Antichain catching at ω_1 versus antichain catching at ω_2

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Outline

Normal ideals

Assume

- κ regular uncountable
- $\mathcal{I}$ normal ideal

Then

$$\mathbb{B}_{\mathcal{I}} := (\wp(\kappa)/\mathcal{I}, \leq_{\mathcal{I}})$$

is a κ^+-complete boolean algebra

EX:

- $\kappa = \omega_1$
- $\mathcal{I} = NS_{\omega_1}$

Generic ultrapower using normal ideal

G generic for $\mathbb{B}_{\mathcal{I}} \implies G$ "is" a V-normal ultrafilter on $\wp(\kappa) \cap V$

In $V[G]$ define

$$j_G : V \to_G ult(V, G)$$

Large-cardinal like behavior

$\mathcal{I}$ is

1. **precipitous** iff $\Vdash_{\mathbb{B}_{\mathcal{I}}}$ "*ult*$(V, \dot{G})$ is wellfounded"
2. **strong** iff
   - $\mathcal{I}$ is precipitous AND
   - $\Vdash_{\mathbb{B}_{\mathcal{I}}} j(\kappa) = \kappa^{+V}$ (NOTE: $\supseteq$ always holds)
3. **saturated** iff $\mathbb{B}_{\mathcal{I}}$ has κ^+-cc

We'll consider a property called *StatCatch*$(\mathcal{I})$

Layered posets and ideals

A boolean algebra $\mathbb{B}$ is **stationarily layered** iff

$$\{\mathbb{C} \in [\mathbb{B}]^{<|\mathbb{B}|} \mid \mathbb{C} \text{ is a regular subalgebra of } \mathbb{B}\} \tag{1}$$

is stationary in $[\mathbb{B}]^{<|\mathbb{B}|}$.

Lemma

$\mathbb{B}$ *stat-layered* $\implies$ $\mathbb{B}$ *has the* $|\mathbb{B}|$*-cc*

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Can also consider

- “club-layered”
- “cofinally layered”

Layered posets and ideals, cont.

$\mathcal{I}$ is layered $:\iff \mathbb{B}_{\mathcal{I}}$ is layered.

Corollary (to previous lemma)

$\mathcal{I}$ stat-layered $\implies \mathcal{I}$ is saturated.

Theorem (Shelah)

If $\mathcal{I}$ is (club-)layered then it has the lifting property

(by a theorem of Kunen-Szymanski-Tall, the latter implies existence of Baire irresolvable space of size κ)

Outline

Normal measure derived from Mostowski collapse

Assume $\theta >> \kappa$ and

- $M \prec (H_\theta, \in, \{\kappa\} \dots)$
- $\alpha_M := M \cap \kappa \in \kappa$
- $\sigma_M : H_M \to M \prec H_\theta$ inverse of Most. collapse

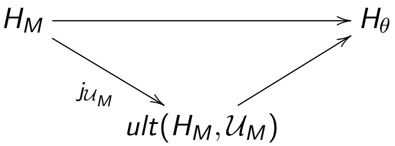
Note $\alpha_M = crit(\sigma_M)$. Define

$$\mathcal{U}_M := \{A \in \wp(\alpha_M) \cap H_M \mid \alpha_M \in \sigma_M(A)\}$$

Normal measure derived from Mostowski collapse, cont.

Then

- $\mathcal{U}_M$ is a H_M-normal ultrafilter on $P^{H_M}(\alpha_M)$
- for any normal ideal $\mathcal{I} \in M$ on κ: if M is $\mathcal{I}$-good then letting $\mathcal{I}_M := \sigma^{-1}(\mathcal{I})$ then $\mathcal{U}_M$ "is" a $(H_M, \mathbb{B}_{\mathcal{I}_M})$-normal ultrafilter



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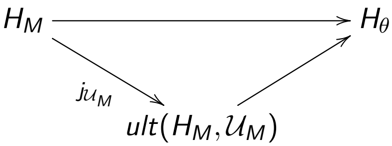

Is $\mathcal{U}_M$ **generic** for $\mathbb{B}_{\mathcal{I}_M}$ over H_M?

Not in L

If $\mathcal{U}_M$ is generic then in particular $ult(H_M, \mathcal{U}_M)$ has the same ordinal height as H_M.

Also $ult(H_M, \mathcal{U}_M)$ collapses α_M.

So they cannot both be levels of L.

