Degrees of Streams

Jörg Endrullis   Dimitri Hendriks   Jan Willem Klop

Vrije Universiteit Amsterdam

Challenges in Combinatorics on Words

Fields Institute, Toronto
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Comparing Streams

Goal

Measure the complexity of streams in terms of their infinite pattern.

Measure should be invariant under

- insertion/removal of finitely many elements
- change of alphabet

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Shortcomings of existing complexity measures:

- Recursion theoretic degrees of unsolvability
  Comparison of streams via transformability by Turing machines.

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All computable streams are identified.

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Shortcomings of existing complexity measures:

- Recursion theoretic degrees of unsolvability

  All computable streams are identified.

- Kolmogorov complexity

  Size of the shortest program computing the stream.

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- Subword complexity

  $\xi_\sigma : \mathbb{N} \to \mathbb{N}$ where $\xi_\sigma(n)$ number of subwords of length n in σ.

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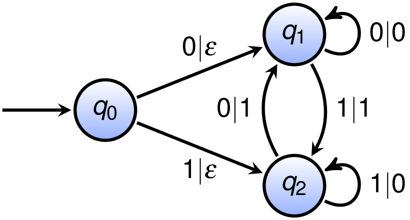
$u = 0\ 1\ 0\ 0\ 0\ 1\ 1\ \ldots$ w contains u

$w = 0\ 2\ 1\ 2\ 2\ 0\ 2\ 2\ 2\ 2\ 0\ \ldots$ but w has trivial complexity

Finite State Transducers

We propose: comparison via finite state transducers (FSTs).

Example: FST computing the difference of consecutive elements



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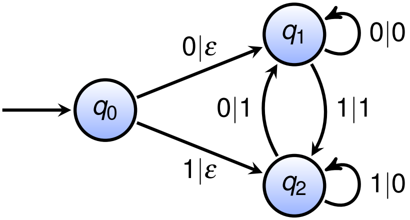

input letter | output word along the edges

Transduces Thue-Morse sequence to period doubling sequence:

$$0\,1\,1\,0\,1\,0\,0\,1\ldots$$
$$\to\ 1\,0\,1\,1\,1\,0\,1\ldots$$

Degrees of Streams

Principle: M is at least as complex as N if it can be transformed to N

$$M \rhd N \iff \text{there exists an FST transforming } M \text{ into } N$$

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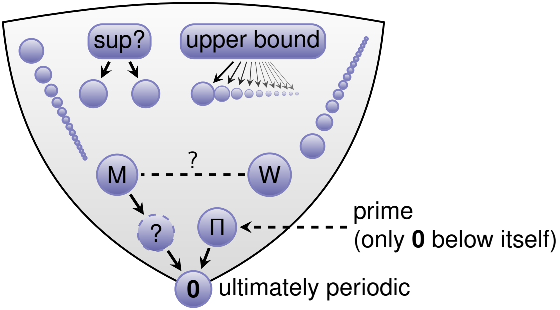
$$M \triangleright N \iff \text{there exists an FST transforming } M \text{ into } N$$



Partial order of degrees induced by $\triangleright$.

(degree is class of streams that can be transformed into each other)

Initial Observations

Theorem

Every degree is countable.

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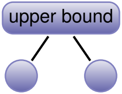

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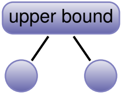

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A set of degrees has an upper bound $\iff$ the set is countable.

$$\mathrm{zip}(w_0, \mathrm{zip}(w_1, \mathrm{zip}(w_2, \ldots))),$$

$w_0(0)$	$w_1(0)$	$w_0(1)$	$w_2(0)$	$w_0(2)$	$w_1(1)$	$w_0(3)$	$w_3(0)$	$w_0(4)$	$w_1(2)$	$w_0(5)$	$w_2(1)$	$\cdots$

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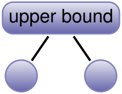

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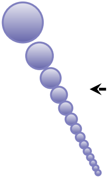
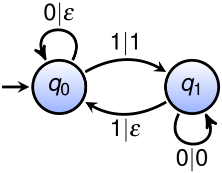
Theorem

There are no maximal degrees.

