smooth manifold, the K-theory Lefschetz formula offers the possibility of improving the classical Lefschetz fixed-point formula by considering more general

$$f \in \mathrm{KK}(C(X), C(X))$$

than just maps.

A way of describing such Kasparov morphisms is by the theory of *correspondences*. The theory of correspondences also works equivariantly with respect to an action of a compact group G.

K-oriented maps

G is a compact group. A G-manifold is a manifold with a smooth action of G.

Definition

A G-equivariant K-orientation on a smooth G-equivariant map $f\colon X \to Y$, where X and Y are smooth G-manifolds, is a G-equivariant K-orientation on the real G-equivariant vector bundle

$$N_f := f^*(TY) \oplus TX$$

over X.

Example

- If X is a smooth manifold, a K-orientation on a map $X \to \mathrm{pnt}$ is the same as a K-orientation on X.
- The identity map $\mathrm{id}\colon X \to X$ is canonically G-equivariantly K-oriented using the obvious complex structure on $N_{\mathrm{id}} = TX \oplus TX$.

An embedding theorem

G is a compact group.

Theorem

(Mostow) Let $f\colon X \to Y$ be a smooth, G-equivariant map between two smooth G-manifolds of finite orbit type. Then there is

- *A smooth G-equivariant vector bundle V over X,*
- *An orthogonal representation of G on some Euclidean space E,*
- *A smooth G-equivariant open embedding $\varphi\colon V \to Y \times E$*

such that $f = \mathrm{pr}_Y \circ \varphi \circ \zeta_V$, where $\zeta\colon X \to V$ is the zero section, $\mathrm{pr}_Y\colon Y \times E \to Y$ the projection.

Moreover, if f is G-equivariantly K-oriented, V and E may be taken to be G-equivariantly K-oriented vector bundles (over X and a point, respectively).

Kasparov morphism from a K-oriented smooth map

Let $f\colon X \to Y$ be a smooth G-equivariantly K-oriented map factoring as $f = \mathrm{pr}_Y \circ \varphi \circ \zeta_V$ as on the previous slide.
Associated to this data is

- The Thom isomorphism class $\alpha_V \in \mathrm{KK}^G_{\dim V}(C_0(X), C_0(V))$.
- The class $[\varphi_!] \in \mathrm{KK}^G_0(C_0(V), C_0(Y \times E))$ associated to the *-homomorphism $\varphi_!\colon C_0(V) \to C_0(Y \times E)$ induced by φ.
- The Thom isomorphism class $\beta_{Y\times E} \in \mathrm{KK}_{-\dim E}(C_0(Y \times E), C_0(Y))$ for $Y \times E$.

We set

$$f! := \zeta_V! \otimes_{C_0(V)} [\varphi^!] \otimes_{C_0(Y\times E)} \mathrm{pr}_Y! \\ \in \mathrm{KK}_{\dim Y - \dim X}(C_0(X), C_0(Y)).$$

It is independent of the chosen factorization and is purely topologically defined.

Definition of a smooth correspondence

Definition

Let X and Y be smooth manifolds and G a compact group. A *smooth G-equivariant correspondence from X to Y* is a diagram

$$X \xleftarrow{b} (M, \xi) \xrightarrow{f} Y$$

where $b\colon M \to X$ is an equivariant smooth map, $f\colon M \to Y$ is an equivariant K-oriented smooth map, and ξ is an equivariant K-theory class which is compactly supported along the fibres of b.

A correspondence determines a morphism

$$b^*(\xi \cdot f!) \in \mathrm{KK}^G_{\dim Y - \dim X}(C_0(X), C_0(Y))$$

by twisting $f!$ by ξ and pulling back by b.

The correspondence formulation of the Index Theorem

If X is a smooth compact equivariantly K-oriented G-manifold and $f: X \to \mathsf{pnt}$ is the map to a point, then the Atiyah-Singer Index theorem says, literally:

Theorem

If X is a smooth compact equivariantly K-oriented G-manifold then the class in $\mathrm{KK}^G(\mathbb{C}, \mathbb{C}) \cong \mathrm{Rep}(G)$ of the G-equivariant correspondence

$$\mathsf{pnt} \leftarrow (X, \xi) \to \mathsf{pnt}$$

is the analytic G-index of the Dirac operator on X twisted by ξ.

Indeed, the element of $\mathrm{Rep}(G)$ determined as we've defined it by the correspondence

$$\mathsf{pnt} \leftarrow (X, \xi) \to \mathsf{pnt}$$

is exactly the topological index of Atiyah and Singer of the Dirac operator on X, twisted by ξ.