Self-genericity

M is called $\mathcal{I}$-**self-generic** iff $\mathcal{U}_M$ is $(H_M, \mathbb{B}_{\mathcal{I}_M})$-generic.

$$S_{\mathcal{I}}^{\text{SelfGen}} := \{M \in P_\kappa(H_\theta) \mid M \text{ is } \mathcal{I}\text{-self generic}\} \quad (2)$$

$(\theta \geq (2^\kappa)^+)$

Theorem (Foreman)

Let $\mathcal{I}$ be a normal ideal on any regular uncountable κ. TFAE:

- *$\mathcal{I}$ is saturated*
- *The set $S_{\mathcal{I}}^{\text{SelfGen}}$ is "club" in $P_\kappa(H_\theta)$. (for all $\theta \geq (2^\kappa)^+$)*

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Question

What if we weaken "club" to, e.g., "stationary"?

ClubCatch, *ProjectiveCatch*, *StatCatch*

Given $\mathcal{I}$ on regular uncountable κ, define:

$$\mathit{ClubCatch}(\mathcal{I}) :\iff S_{\mathcal{I}}^{\mathit{SelfGen}} \text{ is club} \tag{3a}$$

$$\mathit{ProjectiveCatch}(\mathcal{I}) :\iff S_{\mathcal{I}}^{\mathit{SelfGen}} \text{ is } \mathcal{I}\text{-projective} \tag{3b}$$

$$\mathit{StatCatch}(\mathcal{I}) :\iff S_{\mathcal{I}}^{\mathit{SelfGen}} \text{ is stationary} \tag{3c}$$

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Recall: *ClubCatch*($\mathcal{I}$) is equivalent to saturation of $\mathcal{I}$.

What about the others?

Answer: depends heavily on the completeness of $\mathcal{I}$

Theorem (Folklore)

Let $\mathcal{I}$ be on a regular uncountable κ. Then

$$\text{StatCatch}(\mathcal{I}) \implies \mathcal{I} \text{ is somewhere precipitous} \tag{4}$$

$$\text{ProjectiveCatch}(\mathcal{I}) \implies \mathcal{I} \text{ is precipitous} \tag{5}$$

Theorem (Schindler; Ketchersid-Larson-Zapletal)

If $\kappa = \omega_1$ then the converses also hold.

characterization of precipitousness of NS_{ω_1}

Corollary

TFAE:

- NS_{ω_1} *is precipitous*
- $S^{SelfGen}_{NS_{\omega_1}}$ *is projective stationary in the sense of Feng-Jech*

(equiconsistent with a measurable cardinal)

Sketch: precipitousness implies $StatCatch(\mathcal{I})$ when $\kappa = \omega_1$

Let $\mathcal{A} = (H_\theta, \in, \dots)$. Let $\mathcal{I} := NS_{\omega_1}$.

Need to find some countable $M \prec \mathcal{A}$ which is $\mathcal{I}$-self-generic.

Let G be $(V, \mathbb{B}_{\mathcal{I}})$-generic and $j : V \to_G N$ the ultrapower.

Set $\mu := \omega_1^V$. There is a tree T_μ of height ω—defined inside N—such that T_μ has a cofinal branch iff there is a $j(\mathcal{I})$-self generic structure whose intersection with $j(\mu)$ is μ.

$V[G]$ believes:

- $j[H_\theta^V] \prec j(\mathcal{A})$
- $j[H_\theta^V]$ is $j(\mathcal{I})$-self-generic
- $j[H_\theta^V] \cap j(\mu) = \mu$

So $V[G]$ believes T_μ has a cofinal branch; by wellfoundedness of N, N believes this too and thus

$$N \models (\exists M)(M \prec j(\mathcal{A}) \text{ and } M \text{ is } j(\mathcal{I})\text{-self generic}) \qquad (6)$$

Much stronger when completeness is ω_2

Theorem (Cox-Zeman)

Suppose $\mathcal{I}$ is a normal ideal on ω_2 such that:

- $cof(\omega_1) \in Dual(\mathcal{I})$
- $StatCatch^*(\mathcal{I})$ *holds*

Then there is an inner model with a Woodin cardinal.

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Theorem (Cox-Zeman)

Suppose $\mathcal{I}$ is a normal ideal on ω_2 such that:

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The proof is very different from:

Theorem (Claverie-Schindler)

If there is a strong ideal (on any regular uncountable successor cardinal κ) then there is inner model with a Woodin.

Question

Is our theorem just a special case of Claverie-Schindler's? No.

