Weyl modules and subalgebras

Ghislain Fourier

Universität zu Köln

Fields Institute, March 26th, 2012
Geometric Methods in Infinite-dimensional Lie Theory

We fix

- $\mathfrak{g}$ a simple, complex Lie algebra (classical type)

We fix

- $\mathfrak{g}$ a simple, complex Lie algebra (classical type)
- $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$

Simple, finite-dimensional $\mathfrak{g}$-modules are parametrized by dominant integral weights

$$V(\lambda) \leftrightarrow \lambda \in P^+.$$

Simple, finite-dimensional $\mathfrak{g}$-modules are parametrized by dominant integral weights

$$V(\lambda) \leftrightarrow \lambda \in P^+.$$

$V(\lambda)$ decomposes into $\mathfrak{h}$-weight space (denoted $V(\lambda)_\mu$)

Simple, finite-dimensional $\mathfrak{g}$-modules are parametrized by dominant integral weights

$$V(\lambda) \leftrightarrow \lambda \in P^+.$$

$V(\lambda)$ decomposes into $\mathfrak{h}$-weight space (denoted $V(\lambda)_\mu$)

$$\dim V(\lambda)_\lambda = 1 \text{ and } V(\lambda) = U(\mathfrak{n}^-).V(\lambda)_\lambda.$$

Simple, finite-dimensional $\mathfrak{g}$-modules are parametrized by dominant integral weights

$$V(\lambda) \leftrightarrow \lambda \in P^+.$$

$V(\lambda)$ decomposes into $\mathfrak{h}$-weight space (denoted $V(\lambda)_\mu$)

$$\dim V(\lambda)_\lambda = 1 \text{ and } V(\lambda) = U(\mathfrak{n}^-).V(\lambda)_\lambda.$$

Integrability condition:

$$(x_\alpha^-)^{\lambda(h_\alpha)+1}.v_\lambda = 0$$

for $v_\lambda \in V(\lambda)_\lambda$.

Let $\mathfrak{a} \subseteq \mathfrak{g}$ a simple subalgebra, such that

$$\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{n}^{+}) \oplus (\mathfrak{a} \cap \mathfrak{h}) \oplus (\mathfrak{a} \cap \mathfrak{n}^{-})$$

is a triangular decomposition of $\mathfrak{a}$.

Let $\mathfrak{a} \subseteq \mathfrak{g}$ a simple subalgebra, such that

$$\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{n}^{+}) \oplus (\mathfrak{a} \cap \mathfrak{h}) \oplus (\mathfrak{a} \cap \mathfrak{n}^{-})$$

is a triangular decomposition of $\mathfrak{a}$.
We further want compatibility for the root spaces:

$$\mathfrak{a} = \mathfrak{h} \cap \mathfrak{a} \oplus \bigoplus_{\alpha} \mathfrak{g}_{\alpha} \cap \mathfrak{a}$$

Let $\mathfrak{a} \subseteq \mathfrak{g}$ a simple subalgebra, such that

$$\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{n}^+) \oplus (\mathfrak{a} \cap \mathfrak{h}) \oplus (\mathfrak{a} \cap \mathfrak{n}^-)$$

is a triangular decomposition of $\mathfrak{a}$.
We further want compatibility for the root spaces:

$$\mathfrak{a} = \mathfrak{h} \cap \mathfrak{a} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha \cap \mathfrak{a}$$

Let $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$ and

$$\mathfrak{a} = \begin{pmatrix} a & 0 & * & * \\ 0 & 0 & 0 & 0 \\ * & 0 & b & * \\ * & 0 & * & c \end{pmatrix}$$

such that $a + b + c = 0$.

Let $\mathfrak{a} \subseteq \mathfrak{g}$ a simple subalgebra, such that

$$\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{n}^+) \oplus (\mathfrak{a} \cap \mathfrak{h}) \oplus (\mathfrak{a} \cap \mathfrak{n}^-)$$

is a triangular decomposition of $\mathfrak{a}$.
We further want compatibility for the root spaces:

$$\mathfrak{a} = \mathfrak{h} \cap \mathfrak{a} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha \cap \mathfrak{a}$$

Let $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{C})$ and

$$\mathfrak{a} = \begin{pmatrix} a & 0 & * & * \\ 0 & 0 & 0 & 0 \\ * & 0 & b & * \\ * & 0 & * & c \end{pmatrix}$$

such that $a + b + c = 0$. Then $\mathfrak{a}$ is of type A_2.

Any $V(\lambda)$ decomposes into a direct sum of simple $\mathfrak{a}$-modules

$$V(\lambda) \cong_{\mathfrak{a}} \bigoplus_{\tau \in P_{\mathfrak{a}}^{+}} V^{\mathfrak{a}}(\tau)^{m_{\lambda,\tau}}.$$

Any $V(\lambda)$ decomposes into a direct sum of simple $\mathfrak{a}$-modules

$$V(\lambda) \cong_{\mathfrak{a}} \bigoplus_{\tau \in P_{\mathfrak{a}}^{+}} V^{\mathfrak{a}}(\tau)^{m_{\lambda,\tau}}.$$

Especially

$$U(\mathfrak{a}).v_\lambda = U(\mathfrak{a} \cap \mathfrak{n}^-).v_\lambda$$

is a simple $U(\mathfrak{a})$-module of highest weight $\lambda|_{\mathfrak{a}\cap\mathfrak{h}}$.