Topological KK^G-theory

An appropriate equivalence relation on correspondences generated by bordism, equivalence of K-oriented maps, and Thom modification determines a theory $\widehat{\mathrm{KK}}^G_*(X, Y)$ and a map

$$\widehat{\mathrm{KK}}^G_*(X, Y) \to \mathrm{KK}^G_*(C_0(X), C_0(Y)). \tag{0.2}$$

Theorem

(Emerson-Meyer) The map (0.2) *is an isomorphism for any compact group, any locally compact G-space Y, and any smooth compact G-manifold X.*

Thus $\mathrm{KK}^G(X, Y)$ has a purely topological description when X is a smooth G-manifold, Y is a locally compact G-space.

Poincaré duality via correspondences

Let X is a smooth, compact G-manifold, TX the tangent bundle of X with its induced G-action. Let $\zeta\colon TX \to X \times TX$ be the zero section $\zeta(x) := \big(x, (x, 0)\big)$. Then ζ admits a canonical K-orientation and the diagrams

$$X \times TX \xleftarrow{\pi\times\mathrm{id}} TX \to \mathrm{pnt}, \quad \mathrm{pnt} \leftarrow X \xrightarrow{\zeta} X \times TX$$

are smooth G-equivariant correspondences from $X \times TX$ to a point and from a point to $X \times TX$, yielding classes

$$\Delta \in \mathrm{KK}_0^G(C(X) \otimes C_0(TX), \mathbb{C}), \quad \widehat{\Delta} \in \mathrm{KK}_0^G(\mathbb{C}, C(X) \otimes C_0(TX)).$$

Theorem

Δ and $\widehat{\Delta}$ are the unit and co-unit for a G-equivariant Poincaré duality between X and TX in KK^G.

..works exactly the same way, if G is a compact group.

Definition

Two G-C*-algebras A and B are *G-equivariantly Poincaré dual* if there exist classes

$\Delta \in \mathrm{KK}^G(A \otimes B, \mathbb{C})$ (the 'unit'), $\widehat{\Delta} \in \mathrm{KK}^G(\mathbb{C}, A \otimes B)$ ('co-unit')

such that $\widehat{\Delta} \otimes_A \Delta = 1_B$, $\widehat{\Delta} \otimes_A \Delta = 1_A$ in KK^G.

As in in the non-equivariant case,

$$\Delta \cap \cdot : \mathrm{K}^G_*(A) \to \mathrm{K}^*_G(B), \;\; \Delta \cap a := (a \otimes_{\mathbb{C}} 1_B) \otimes_{A \otimes B} \Delta$$

is an isomorphism interchanging the G-equivariant K-theory of A and the G-equivariant K-homology of B.

Formal consequence of G-equivariant Poincaré duality

While ordinary KK-theory involves abelian groups, G-equivariant KK-theory involves $\mathrm{KK}^G(\mathbb{C}, \mathbb{C}) \cong \mathrm{Rep}(G)$-modules.

Example

If $G = T$ is the circle, $\mathrm{Rep}(G) = \mathbb{Z}[X, X^{-1}]$ is the ring of Laurent polynomials with integer coefficients. If A is a T-C*-algebra, $\mathrm{K}_*^T(A)$ is thus a module over $\mathbb{Z}[X, X^{-1}]$.

The equivariant version of the theorem previously stated about dualizable A having finite rank K-theory is:

Proposition

If G is a compact group and A is a G-equivariantly dualizable G-C-algebra, then $\mathrm{K}_*^G(A)$ has finite rank as a $\mathrm{Rep}(G)$-module.*

For example $\mathrm{K}_G^*(X)$ has finite rank as a $\mathrm{Rep}(G)$-module, for any smooth compact G-manifold X.

The geometric trace of a correspondence

Theorem

Let X be a smooth compact manifold, $\Lambda \in \mathrm{KK}(C(X), C(X))$ the class of a smooth correspondence $X \xleftarrow{b} (M, \xi) \xrightarrow{f} X$ from X to X. Assume that the map $(b, f)\colon M \to X \times X$ is transverse to the diagonal $X \to X \times X$. Then the intersection space

$$Q_{b,f} := \{m \in M \mid b(m) = f(m)\}$$

admits a canonical smooth structure and equivariant K-orientation, and the geometric trace of Λ is the class of the smooth correspondence $\mathrm{pnt} \leftarrow (Q_{b,f}, \xi|_{Q_{b,f}}) \to \mathrm{pnt}$.
That is: the geometric trace of Λ is the index of the Dirac operator on $Q_{b,f}$ twisted by ξ.