An Infinite Descending Chain



descending sequence of degrees

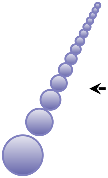


Theorem

The following is an infinite descending sequence:

$$
\begin{aligned}
& D_0 = 10^{2^0} 10^{2^1} 10^{2^2} 10^{2^3} 10^{2^4} 10^{2^5} 10^{2^6} \dots \\
\mathrel{\underset{\neq}{\triangleright}}\ & D_1 = 10^{2^0} 10^{2^2} 10^{2^4} 10^{2^6} 10^{2^8} 10^{2^{10}} 10^{2^{12}} \dots \\
\mathrel{\underset{\neq}{\triangleright}}\ & D_2 = 10^{2^0} 10^{2^4} 10^{2^8} 10^{2^{12}} 10^{2^{16}} 10^{2^{20}} 10^{2^{24}} \dots \\
\mathrel{\underset{\neq}{\triangleright}}\ & \dots
\end{aligned}
$$

An Infinite Ascending Chain



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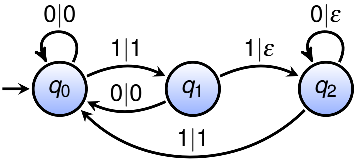
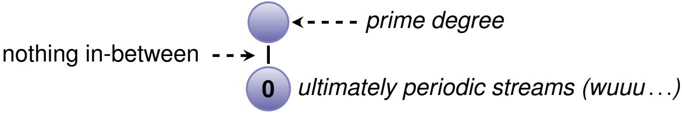


Theorem

The following is an infinite ascending sequence:

$$\begin{array}{l} \vdots \\ \underset{\neq}{\triangleright}\ A_3 = \color{red}{1}(10)^3\,\color{red}{1}(100)^3\,\color{red}{1}(10000)^3\,\color{red}{1}(100000000)^3\ldots \\ \underset{\neq}{\triangleright}\ A_2 = \color{red}{1}(10)^2\,\color{red}{1}(100)^2\,\color{red}{1}(10000)^2\,\color{red}{1}(100000000)^2\ldots \\ \underset{\neq}{\triangleright}\ A_1 = \color{red}{1}10\,\color{red}{1}100\,\color{red}{1}10000\,\color{red}{1}100000000\ldots \\ \underset{\neq}{\triangleright}\ A_0 = \color{red}{111111}\ldots \end{array}$$

Prime Degrees



Definition

A degree $M \neq \mathbf{0}$ is prime if there is no N between M and $\mathbf{0}$:

$$\neg\exists N.\; M \mathrel{\underset{\neq}{\trianglerighteq}} N \mathrel{\underset{\neq}{\trianglerighteq}} \mathbf{0}$$

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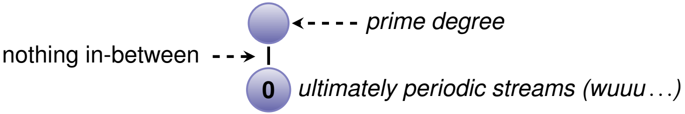

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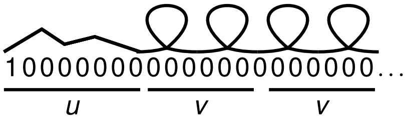
Theorem

The degree of the following stream is prime:

$$\Pi = 10\ 100\ 1000\ 10000\ 100000\ 1\ldots$$
$$= 1\,0^1\ 1\,0^2\ 1\,0^3\ 1\,0^4\ 1\,0^5\ 1\,0^6\ 1\ldots$$

A Prime: $\Pi = 110100100010000100000 1\ldots$



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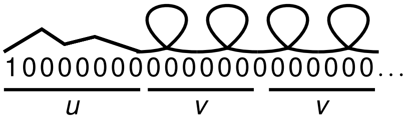

Let Z be the least common multiple of lengths of 0-loops in the FST.

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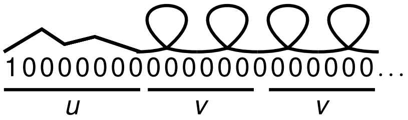

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Lemma

For all $q \in Q$, $n > |Q|$, there exist $u, v \in \Gamma^*$ s.t. for all $i \in \mathbb{N}$:

$$\delta(q, 10^{n+i \cdot Z}) = \delta(q, 10^n) \qquad \delta = \text{state transition function}$$

$$\lambda(q, 10^{n+i \cdot Z}) = u\,v^i \qquad \lambda = \text{output function}$$

Proof.