Any $V(\lambda)$ decomposes into a direct sum of simple $\mathfrak{a}$-modules

$$V(\lambda) \cong_{\mathfrak{a}} \bigoplus_{\tau \in P_{\mathfrak{a}}^{+}} V^{\mathfrak{a}}(\tau)^{m_{\lambda,\tau}}.$$

Especially

$$U(\mathfrak{a}).v_\lambda = U(\mathfrak{a} \cap \mathfrak{n}^-).v_\lambda$$

is a simple $U(\mathfrak{a})$-module of highest weight $\lambda|_{\mathfrak{a}\cap\mathfrak{h}}$.
Let $\lambda = \sum_{i=1}^{3} m_i \omega_i$

Any $V(\lambda)$ decomposes into a direct sum of simple $\mathfrak{a}$-modules

$$V(\lambda) \cong_{\mathfrak{a}} \bigoplus_{\tau \in P^{+}_{\mathfrak{a}}} V^{\mathfrak{a}}(\tau)^{m_{\lambda,\tau}}.$$

Especially

$$U(\mathfrak{a}).v_{\lambda} = U(\mathfrak{a} \cap \mathfrak{n}^{-}).v_{\lambda}$$

is a simple $U(\mathfrak{a})$-module of highest weight $\lambda|_{\mathfrak{a}\cap\mathfrak{h}}$.
Let $\lambda = \sum_{i=1}^{3} m_i \omega_i$, then

$$U(\mathfrak{a}).v_{\lambda} \cong_{\mathfrak{a}} V^{\mathfrak{a}}((m_1 + m_2)\omega_1^{\mathfrak{a}} + m_3 \omega_2^{\mathfrak{a}})$$

Changing the Lie algebras:

Changing the Lie algebras:
Let $\mathfrak{g} \otimes \mathbb{C}[t]$ be the current algebra of $\mathfrak{g}$ with bracket

$$[x \otimes p(t), y \otimes q(t)] := [x, y]_{\mathfrak{g}} \otimes p(t)q(t).$$

Changing the Lie algebras:
Let $\mathfrak{g} \otimes \mathbb{C}[t]$ be the current algebra of $\mathfrak{g}$ with bracket

$$[x \otimes p(t), y \otimes q(t)] := [x, y]_{\mathfrak{g}} \otimes p(t)q(t).$$

Any $c \in \mathbb{C}$ induces a map

$$\mathrm{ev}_c : \mathfrak{g} \otimes \mathbb{C}[t] \longrightarrow \mathfrak{g}\,;\, x \otimes p(t) \mapsto p(c)x$$

Changing the Lie algebras:
Let $\mathfrak{g} \otimes \mathbb{C}[t]$ be the current algebra of $\mathfrak{g}$ with bracket

$$[x \otimes p(t), y \otimes q(t)] := [x, y]_{\mathfrak{g}} \otimes p(t)q(t).$$

Any $c \in \mathbb{C}$ induces a map

$$\mathrm{ev}_c : \mathfrak{g} \otimes \mathbb{C}[t] \longrightarrow \mathfrak{g}\,;\; x \otimes p(t) \mapsto p(c)x$$

Let $\lambda \in P^+$, then $V(\lambda)$ can be equipped with a $\mathfrak{g} \otimes \mathbb{C}[t]$-module structure:

Changing the Lie algebras:
Let $\mathfrak{g} \otimes \mathbb{C}[t]$ be the current algebra of $\mathfrak{g}$ with bracket

$$[x \otimes p(t), y \otimes q(t)] := [x, y]_{\mathfrak{g}} \otimes p(t)q(t).$$

Any $c \in \mathbb{C}$ induces a map

$$\mathrm{ev}_c : \mathfrak{g} \otimes \mathbb{C}[t] \longrightarrow \mathfrak{g}\,;\ x \otimes p(t) \mapsto p(c)x$$

Let $\lambda \in P^+$, then $V(\lambda)$ can be equipped with a $\mathfrak{g} \otimes \mathbb{C}[t]$-module structure: Let $v \in V(\lambda)$

$$x \otimes p(t).v := \mathrm{ev}_c(x \otimes p(t)).v = p(c)x.v$$

Changing the Lie algebras:
Let $\mathfrak{g} \otimes \mathbb{C}[t]$ be the current algebra of $\mathfrak{g}$ with bracket

$$[x \otimes p(t), y \otimes q(t)] := [x, y]_{\mathfrak{g}} \otimes p(t)q(t).$$

Any $c \in \mathbb{C}$ induces a map

$$\mathrm{ev}_c : \mathfrak{g} \otimes \mathbb{C}[t] \longrightarrow \mathfrak{g}\,;\, x \otimes p(t) \mapsto p(c)x$$

Let $\lambda \in P^+$, then $V(\lambda)$ can be equipped with a $\mathfrak{g} \otimes \mathbb{C}[t]$-module structure: Let $v \in V(\lambda)$

$$x \otimes p(t).v := \mathrm{ev}_c(x \otimes p(t)).v = p(c)x.v$$

$\mathrm{ev}_c(V(\lambda))$ is a simple $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