Hence, by the KK-Lefschetz theorem, the graded trace of $\Lambda_*\colon \mathrm{K}^*(X) \to \mathrm{K}^*(X)$ acting on K-theory is determined by the topology of the intersection manifold $Q_{b,f}$.

Since for smooth manifolds one has *equivariant* duality, the obvious notion of ‘geometric trace’ still makes sense.

Definition

Let G be a compact group. If a G-C*-algebra A is dual in KK^G to B with unit and co-unit Δ and $\widehat{\Delta}$, we define the *geometric trace* of f to be

$$(f \otimes 1_B)_*(\widehat{\Delta}) \otimes_{A \otimes B} \Delta \in \mathrm{KK}^G(\mathbb{C}, \mathbb{C}) \cong \mathrm{Rep}(G)$$

as before.

The geometric trace of a correspondence

The previous computation with correspondences goes through equivariantly.

Theorem

Let $\Lambda \in \mathrm{KK}^G(C(X), C(X))$ be the class of a smooth G-equivariant correspondence $X \xleftarrow{b} (M, \xi) \xrightarrow{f} X$ from X to X. Assume that the map $(b, f)\colon M \to X \times X$ is transverse to the diagonal $X \to X \times X$. Then the intersection space

$$Q_{b,f} := \{m \in M \mid b(m) = f(m)\}$$

admits a canonical G-equivariant smooth structure and K-orientation, and the geometric trace of Λ is the Atiyah-Singer G-index of the Dirac operator on $Q_{b,f}$ twisted by ξ.

Example - the case of (equivariant) maps

A smooth G-equivariant map $b\colon X \to X$ is encoded by the correspondence

$$X \xleftarrow{b} X \xrightarrow{\mathrm{id}} X$$

and the transversality assumption that (b, id) is transverse to the diagonal is the traditional general position assumption of the Lefschetz fixed-point theorem. Moreover,

$$Q_{b,\mathrm{id}} = \{x \in X \mid b(x) = x\}$$

is the fixed-point set of b, with a suitable G-equivariant K-orientation – *i.e.* a suitable G-equivariant $\mathbb{Z}/2$-graded complex line bundle L on Q (next slide).

The equivariant K-orientation on $Q_{\mathrm{id},b}$

(continuing the case of maps...)

$Q_{\mathrm{id},b}$ is a finite, G-invariant set of points of X.

Choose $q \in Q_{\mathrm{id},b}$, let $H := \mathrm{Stab}_G(q)$. The function

$$\chi_q \colon H \to \{\pm 1\}, \quad \chi_q(h) := \mathrm{sign}\,\det(\mathrm{id} - D_q b|_{\mathrm{Fixed}(h)})$$

is $\pm$ a character of H, corresponding to $\pm$ a one-dimensional representation V_q of H, and

$$L|_{Gq} = \mathrm{ind}_H^G(V_q) := G \times_H V_q$$

describes the K-orientation L along the orbit Gq.

Is there a homological trace in the equivariant case?

Let G be a compact group, A and B G-C*-algebras which are G-equivariantly dual with unit and co-units Δ and $\widehat{\Delta}$. So the *geometric* trace of any $f \in \mathrm{KK}^G(A, A)$ is defined.

Problem

Is there a G-equivariant analogue of the *homological* trace of $f \in \mathrm{KK}^G(A, A)$ and a corresponding *equivariant* analogue of the Lefschetz fixed-point theorem?

Various problems

- While ordinary KK-theory involves abelian groups and abelian group homomorphisms, G-equivariant KK-theory involves $\mathrm{Rep}(G)$-modules and $\mathrm{Rep}(G)$-module homomorphisms. Although an obvious guess for an ‘equivariant Lefschetz fixed-point formula’ would involve the *module* trace of a module map, since not all modules are free, there is no well-defined notion of ‘trace’ (nor even of ‘rank’).
- (Worse) There is finite group G and two elements f with different geometric traces but which induce the *same map* on equivariant K-theory!

Hodgin groups

Definition

A *Hodgkin group* is a compact group which is connected with torsion-free fundamental group.

Example: tori, SU_n..

Lemma

If G is a Hodgkin group then $\mathrm{Rep}(G)$ is an integral domain.