Outline

StatCatch from supercompact tower

Theorem (C.-Zeman)

Suppose κ is δ-supercompact and that δ is the **least** *inaccessible cardinal above κ.*

Then in $V^{Col(\omega_1,<\kappa)*Col(\kappa,<\delta)}$ there is a normal ideal $\mathcal{I}$ whose dual concentrates on $\omega_2 \cap cof(\omega_1)$ such that:

- *StatCatch$^*(\mathcal{I})$ holds (in fact ProjectiveCatch$^*(\mathcal{I})$)*
- *$\mathcal{I}$ is not strong; i.e.*

$$\Vdash_{\mathbb{B}_{\mathcal{I}}} j(\omega_2) > \omega_3^V$$

(in particular $\mathbb{B}_{\mathcal{I}}$ collapses ω_3)

Remark about Forcing Axiom for $\mathbb{B}_{\mathcal{I}}$

Remark

If $\kappa = \omega_2$ *then*

$$StatCatch(\mathcal{I}) \implies FA_{\omega_1}(\mathbb{B}_{\mathcal{I}}) \tag{7}$$

$FA_{\omega_1}(\mathbb{B}_{\mathcal{I}})$ is much easier to obtain, and only requires a measurable cardinal.

Supercompact towers

Let $\kappa < \delta$ be inaccessibles. A sequence $\vec{U} = \langle \mathcal{U}_\lambda \mid \lambda < \delta \rangle$ is a (supercompact) tower iff for all $\lambda < \lambda' < \delta$:

- $\mathcal{U}_\lambda$ is a normal measure on $P_\kappa(\lambda)$
- $\mathcal{U}_\lambda$ is the projection of $\mathcal{U}_{\lambda'}$ to $P_\kappa(\lambda)$.

EXAMPLES:

- Project a normal measure on $P_\kappa(\delta)$ downwards.
- Almost huge embeddings

Supercompact towers, cont.

$\vec{U} = \langle U_\lambda \mid \lambda < \delta \rangle$ gives rise to directed system of ultrapowers and a direct limit map

$$j_{\vec{U}} : V \to_{\vec{U}} N_{\vec{U}}$$

Facts:

1. $N_{\vec{U}}$ is closed under $< \delta$ sequences
2. $j(\kappa) \geq \delta$
3. $j(\kappa) = \delta \iff j$ is an almost huge embedding

Saturated ideals from tower embeddings

Theorem (Magidor, building on work of Kunen)

Suppose $\vec{U}$ is an **almost huge** *tower of height δ and critical point κ. Then for any regular $\mu < \kappa$ there is a μ-closed $\mathbb{P}$ such that*

$$V^{\mathbb{P} * Col^{V^{\mathbb{P}}}(\kappa, <\delta)} \models \text{ there is a saturated ideal on } \kappa = \mu^+$$

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If δ is Mahlo then the ideal is also **layered**
(Foreman-Magidor-Shelah)

Only known method

These are essentially the only known ways to obtain saturated ideals on ω_2.

Only a Woodin cardinal is needed to get saturated ideals on ω_1 (Shelah; RCS iteration)

Separating *StatCatch* from strongness

Question

How much of Magidor's argument can be salvaged if $\vec{U}$ is **not** *almost huge?*

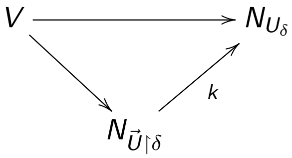
Answer: a lot.

(To separate *StatCatch* from strongness using a Magidor-style argument, you **cannot** start with an almost huge embedding.)

Separating *StatCatch* from strongness, cont.

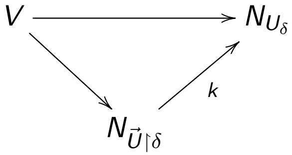
MAIN IDEA: Start with a tower $\vec{U}$ and use it to define an ideal $\mathcal{I}$ in $V^{\mathbb{P} * Col(\kappa, <\delta)}$

1. Assume $\vec{U}$ is **not** almost huge, so that $\mathcal{I}$ will **not** be strong
   - Requires generalizing Magidor argument to non-almost-huge setting
2. Assume $\vec{U}$ is the projection of a normal measure on $P_\kappa(\delta)$ (or a taller tower); somehow use the measure at the top to arrange $StatCatch(\mathcal{I})$

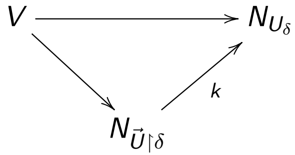


Separating *StatCatch* from strongness, cont.