Even more, let $c_1, \ldots, c_k \in \mathbb{C}$ pairwise distinct and $\lambda_1, \ldots, \lambda_k \in P^+$

Even more, let $c_1, \ldots, c_k \in \mathbb{C}$ pairwise distinct and $\lambda_1, \ldots, \lambda_k \in P^+$, then

$$V_{\underline{c}}(\underline{\lambda}) := V_{c_1}(\lambda_1) \otimes \ldots \otimes V_{c_k}(\lambda_k)$$

is a simple $\mathfrak{g} \otimes \mathbb{C}[t]/(\prod(t - c_i))$-module, hence a simple $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

Even more, let $c_1, \ldots, c_k \in \mathbb{C}$ pairwise distinct and $\lambda_1, \ldots, \lambda_k \in P^+$, then

$$V_{\underline{c}}(\underline{\lambda}) := V_{c_1}(\lambda_1) \otimes \ldots \otimes V_{c_k}(\lambda_k)$$

is a simple $\mathfrak{g} \otimes \mathbb{C}[t]/(\prod(t - c_i))$-module, hence a simple $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

Theorem (Chari-Pressley)

Every simple, finite-dimensional $\mathfrak{g} \otimes \mathbb{C}[t]$-module is isomorphic to such a tensor product.

Even more, let $c_1, \ldots, c_k \in \mathbb{C}$ pairwise distinct and $\lambda_1, \ldots, \lambda_k \in P^+$, then

$$V_{\underline{c}}(\underline{\lambda}) := V_{c_1}(\lambda_1) \otimes \ldots \otimes V_{c_k}(\lambda_k)$$

is a simple $\mathfrak{g} \otimes \mathbb{C}[t]/(\prod(t - c_i))$-module, hence a simple $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

Theorem (Chari-Pressley)

Every simple, finite-dimensional $\mathfrak{g} \otimes \mathbb{C}[t]$-module is isomorphic to such a tensor product.

Even more, let $c_1, \ldots, c_k \in \mathbb{C}$ pairwise distinct and $\lambda_1, \ldots, \lambda_k \in P^+$, then

$$V_{\underline{c}}(\underline{\lambda}) := V_{c_1}(\lambda_1) \otimes \ldots \otimes V_{c_k}(\lambda_k)$$

is a simple $\mathfrak{g} \otimes \mathbb{C}[t]/(\prod(t - c_i))$-module, hence a simple $\mathfrak{g} \otimes \mathbb{C}[t]$-module.

Theorem (Chari-Pressley)

Every simple, finite-dimensional $\mathfrak{g} \otimes \mathbb{C}[t]$-module is isomorphic to such a tensor product.

We have further for $h \otimes p(t) \in \mathfrak{h} \otimes \mathbb{C}[t]$:

$$h \otimes p(t).v_{\lambda_1} \otimes \ldots \otimes v_{\lambda_k} = \left(\sum_{i=1}^{k} \lambda_i(h) p(c_i) \right) v_{\lambda_1} \otimes \ldots \otimes v_{\lambda_k}$$

Let $\mathfrak{a}$ as before, and we consider the restriction of $V_{\underline{c}}(\underline{\lambda})$ to $\mathfrak{a} \otimes \mathbb{C}[t]$.

Let $\mathfrak{a}$ as before, and we consider the restriction of $V_{\underline{c}}(\underline{\lambda})$ to $\mathfrak{a} \otimes \mathbb{C}[t]$.
Again, the operation factors through $\mathfrak{a} \otimes (\prod(t - c_i))$

Let $\mathfrak{a}$ as before, and we consider the restriction of $V_{\underline{c}}(\underline{\lambda})$ to $\mathfrak{a} \otimes \mathbb{C}[t]$.

Again, the operation factors through $\mathfrak{a} \otimes (\prod(t - c_i))$, hence the restricted module is a module for

$$\mathfrak{a} \otimes \mathbb{C}[t]/(\prod(t - c_i)) \cong \bigoplus \mathfrak{a} \otimes \mathbb{C}[t]/(t - c_i).$$

Let $\mathfrak{a}$ as before, and we consider the restriction of $V_{\underline{c}}(\underline{\lambda})$ to $\mathfrak{a} \otimes \mathbb{C}[t]$.
Again, the operation factors through $\mathfrak{a} \otimes (\prod(t - c_i))$, hence the restricted module is a module for

$$\mathfrak{a} \otimes \mathbb{C}[t]/(\prod(t - c_i)) \cong \bigoplus \mathfrak{a} \otimes \mathbb{C}[t]/(t - c_i).$$

This leads to

Proposition

$V_{\underline{c}}(\underline{\lambda})$ decomposes as a $\mathfrak{a} \otimes \mathbb{C}[t]$-module into a direct sum of simple $\mathfrak{a} \otimes \mathbb{C}[t]$-modules.