Analogous to the pumping lemma for regular languages. □

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Lemma

Every transduct of Π is of the form

$$w \cdot \prod_{i=0}^{\infty} w_i \quad \text{where} \quad w_i = \prod_{j=0}^{n-1} u_j \cdot v_j^i$$

for some $n \in \mathbb{N}$ and finite words w, u_j, v_j.

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Proof.

By the pigeonhole principle we find blocks 10^k and 10^ℓ in Π s.t.:

- $|Q| < k < \ell$
- $k \equiv \ell \bmod Z$
- automaton enters 10^k and 10^ℓ with the same state q

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Define $n = \ell - k$. Then $Z \mid n$ and

- automaton also enters $1\,0^{k+1}$ and $1\,0^{\ell+1}$ in the same state q'
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For all $i \in \mathbb{N}$, the blocks $1\,0^{k+j+i\cdot n}$ are entered in the same state. □

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We consider a transduct T of Π:

$$T = w \cdot \prod_{i=0}^{\omega} w_i \qquad\qquad w_i = \prod_{j=0}^{n} u_j \cdot v_j^i$$

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Removing 'ambiguous' factors, that is, factors $j \leq n$ for which:

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The function $i \mapsto |w_i|$ is linear, so FST can transduce w_i to 10^i. $\square$

Infima and Suprema

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There exist degrees X, Y that have no supremum.

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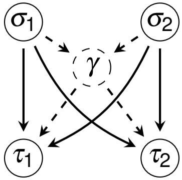
Idea: construct $\sigma_1, \sigma_2, \tau_1, \tau_2$ such that



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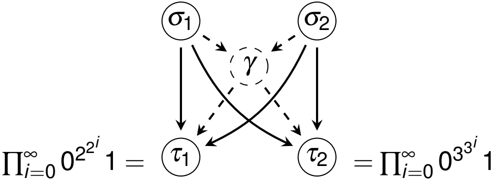

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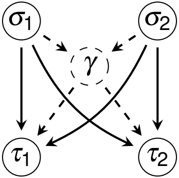
$$\prod_{i=0}^{\infty}(0^{2^{2^i}} 1 0^{3^{3^i}} 1) = \sigma_1 \qquad \sigma_2 = \prod_{i=0}^{\infty}(0^{3^{3^i}} 1 0^{2^{2^i}} 1)$$



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- morphic streams

are closed under finite state transduction.

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Idea: for every Turing machine M define a stream

$$w(M) = x^{s(M,0)} o(M,0)\ x^{s(M,1)} o(M,1)\ x^{s(M,2)} o(M,2)\ \dots$$

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Idea: for every Turing machine M define a stream

$$w(M) = x^{s(M,0)} o(M,0)\ x^{s(M,1)} o(M,1)\ x^{s(M,2)} o(M,2)\ \ldots$$

where x is a fresh symbol and

- $o(M,n) =$ output of M on input n
- $s(M,n) =$ number of steps of M until termination on input n

The Subhierarchy of Computable Streams

It is also interesting to look at subhierarchies. For example

- computable streams
- morphic streams

are closed under finite state transduction.

Theorem

The subhierarchy of computable streams has a top degree.

Shuffling all computable streams does not work
(the resulting stream is not computable).

Idea: for every Turing machine M define a stream

$$w(M) = x^{s(M,0)} o(M,0)\ x^{s(M,1)} o(M,1)\ x^{s(M,2)} o(M,2)\ \ldots$$

where x is a fresh symbol and

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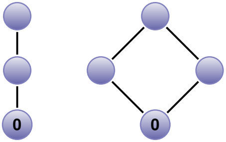
The shuffling of these streams is computable.

Open questions

- How to prove non-transducibility (e.g. for morphic streams)?
- Are Thue-Morse and Mephisto Walz transducible to each other?
- How many prime degrees are out there?
- Is Thue-Morse prime?
- Are there degrees forming the following structures?



- When does a set of degrees have a supremum?
- What is the structure of the subhierarchy of computable streams?
- What is the structure of the subhierarchy of morphic streams?
- …