Application of entropy compression in pattern avoidance

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Definitions

- p : a pattern over the alphabet $\{A, B, C, \ldots\}$
- w : a word over the alphabet $\Sigma_k = \{0, 1, \ldots, k-1\}$

$$p = ABA \qquad w = 0112101120$$

- $\lambda(p)$: avoidability index of p

$$\lambda(AA) = 3 \qquad \lambda(AAA) = 2$$

Results

Theorem (Bell & Goh, 2007 ; Rampersad, 2011)

Let p be a pattern over k variables

- *If* $|p| \geq 2^k$*, then* $\lambda(p) \leq 4$ *[Bell & Goh, 2007]*
- *If* $|p| \geq 3^k$*, then* $\lambda(p) \leq 3$ *[Rampersad, 2011]*
- *If* $|p| \geq 4^k$*, then* $\lambda(p) = 2$ *[Rampersad, 2011]*

Theorem (O. & Pinlou, 2013; Blanchet-Sadri & Woodhouse, 2013)

Let p be a pattern over k variables

1. *If* $|p| \geq 2^k$*, then* $\lambda(p) \leq 3$
2. *If* $|p| \geq 3 \times 2^{k-1}$*, then* $\lambda(p) = 2$

Optimality

Let p be a pattern over k variables

1. If $|p| \geq 2^k$, then $\lambda(p) \leq 3$
2. If $|p| \geq 3 \times 2^{k-1}$, then $\lambda(p) = 2$

$\forall k \geq 1$:

1. there exists an unavoidable pattern of size $2^k - 1$
   $\{A, ABA, ABACABA, ABACABADABACABA, \ldots\}$

2. there exists a 2-unavoidable pattern of size $3 \times 2^{k-1} - 1$
   $\{AA, AABAA, AABAACAABAA, AABAACAABAADAABAACAABA$

Known results

Let p be a pattern over k variables

1. If $|p| \geq 2^k$, then $\lambda(p) \leq 3$
2. If $|p| \geq 3 \times 2^{k-1}$, then $\lambda(p) = 2$

Patterns with at most 3 variables

- $k = 1$: $\lambda(AA) = 3$ et $\lambda(AAA) = 2$
- $k = 2$: For a pattern $p \in \{A, B\}^*$
  - if $|p| \geq 4$, then p contains a square, so $\lambda(p) \leq 3$
  - if $|p| \geq 6$, then $\lambda(p) = 2$ (Roth, 1992)
- $k = 3$: For a pattern $p \in \{A, B, C\}^*$
  - if $|p| \geq 8$, then $\lambda(p) \leq 3$
  - if $|p| \geq 12$, then $\lambda(p) = 2$

pattern, occurrence, factor

- An occurrence y of a pattern p forms a factor

  Example : $p = ABA$
  $y = (A = 00; B = 1)$ $\rightarrow$ forms the factor 00100

pattern, occurrence, factor

- An occurrence y of a pattern p forms a factor

  Example : $p = ABA$
  $y = (A = 00; B = 1) \quad \rightarrow \quad$ forms the factor 00100
  $y = (A = 0; B = 010) \quad \rightarrow \quad$ forms the factor 00100

Preliminary result

doubled pattern: every variable appears at least twice.

balanced pattern: every variable appears both in the prefix and the suffix of length $\left\lfloor \frac{|p|}{2} \right\rfloor$.

Proposition

For every pattern p on k variables and every $a \geq 2$, if $|p| \geq a \times 2^{k-1}$, then p contains a balanced pattern p' with $k' \geq 1$ variables such that $|p'| \geq a \times 2^{k'-1}$.

Proof by contradiction

Let p be pattern over $k \geq 4$ variables

1. If $|p| \geq 2^k$, then $\lambda(p) \leq 3$
2. If $|p| \geq 3 \times 2^{k-1}$, then $\lambda(p) = 2$

We show that:
If $|p| \geq 3 \times 2^{k-1}$ and p is doubled, then $\lambda(p) = 2$.

Suppose that p is a doubled pattern with k variables,
$|p| \geq 3 \times 2^{k-1}$, and $\lambda(p) > 2$.

$\Rightarrow$ There exists n such that every word in Σ_2^n contains p.

Algorithm

Algorithm 1: AVOIDP

Input : $V = \{0, 1\}^t$.