Let $\mathfrak{a}$ as before, and we consider the restriction of $V_{\underline{c}}(\underline{\lambda})$ to $\mathfrak{a} \otimes \mathbb{C}[t]$.
Again, the operation factors through $\mathfrak{a} \otimes (\prod(t - c_i))$, hence the restricted module is a module for

$$\mathfrak{a} \otimes \mathbb{C}[t]/(\prod(t - c_i)) \cong \bigoplus \mathfrak{a} \otimes \mathbb{C}[t]/(t - c_i).$$

This leads to

Proposition

$V_{\underline{c}}(\underline{\lambda})$ decomposes as a $\mathfrak{a} \otimes \mathbb{C}[t]$-module into a direct sum of simple $\mathfrak{a} \otimes \mathbb{C}[t]$-modules.

Let $\mathfrak{a}$ as before, and we consider the restriction of $V_{\underline{c}}(\underline{\lambda})$ to $\mathfrak{a} \otimes \mathbb{C}[t]$.
Again, the operation factors through $\mathfrak{a} \otimes (\prod(t - c_i))$, hence the restricted module is a module for

$$\mathfrak{a} \otimes \mathbb{C}[t]/(\prod(t - c_i)) \cong \bigoplus \mathfrak{a} \otimes \mathbb{C}[t]/(t - c_i).$$

This leads to

Proposition

$V_{\underline{c}}(\underline{\lambda})$ decomposes as a $\mathfrak{a} \otimes \mathbb{C}[t]$-module into a direct sum of simple $\mathfrak{a} \otimes \mathbb{C}[t]$-modules.

Knowing $m_{\lambda_i,\tau}$

Let $\mathfrak{a}$ as before, and we consider the restriction of $V_{\underline{c}}(\underline{\lambda})$ to $\mathfrak{a} \otimes \mathbb{C}[t]$.
Again, the operation factors through $\mathfrak{a} \otimes (\prod(t-c_i))$, hence the restricted module is a module for

$$\mathfrak{a} \otimes \mathbb{C}[t]/(\prod(t-c_i)) \cong \bigoplus \mathfrak{a} \otimes \mathbb{C}[t]/(t-c_i).$$

This leads to

Proposition

$V_{\underline{c}}(\underline{\lambda})$ decomposes as a $\mathfrak{a} \otimes \mathbb{C}[t]$-module into a direct sum of simple $\mathfrak{a} \otimes \mathbb{C}[t]$-modules.

Knowing $m_{\lambda_i,\tau} \implies$ decomposition formula for $V_{\underline{c}}(\underline{\lambda})$

Let $\mathfrak{a}$ as before, and we consider the restriction of $V_{\underline{c}}(\underline{\lambda})$ to $\mathfrak{a} \otimes \mathbb{C}[t]$.
Again, the operation factors through $\mathfrak{a} \otimes (\prod(t - c_i))$, hence the restricted module is a module for

$$\mathfrak{a} \otimes \mathbb{C}[t]/(\prod(t - c_i)) \cong \bigoplus \mathfrak{a} \otimes \mathbb{C}[t]/(t - c_i).$$

This leads to

Proposition

$V_{\underline{c}}(\underline{\lambda})$ decomposes as a $\mathfrak{a} \otimes \mathbb{C}[t]$-module into a direct sum of simple $\mathfrak{a} \otimes \mathbb{C}[t]$-modules.

Knowing $m_{\lambda_i,\tau} \implies$ decomposition formula for $V_{\underline{c}}(\underline{\lambda})$

Note that

$$U(\mathfrak{a} \otimes \mathbb{C}[t]).v_{\lambda_1} \otimes \ldots \otimes v_{\lambda_k}$$

is a simple $\mathfrak{a} \otimes \mathbb{C}[t]$-module.

We turn to Weyl modules:

We turn to Weyl modules:

- To $V_{\underline{c}}(\underline{\lambda})$ we associate a "largest" integrable module $W_{\underline{c}}(\underline{\lambda})$.

We turn to Weyl modules:

- To $V_{\underline{c}}(\underline{\lambda})$ we associate a ”largest” integrable module $W_{\underline{c}}(\underline{\lambda})$.

We turn to Weyl modules:

- To $V_{\underline{c}}(\underline{\lambda})$ we associate a ”largest” integrable module $W_{\underline{c}}(\underline{\lambda})$.
- $W_{\underline{c}}(\underline{\lambda})$ is generated through a highest weight vector w, s.t. $\mathfrak{h} \otimes \mathbb{C}[t]$ acts by evaluation (as before) on w.

We turn to Weyl modules:

- To $V_{\underline{c}}(\underline{\lambda})$ we associate a "largest" integrable module $W_{\underline{c}}(\underline{\lambda})$.
- $W_{\underline{c}}(\underline{\lambda})$ is generated through a highest weight vector w, s.t. $\mathfrak{h} \otimes \mathbb{C}[t]$ acts by evaluation (as before) on w.