In this case $\mathrm{Rep}(G)$ embeds in its field of fractions F_G and any $\mathrm{Rep}(G)$-module (*i.e.* $\mathrm{KK}^G(A, B)$ for any A, B), can be made into an F_G-vector space by replacing it by

$$\mathrm{KK}^G(A, B) \otimes_{\mathrm{Rep}(G)} F_G.$$

This construction is 'natural' and so if $f \in \mathrm{KK}^G(A, A)$ then f induces a canonical vector space map on $\mathrm{K}_*^G(A) \otimes_{\mathrm{Rep}(G)} F_G$.

The homological trace for Hodgkin groups

Definition

If A is a dualizable object of KK^G for a Hodgkin group G, and $f \in \mathrm{KK}^G(A, A)$, the *homological trace of f* is defined to be the (graded) vector space trace $\mathrm{tr}_s(f_*)$ of f acting on $\mathrm{K}_*^G(A) \otimes_{\mathrm{Rep}(G)} F_G$.

Remark

Such homological traces are very difficult to compute in general.

Example of the module structure

Example

Let $G = \mathrm{SU}_n$ and $T \subset G$ its maximal torus. Then the $\mathrm{Rep}(G)$-module $\mathrm{K}^*_G(G/T)$ is roughly the ring of integer Laurent polynomials in $n-1$-variables viewed as a module over the subring of symmetric Laurent polynomials. It is a classical (and non-trivial) theorem of Chevalley that it is free of rank the cardinality of the Weyl group $(n-1)!$. But it is not easy to construct a free basis to compute traces with.

The main theorem...

Theorem

(Emerson, Meyer, Dell'Ambrogio) If G is a Hodgkin group and A a dualizable object of KK^G, then the homological trace $\mathrm{tr}_s(f_*)$ of f lies in the image of $\mathrm{Rep}(G) \to F_G$ and agrees with the geometric trace of f.

Corollary

Let X be a smooth compact manifold with a smooth action of a Hodgkin group G. Let $\Lambda \in \mathrm{KK}^G(C(X), C(X))$ be the class of a smooth equivariant correspondence $X \xleftarrow{b} (M, \xi) \xrightarrow{f} X$ from X to X with $(b, f)\colon M \to X \times X$ transverse to the diagonal.
Let $Q_{b,f}$ be the corresponding K-oriented ntersection manifold and $D_{b,f} \cdot \xi$ be the G-equivariant Dirac operator on $Q_{b,f}$ twisted by the equivariant K-theory class ξ. Let $\mathrm{tr}_s(f_*) \in \mathrm{Rep}(G)$ be the homological trace defined on the previous slide.
Then

$$\mathrm{tr}_s(\Lambda_*) = \mathrm{ind}_G(D_{b,f} \cdot \xi) \in \mathrm{Rep}(G).$$

The case of compact Lie groups

- For *connected* groups G: KK^G embeds in $\mathrm{KK}^{\tilde{G}}$ for an appropropriate finite cover $\tilde{G} \to G$, where $\tilde{G}$ is Hodgkin.
- For general compact lie groups the total ring of fractions of $\mathrm{Rep}(G)$ (obtained by inverting all elements which are not zero divisors) is a finite product of fields parameterized by conjugacy classes of Cartan subgroups H. To each such H corresponds a minimal prime ideal I_H in $\mathrm{Rep}(G)$ and $\mathrm{Rep}(G)/I_H$ is an integral domain, which thus embeds in a field of fractions F_H. For any A we consider

$$\mathrm{K}^*_G(A) \otimes_{\mathrm{Rep}(G)} F_H.$$

Any $f \in \mathrm{KK}^G(A, A)$ acts on $\mathrm{K}^*_G(A) \otimes_{\mathrm{Rep}(G)} F_H$. and we can compute its trace there. It agrees with the image of the geometric trace under the map $\mathrm{Rep}(G) \to F_H$.

Summary, problems..

The geometric equivariant Lefschetz fixed-point formula we have presented here generalizes the classical formula in two ways: it is equivariant, and applies to correspondences, not just maps.

Problem

Find applications and/or examples of interesting equivariant correspondences of smooth G-manifolds where the homological trace is of interest.

Problem

Develop a correspondence theory for orbifolds, *i.e.* of proper actions $C_0(X) \rtimes G$, and extend the orbifold Lefschetz fixed-point formula of E-E-K to these.

Problem

Use the Lefshchetz theorem to find new analogues of the classical theorem for some of the standard interesting noncommutative examples of Poincaré duality, *e.g.* A_θ, O_A...