Output: w (a word avoiding p) and R (a data-structure recording stuff).

```
w ← ε
R ← ∅
for i ← 1 to t do
    Append V[i] to w
    Record in R that a letter has been added to w
    if w contains an occurrence y of p then
        Record y in R
        Erase the factor f corresponding to y in w
        Record in R that |y| letters have been erased in w
return w, R
```

Execution

- $k = 3$
- Arbitrary order on the variables : (A, B, C)
- $p = ACBBCBBABCAB$
- $V = [0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0]$

- $w_0 = \varepsilon$
- $R_0 = \begin{cases} D & = & \varepsilon \\ L & = & [\,] \\ X & = & \varepsilon \end{cases}$

Execution

- $k = 3$
- Arbitrary order on the variables : (A, B, C)
- $p = ACBBCBBABCAB$
- $V = [\underset{\uparrow}{0}, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0]$

- $w_1 = 0$
- $R_1 = \begin{cases} D & = & 0 \\ L & = & [\,] \\ X & = & \varepsilon \end{cases}$

Execution

- $k = 3$
- Arbitrary order on the variables : (A, B, C)
- $p = ACBBCBBABCAB$
- $V = [0, \underset{\uparrow}{0}, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0]$

- $w_2 = 00$
- $R_2 = \begin{cases} D & = & 00 \\ L & = & [\,] \\ X & = & \varepsilon \end{cases}$

Execution

- $k = 3$
- Arbitrary order on the variables : (A, B, C)
- $p = ACBBCBBABCAB$
- $V = [0, 0, \underset{\uparrow}{1}, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0]$

- $w_3 = 001$
- $R_3 = \begin{cases} D & = & 000 \\ L & = & [\,] \\ X & = & \varepsilon \end{cases}$

Execution

- $k = 3$
- Arbitrary order on the variables : (A, B, C)
- $p = ACBBCBBABCAB$
- $V = [0, 0, 1, \underset{\uparrow}{0}, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0]$

- $w_4 = 0010$

- $R_4 = \begin{cases} D = 0000 \\ L = [\,] \\ X = \varepsilon \end{cases}$

Execution

- $k = 3$
- Arbitrary order on the variables : (A, B, C)
- $p = ACBBCBBABCAB$
- $V = [0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, \underset{\uparrow}{1}, 1, 0]$

- $w_{24} = 001001100111001101110001$

- $R_{24} = \begin{cases} D & = & 000000000000000000000000 = 0^{24} \\ L & = & [\,] \\ X & = & \varepsilon \end{cases}$

Execution

- $k = 3$
- Arbitrary order on the variables : (A, B, C)
- $p = ACBBCBBABCAB$
- $V = [0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, \underset{\uparrow}{1}, 0]$

- $w_{25} = 001001100111001101110001\mathbf{1}$

- $R_{25} = \begin{cases} D = 000000000000000000000000\mathbf{0} = 0^{25} \\ L = [\,] \\ X = \varepsilon \end{cases}$

Execution

- $k = 3$
- Arbitrary order on the variables : (A, B, C)
- $p = ACBBCBBABCAB$
- $V = [0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, \underset{\uparrow}{1}, 0]$

- $w_{25} = 0010\,\underbrace{011001110011011100011}_{\ell=21}$
- $R_{25} = \begin{cases} D &= 0000000000000000000000000 = 0^{25} \\ L &= [\,] \\ X &= \varepsilon \end{cases}$

  Occurrence $y = (A = 01; B = 1; C = 100)$ of p

Execution

- $k = 3$
- Arbitrary order on the variables : (A, B, C)
- $p = ACBBCBBABCAB$
- $V = [0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, \underset{\uparrow}{1}, 0]$

- $w_{25} = 0010$~~011001110011011100011~~

- $R_{25} = \begin{cases} D & = & 0^{25}1^{21} \\ L & = & [\{|A|; |A \cdot B|\}] \\ X & = & A \cdot B \cdot C \end{cases}$

  Occurrence $y = (A = 01; B = 1; C = 100)$ of p

Execution

- $k = 3$
- Arbitrary order on the variables : (A, B, C)
- $p = ACBBCBBABCAB$
- $V = [0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, \underset{\uparrow}{1}, 0]$

- $w_{25} = 0010$~~011001110011011100011~~$= 0010$

- $R_{25} = \begin{cases} D &= 0^{25}1^{21} \\ L &= [\{|A|; |A \cdot B|\}] = [\{2, 3\}] \\ X &= A \cdot B \cdot C \end{cases}$

  Occurrence $y = (A = 01; B = 1; C = 100)$ of p

Execution

- $k = 3$
- Arbitrary order on the variables : (A, B, C)
- $p = ACBBCBBABCAB$
- $V = [0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, \underset{\uparrow}{1}, 0]$

- $w_{25} = 0010$
- $R_{25} = \begin{cases} D & = & 0^{25}1^{21} \\ L & = & [\{2, 3\}] \\ X & = & \color{red}{01}\color{blue}{1}\color{green}{100} \end{cases}$