We turn to Weyl modules:

- To $V_{\underline{c}}(\underline{\lambda})$ we associate a "largest" integrable module $W_{\underline{c}}(\underline{\lambda})$.
- $W_{\underline{c}}(\underline{\lambda})$ is generated through a highest weight vector w, s.t. $\mathfrak{h} \otimes \mathbb{C}[t]$ acts by evaluation (as before) on w.
- $W_{\underline{c}}(\underline{\lambda})$ is a module for

$$\mathfrak{g} \otimes \mathbb{C}[t] / \prod (t - c_i)^{n_i}$$

for some n_i

We turn to Weyl modules:

- To $V_{\underline{c}}(\underline{\lambda})$ we associate a "largest" integrable module $W_{\underline{c}}(\underline{\lambda})$.
- $W_{\underline{c}}(\underline{\lambda})$ is generated through a highest weight vector w, s.t. $\mathfrak{h} \otimes \mathbb{C}[t]$ acts by evaluation (as before) on w.
- $W_{\underline{c}}(\underline{\lambda})$ is a module for

$$\mathfrak{g} \otimes \mathbb{C}[t] / \prod (t - c_i)^{n_i}$$

for some n_i

We turn to Weyl modules:

- To $V_{\underline{c}}(\underline{\lambda})$ we associate a "largest" integrable module $W_{\underline{c}}(\underline{\lambda})$.
- $W_{\underline{c}}(\underline{\lambda})$ is generated through a highest weight vector w, s.t. $\mathfrak{h} \otimes \mathbb{C}[t]$ acts by evaluation (as before) on w.
- $W_{\underline{c}}(\underline{\lambda})$ is a module for

$$\mathfrak{g} \otimes \mathbb{C}[t] / \prod (t - c_i)^{n_i}$$

for some $n_i \Longrightarrow$ not a semi-simple Lie algebra.

We turn to Weyl modules:

- To $V_{\underline{c}}(\underline{\lambda})$ we associate a ”largest” integrable module $W_{\underline{c}}(\underline{\lambda})$.
- $W_{\underline{c}}(\underline{\lambda})$ is generated through a highest weight vector w, s.t. $\mathfrak{h} \otimes \mathbb{C}[t]$ acts by evaluation (as before) on w.
- $W_{\underline{c}}(\underline{\lambda})$ is a module for

$$\mathfrak{g} \otimes \mathbb{C}[t] / \prod (t - c_i)^{n_i}$$

  for some $n_i \implies$ not a semi-simple Lie algebra.
- These local Weyl modules are finite-dimensional.

Theorem (Chari-Pressley)

$$W_{\underline{c}}(\underline{\lambda}) \cong W_{c_1}(\lambda_1) \otimes \ldots \otimes W_{c_k}(\lambda_k)$$

Theorem (Chari-Pressley)

$$W_{\underline{c}}(\underline{\lambda}) \cong W_{c_1}(\lambda_1) \otimes \ldots \otimes W_{c_k}(\lambda_k)$$

Theorem (Chari-Pressley)

$$W_{\underline{c}}(\underline{\lambda}) \cong W_{c_1}(\lambda_1) \otimes \ldots \otimes W_{c_k}(\lambda_k)$$

The $\mathfrak{g}$-decomposition of $W_c(\lambda)$ is isomorphic to the tensor product of "fundamental" Weyl modules.

Theorem (Chari-Pressley)

$$W_{\underline{c}}(\underline{\lambda}) \cong W_{c_1}(\lambda_1) \otimes \ldots \otimes W_{c_k}(\lambda_k)$$

The $\mathfrak{g}$-decomposition of $W_c(\lambda)$ is isomorphic to the tensor product of ”fundamental” Weyl modules.
The $\mathfrak{g}$-decomposition of these fundamental Weyl modules is known due to Chari and Kleber.

Recall the previous example with $\mathfrak{g}$ of type A_3 and $\mathfrak{a}$ of type A_2.

Recall the previous example with $\mathfrak{g}$ of type A_3 and $\mathfrak{a}$ of type A_2. Let $\lambda = \omega_1 + \omega_2$ and $c = 0$, then

$$W_0(\lambda) \cong_{\mathfrak{g}} V(\omega_1) \otimes V(\omega_2)$$

Recall the previous example with $\mathfrak{g}$ of type A_3 and $\mathfrak{a}$ of type A_2. Let $\lambda = \omega_1 + \omega_2$ and $c = 0$, then

$$W_0(\lambda) \cong_{\mathfrak{g}} V(\omega_1) \otimes V(\omega_2)$$

and

$$U(\mathfrak{a} \otimes \mathbb{C}[t]).(W_0(\lambda))_\lambda \cong_{\mathfrak{a}} V^{\mathfrak{a}}(2\omega_1) \oplus V^{\mathfrak{a}}(\omega_2) \cong_{\mathfrak{a}} W_0^{\mathfrak{a}}(2\omega_1)$$

Recall the previous example with $\mathfrak{g}$ of type A_3 and $\mathfrak{a}$ of type A_2. Let $\lambda = \omega_1 + \omega_2$ and $c = 0$, then

$$W_0(\lambda) \cong_{\mathfrak{g}} V(\omega_1) \otimes V(\omega_2)$$

and

$$U(\mathfrak{a} \otimes \mathbb{C}[t]).(W_0(\lambda))_\lambda \cong_{\mathfrak{a}} V^{\mathfrak{a}}(2\omega_1) \oplus V^{\mathfrak{a}}(\omega_2) \cong_{\mathfrak{a}} W_0^{\mathfrak{a}}(2\omega_1)$$

On the other hand, let $\mathfrak{g}$ be of type C_3 and $\mathfrak{a}$ be of type A_1 corresponding to the positive root $\alpha_1 + 2\alpha_2 + \alpha_3$.