- F-M-S construction of (stationarily) layered ideal uses nice lifting behavior of k, assuming $\vec{U} \restriction \delta$ is almost huge.
- If $\vec{U} \restriction \delta$ is **not** almost huge then k behaves badly.
  - **Cannot** lift k to the relevant generic extensions



Problem: Critical point of k when tower is not almost huge



N_{U_δ} computes δ^+ correctly, whereas $N_{\vec{U}\restriction\delta}$ does not. So only 2 possibilities:

- $crit(k) = \delta$
- $crit(k) = \delta^{+N_{\vec{U}\restriction\delta}}$

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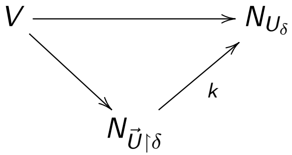

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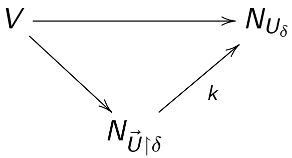
- $crit(k) = \delta$
- $crit(k) = \delta^{+N_{\vec{U}\restriction\delta}}$

In either case, non-almost-hugeness of $\vec{U}$ implies $crit(k) < j_{\vec{U}}(\kappa)$.

But then $crit(k)$ is not even a cardinal in $(N_{\vec{U}\restriction\delta})^{j_{\vec{U}\restriction\delta}(\mathbb{P})}$

Interpolation between $j_{\vec{U}\restriction\delta}(\mathbb{P})$ and $j_{U_\delta}(\mathbb{P})$



Key point: **if** $crit(k) > \delta$ then the following two ideals—both defined in $V^{\mathbb{P}*Col(\kappa,<\delta)}$—are exactly the same:

1. The ideal on κ derived from liftings of $j_{\vec{U}\restriction\delta}$
2. The ideal on κ derived from liftings of j_{U_δ}.

Then letting $\mathcal{I}$ be this ideal:

- Characterization 1, along with non-almost-hugeness of $\vec{U}$, tells us $\mathcal{I}$ is **not** strong
- Characterization 2 tells us $StatCatch(\mathcal{I})$ holds.

A curiosity

Theorem (C.)

Suppose κ is supercompact up to δ where δ is the least inaccessible limit of inaccessibles above κ. Then in $V^{Col(\mu,<\kappa)*Col(\kappa,<\delta)}$: there is an ideal $\mathcal{I}$ on μ^+ such that $\mathbb{B} := \mathbb{B}_{\mathcal{I}}$ is:

1. *a μ^{++}-complete b.a.*
2. *forcing equivalent to Col(μ, μ^{++})*
3. *cofinally layered*
    - *i.e. for every $Z \subset \mathbb{B}$ of size $< \mu^{++}$ there is a $< \mu^{++}$-sized regular subalgebra containing Z.*

Question

Does existence of a $\mathbb{B}$ satisfying 1 through 3 have any large cardinal strength?

Questions

Famous open problem: Can $NS \restriction S^2_1$ be saturated?

Question

Can StatCatch$(NS \restriction S^2_1)$ *hold?*

Shelah: $NS \restriction S^2_0$ cannot be saturated.

Question

Can StatCatch$(NS \restriction S^2_0)$ *hold?*

Question

Any other examples of a cofinally-layered, μ^{++}*-complete* $\mathbb{B}$ *such that* $\mathbb{B} \sim Col(\mu, \mu^{++})$*?*
Does existence of such a $\mathbb{B}$ *have large cardinal strength?*

(Proof of harder direction of Foreman's theorem)
For simplicity assume $\mathcal{I} = NS_{\omega_1}$. Suppose there is some $\mathcal{B} = (H_\theta, \in, \dots)$ such that every ctble elementary substructure of $\mathcal{B}$ is $\mathcal{I}$-self-generic. Let A be a maximal antichain for NS_{ω_1}. Let C^A be a club of $\alpha < \omega_1$ such that $Sk^{\mathcal{B}^\frown(\{A\})}(\alpha) \cap \omega_1 = \alpha$ for all $\alpha \in C^A$.
$A \in Sk^{\mathcal{B}^\frown(\{A\})}(\alpha)$ for each $\alpha \in C^A$, so by self-genericity there is a (unique) $T_\alpha \in A \cap Sk^{\mathcal{B}^\frown(\{A\})}(\alpha)$ such that $\alpha \in T_\alpha$.
CLAIM: $A = \{T_\alpha \mid \alpha \in C^A\}$ (note the RHS has size ω_1).
PROOF: Let $S \in A$; we need to show there is some α such that $S \cap T_\alpha$ is stationary (then $S = T_\alpha$ b/c they're both in the antichain).
The map $\alpha \mapsto T_\alpha$ is essentially regressive (by considering the Skolem term that yields T_α). So there is a fixed T^* such that for stationarily many $\alpha \in S \cap C^A$: $T^* = T_\alpha$. So there are stationarily many $\alpha \in S \cap C^A$ such that $\alpha \in T^*$ (because $\alpha \in T_\alpha$ for **all** $\alpha \in C^A$). In particular $S \cap T^*$ is stationary.