Recall the previous example with $\mathfrak{g}$ of type A_3 and $\mathfrak{a}$ of type A_2.
Let $\lambda = \omega_1 + \omega_2$ and $c = 0$, then

$$W_0(\lambda) \cong_{\mathfrak{g}} V(\omega_1) \otimes V(\omega_2)$$

and

$$U(\mathfrak{a} \otimes \mathbb{C}[t]).(W_0(\lambda))_\lambda \cong_{\mathfrak{a}} V^{\mathfrak{a}}(2\omega_1) \oplus V^{\mathfrak{a}}(\omega_2) \cong_{\mathfrak{a}} W_0^{\mathfrak{a}}(2\omega_1)$$

On the other hand, let $\mathfrak{g}$ be of type C_3 and $\mathfrak{a}$ be of type A_1 corresponding to the positive root $\alpha_1 + 2\alpha_2 + \alpha_3$.
Let $\lambda = \omega_2$ and $c = 0$, then

$$W_0(\omega_2) \cong_{\mathfrak{g}} V(\omega_2)$$

and

$$U(\mathfrak{a} \otimes \mathbb{C}[t]).(W_0(\omega_2))_{\omega_2} \cong_{\mathfrak{a}} V^{\mathfrak{a}}(2\omega) \subsetneq_{\mathfrak{a}} W_0^{\mathfrak{a}}(2\omega) \cong_{\mathfrak{a}} V^{\mathfrak{a}}(2\omega) \oplus \mathbb{C}.$$

Recall the previous example with $\mathfrak{g}$ of type A_3 and $\mathfrak{a}$ of type A_2.
Let $\lambda = \omega_1 + \omega_2$ and $c = 0$, then

$$W_0(\lambda) \cong_{\mathfrak{g}} V(\omega_1) \otimes V(\omega_2)$$

and

$$U(\mathfrak{a} \otimes \mathbb{C}[t]).(W_0(\lambda))_\lambda \cong_{\mathfrak{a}} V^{\mathfrak{a}}(2\omega_1) \oplus V^{\mathfrak{a}}(\omega_2) \cong_{\mathfrak{a}} W_0^{\mathfrak{a}}(2\omega_1)$$

On the other hand, let $\mathfrak{g}$ be of type C_3 and $\mathfrak{a}$ be of type A_1 corresponding to the positive root $\alpha_1 + 2\alpha_2 + \alpha_3$.
Let $\lambda = \omega_2$ and $c = 0$, then

$$W_0(\omega_2) \cong_{\mathfrak{g}} V(\omega_2)$$

and

$$U(\mathfrak{a} \otimes \mathbb{C}[t]).(W_0(\omega_2))_{\omega_2} \cong_{\mathfrak{a}} V^{\mathfrak{a}}(2\omega) \subsetneq_{\mathfrak{a}} W_0^{\mathfrak{a}}(2\omega) \cong_{\mathfrak{a}} V^{\mathfrak{a}}(2\omega) \oplus \mathbb{C}.$$

$\Rightarrow$ Find sufficient criteria for restrictions being local Weyl modules

A pair $(\mathfrak{a}, \lambda = \sum m_\ell \omega_\ell)$ is called *local non-admissible* for $\mathfrak{g}$ if

A pair $(\mathfrak{a}, \lambda = \sum m_\ell \omega_\ell)$ is called *local non-admissible* for $\mathfrak{g}$ if

1. either $\mathfrak{a} \cong \mathfrak{sp}_{2n+1}$, $\epsilon_i + \epsilon_j$ is the unique simple short root of $\mathfrak{a}$ and $m_k \neq 0$ for some $i \leq k \leq n-1$ and $i < j$,

A pair $(\mathfrak{a}, \lambda = \sum m_\ell \omega_\ell)$ is called *local non-admissible* for $\mathfrak{g}$ if

1. either $\mathfrak{a} \cong \mathfrak{sp}_{2n+1}$, $\epsilon_i + \epsilon_j$ is the unique simple short root of $\mathfrak{a}$ and $m_k \neq 0$ for some $i \leq k \leq n-1$ and $i < j$,
2. or $\mathfrak{g} \cong \mathfrak{sp}_n$, $\mathfrak{a} \cong \mathfrak{sl}_{s+1}$, $\mathfrak{g}_{\epsilon_i+\epsilon_j} \subset \mathfrak{a}$ and there exists $k \in I$ with $m_k \neq 0$ and $\omega_k|_{\mathfrak{h}\cap\mathfrak{a}}$ is not a fundamental weight.

A pair $(\mathfrak{a}, \lambda = \sum m_\ell \omega_\ell)$ is called *local non-admissible* for $\mathfrak{g}$ if

1. either $\mathfrak{a} \cong \mathfrak{sp}_{2n+1}$, $\epsilon_i + \epsilon_j$ is the unique simple short root of $\mathfrak{a}$ and $m_k \neq 0$ for some $i \leq k \leq n-1$ and $i < j$,
2. or $\mathfrak{g} \cong \mathfrak{sp}_n$, $\mathfrak{a} \cong \mathfrak{sl}_{s+1}$, $\mathfrak{g}_{\epsilon_i+\epsilon_j} \subset \mathfrak{a}$ and there exists $k \in I$ with $m_k \neq 0$ and $\omega_k|_{\mathfrak{h}\cap\mathfrak{a}}$ is not a fundamental weight.

Theorem

Let $(\mathfrak{a}, \lambda)$ be local admissible, then

$$U(\mathfrak{a} \otimes \mathbb{C}[t]).w \cong W_0^{\mathfrak{a}}(\lambda_{\mathfrak{h}\cap\mathfrak{a}}),$$

e.g. the highest weight component of the restricted local Weyl module is a local Weyl module.

A pair $(\mathfrak{a}, \lambda = \sum m_\ell \omega_\ell)$ is called *local non-admissible* for $\mathfrak{g}$ if

1. either $\mathfrak{a} \cong \mathfrak{sp}_{2n+1}$, $\epsilon_i + \epsilon_j$ is the unique simple short root of $\mathfrak{a}$ and $m_k \neq 0$ for some $i \leq k \leq n-1$ and $i < j$,
2. or $\mathfrak{g} \cong \mathfrak{sp}_n$, $\mathfrak{a} \cong \mathfrak{sl}_{s+1}$, $\mathfrak{g}_{\epsilon_i+\epsilon_j} \subset \mathfrak{a}$ and there exists $k \in I$ with $m_k \neq 0$ and $\omega_k|_{\mathfrak{h}\cap\mathfrak{a}}$ is not a fundamental weight.

Theorem

Let $(\mathfrak{a}, \lambda)$ be local admissible, then

$$U(\mathfrak{a} \otimes \mathbb{C}[t]).w \cong W_0^{\mathfrak{a}}(\lambda_{\mathfrak{h}\cap\mathfrak{a}}),$$

e.g. the highest weight component of the restricted local Weyl module is a local Weyl module.

A pair $(\mathfrak{a}, \lambda = \sum m_\ell \omega_\ell)$ is called *local non-admissible* for $\mathfrak{g}$ if

1. either $\mathfrak{a} \cong \mathfrak{sp}_{2n+1}$, $\epsilon_i + \epsilon_j$ is the unique simple short root of $\mathfrak{a}$ and $m_k \neq 0$ for some $i \leq k \leq n-1$ and $i < j$,
2. or $\mathfrak{g} \cong \mathfrak{sp}_n$, $\mathfrak{a} \cong \mathfrak{sl}_{s+1}$, $\mathfrak{g}_{\epsilon_i+\epsilon_j} \subset \mathfrak{a}$ and there exists $k \in I$ with $m_k \neq 0$ and $\omega_k|_{\mathfrak{h}\cap\mathfrak{a}}$ is not a fundamental weight.

Theorem

Let $(\mathfrak{a}, \lambda)$ be local admissible, then

$$U(\mathfrak{a} \otimes \mathbb{C}[t]).w \cong W_0^{\mathfrak{a}}(\lambda_{\mathfrak{h}\cap\mathfrak{a}}),$$

e.g. the highest weight component of the restricted local Weyl module is a local Weyl module.

About the proof:

A pair $(\mathfrak{a}, \lambda = \sum m_\ell \omega_\ell)$ is called *local non-admissible* for $\mathfrak{g}$ if

1. either $\mathfrak{a} \cong \mathfrak{sp}_{2n+1}$, $\epsilon_i + \epsilon_j$ is the unique simple short root of $\mathfrak{a}$ and $m_k \neq 0$ for some $i \leq k \leq n-1$ and $i < j$,
2. or $\mathfrak{g} \cong \mathfrak{sp}_n$, $\mathfrak{a} \cong \mathfrak{sl}_{s+1}$, $\mathfrak{g}_{\epsilon_i+\epsilon_j} \subset \mathfrak{a}$ and there exists $k \in I$ with $m_k \neq 0$ and $\omega_k|_{\mathfrak{h}\cap\mathfrak{a}}$ is not a fundamental weight.

Theorem

Let $(\mathfrak{a}, \lambda)$ be local admissible, then

$$U(\mathfrak{a} \otimes \mathbb{C}[t]).w \cong W_0^{\mathfrak{a}}(\lambda_{\mathfrak{h}\cap\mathfrak{a}}),$$

e.g. the highest weight component of the restricted local Weyl module is a local Weyl module.

About the proof: Prove the theorem for fundamental weights

A pair $(\mathfrak{a}, \lambda = \sum m_\ell \omega_\ell)$ is called *local non-admissible* for $\mathfrak{g}$ if

1. either $\mathfrak{a} \cong \mathfrak{sp}_{2n+1}$, $\epsilon_i + \epsilon_j$ is the unique simple short root of $\mathfrak{a}$ and $m_k \neq 0$ for some $i \leq k \leq n-1$ and $i < j$,
2. or $\mathfrak{g} \cong \mathfrak{sp}_n$, $\mathfrak{a} \cong \mathfrak{sl}_{s+1}$, $\mathfrak{g}_{\epsilon_i+\epsilon_j} \subset \mathfrak{a}$ and there exists $k \in I$ with $m_k \neq 0$ and $\omega_k|_{\mathfrak{h}\cap\mathfrak{a}}$ is not a fundamental weight.

Theorem

Let $(\mathfrak{a}, \lambda)$ be local admissible, then

$$U(\mathfrak{a} \otimes \mathbb{C}[t]).w \cong W_0^{\mathfrak{a}}(\lambda_{\mathfrak{h}\cap\mathfrak{a}}),$$

e.g. the highest weight component of the restricted local Weyl module is a local Weyl module.

About the proof: Prove the theorem for fundamental weights and then use the realization of $W_0(\lambda)$ as a fusion product of fundamental Weyl modules.

Recall the $(\mathfrak{sp}_3, \mathfrak{sl}_2)$-example:

Recall the $(\mathfrak{sp}_3, \mathfrak{sl}_2)$-example:
There is no local Weyl module of weight $(2m+1)\omega$ that can be obtained by restriction,

Recall the $(\mathfrak{sp}_3, \mathfrak{sl}_2)$-example:
There is no local Weyl module of weight $(2m+1)\omega$ that can be obtained by restriction, same for simple modules.

Recall the $(\mathfrak{sp}_3, \mathfrak{sl}_2)$-example:
There is no local Weyl module of weight $(2m+1)\omega$ that can be obtained by restriction, same for simple modules.
A pair $(\mathfrak{a}, \lambda = \sum m_\ell \omega_\ell)$ is called *global admissible* for $\mathfrak{g}$ if

$$m_\ell \neq 0 \Rightarrow \omega_\ell|_{\mathfrak{h}\cap\mathfrak{a}} \text{ is a fundamental weight}$$

Recall the $(\mathfrak{sp}_3, \mathfrak{sl}_2)$-example:
There is no local Weyl module of weight $(2m+1)\omega$ that can be obtained by restriction, same for simple modules.
A pair $(\mathfrak{a}, \lambda = \sum m_\ell \omega_\ell)$ is called *global admissible* for $\mathfrak{g}$ if

$$m_\ell \neq 0 \Rightarrow \omega_\ell|_{\mathfrak{h}\cap\mathfrak{a}} \text{ is a fundamental weight}$$

Proposition

Let $(\mathfrak{a}, \lambda)$ be global admissible for $\mathfrak{g}$. Then

$$\textit{Simple } \mathfrak{g} \otimes \mathbb{C}[t] - \textit{modules of hw } \lambda / \sim$$
$$\Downarrow_{\textit{Restriction}}$$
$$\textit{Simple } \mathfrak{a} \otimes \mathbb{C}[t] - \textit{modules of hw } \lambda|_{\mathfrak{h}\otimes\mathfrak{a}} / \sim$$

is surjective.

To each $\lambda \in P^+$ we can associate a global Weyl module, the integrable module of this highest weight. All local Weyl modules are quotients.

To each $\lambda \in P^+$ we can associate a global Weyl module, the integrable module of this highest weight. All local Weyl modules are quotients.

What about the restricted module?

To each $\lambda \in P^+$ we can associate a global Weyl module, the integrable module of this highest weight. All local Weyl modules are quotients.
What about the restricted module?
We need all local Weyl modules of $\mathfrak{a} \otimes \mathbb{C}[t]$ as restrictions.

To each $\lambda \in P^+$ we can associate a global Weyl module, the integrable module of this highest weight. All local Weyl modules are quotients.
What about the restricted module?
We need all local Weyl modules of $\mathfrak{a} \otimes \mathbb{C}[t]$ as restrictions.
This leads to

Theorem

Let $(\mathfrak{a}, \lambda)$ be global admissible for $\mathfrak{g}$, then

$$U(\mathfrak{a} \otimes \mathbb{C}[t]).w \cong W^{\mathfrak{a}}(\lambda|_{\mathfrak{h}\cap\mathfrak{a}}),$$

the generator-component of the restricted global Weyl module is the global Weyl module for $\mathfrak{a} \otimes \mathbb{C}[t]$.

Outlook:

Outlook:
Does the restricted local Weyl module decomposes (in the local admissible case) into a direct sum of local Weyl modules?

Outlook:
Does the restricted local Weyl module decomposes (in the local admissible case) into a direct sum of local Weyl modules?

$\Rightarrow$ Necessary/sufficient conditions?

Outlook:
Does the restricted local Weyl module decomposes (in the local admissible case) into a direct sum of local Weyl modules?

$\Rightarrow$ Necessary/sufficient conditions?

Find ”lower rank”-criteria to determine local Weyl modules!

Outlook:
Does the restricted local Weyl module decomposes (in the local admissible case) into a direct sum of local Weyl modules?

$\Rightarrow$ Necessary/sufficient conditions?

Find "lower rank"-criteria to determine local Weyl modules!

local Weyl module for $\mathfrak{g} \otimes \mathbb{C}[t]$ $\Leftrightarrow$ local Weyl module for all $\mathfrak{a} \otimes \mathbb{C}[t]$?

Outlook:
Does the restricted local Weyl module decomposes (in the local admissible case) into a direct sum of local Weyl modules?

$\Rightarrow$ Necessary/sufficient conditions?

Find ”lower rank”-criteria to determine local Weyl modules!

local Weyl module for $\mathfrak{g} \otimes \mathbb{C}[t]$ $\Leftrightarrow$ local Weyl module for all $\mathfrak{a} \otimes \mathbb{C}[t]$?

$\Rightarrow$ Which subalgebras are necessary/sufficient?

Thank you!