Execution

- $k = 3$
- Arbitrary order on the variables : (A, B, C)
- $p = ACBBCBBABCAB$
- $V = [0, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 1, 1, \underset{\uparrow}{0}]$

- $w_{26} = 00100$

- $R_{26} = \begin{cases} D & = & 0^{25}1^{21}0 \\ L & = & [\{2, 3\}] \\ X & = & 011100 \end{cases}$

Sketch of proof

- $\mathcal{V}$: set of entry vectors $V = \{0,1\}^t$
- $\mathcal{R}$: set of records R produced by the algorithm
- $\mathscr{O}$: set of couples (w, R) produced by the algorithm

We have :

- $|w_t| \leq n \Rightarrow 2^n$ possible words
- $|\mathcal{V}| = 2^t$
- $|\mathscr{O}| \leq 2^n \times |\mathcal{R}|$

We will show that :

- $|\mathcal{V}| \leq |\mathscr{O}|$
- $|\mathcal{R}| = o(2^t)$
- $2^t = |\mathcal{V}| \leq |\mathscr{O}| \leq 2^n \times |\mathcal{R}| = o(2^t) \quad \rightarrow \quad$ Contradiction

Let us show that $|\mathcal{V}| \leq |\mathscr{O}|$

Lemme

After i steps, V_i can be recovered from the couple (w_i, R_i).

Proof

- Step 0 : $w_0 = \varepsilon$, $R_0 = (\varepsilon, [\,], [\,])$, $V_0 = \varepsilon$

Let us show that $|\mathcal{V}| \leq |\mathscr{O}|$

Lemme

After i steps, V_i can be recovered from the couple (w_i, R_i).

Proof

- Step 0 : $w_0 = \varepsilon$, $R_0 = (\varepsilon, [\,], [\,])$, $V_0 = \varepsilon$
- Step i :

Let us show that $|\mathcal{V}| \leq |\mathscr{O}|$

Lemme

After i steps, V_i can be recovered from the couple (w_i, R_i).

Proof

- Step 0 : $w_0 = \varepsilon$, $R_0 = (\varepsilon, [\,], [\,])$, $V_0 = \varepsilon$
- Step i :
  - If D ends with 0

    $p = ACBBCBBABCAB$, variable order : (A, B, C)

    $w_{24} = 001001100111001101110001$

    $$R_{24} = \begin{cases} D = & 000000000000000000000000 = 0^{24} \\ L = & [\,] \\ X = & [\,] \end{cases}$$

    By induction, (R_{i-1}, w_{i-1}) gives V_{i-1}.

    $V_i = V_{i-1} \cdot V[i]$

Let us show that $|\mathcal{V}| \leq |\mathscr{O}|$

Lemme

After i steps, V_i can be recovered from the couple (w_i, R_i).

Proof

- Step 0 : $w_0 = \varepsilon$, $R_0 = (\varepsilon, [\,], [\,])$, $V_0 = \varepsilon$
- Step i :
  - If D ends with 0

$$p = ACBBCBBABCAB, \text{ variable order : } (A, B, C)$$
$$w_{23} = 001001100111001101111000\not{1}$$
$$R_{23} = \begin{cases} D = & 00000000000000000000000\not{0} = 0^{23} \\ L = & [\,] \\ X = & [\,] \end{cases}$$

By induction, (R_{i-1}, w_{i-1}) gives V_{i-1}.

$V_i = V_{i-1} \cdot V[i]$

Let us show that $|\mathcal{V}| \leq |\mathscr{O}|$

Lemme

After i steps, V_i can be recovered from the couple (w_i, R_i).

Proof

- Step 0 : $w_0 = \varepsilon$, $R_0 = (\varepsilon, [\,], [\,])$, $V_0 = \varepsilon$
- Step i :
  - If D ends with 0
  - If D ends with 01^ℓ

    $p = ACBBCBBABCAB$, variable order : (A, B, C)
    $w_{25} = 0010$
    $$R_{25} = \begin{cases} D = & 0^{24}01^{21} \\ L = & [\{2,3\}] \\ X = & [011100] \end{cases}$$

    By induction, (w_{i-1}, R_{i-1}) gives V_{i-1}.
    $V_i = V_{i-1} \cdot V[i]$

Let us show that $|\mathcal{V}| \leq |\mathscr{O}|$

Lemme

After i steps, V_i can be recovered from the couple (w_i, R_i).

Proof

- Step 0 : $w_0 = \varepsilon$, $R_0 = (\varepsilon, [\,], [\,])$, $V_0 = \varepsilon$
- Step i :
  - If D ends with 0
  - If D ends with 01^ℓ

    $p = ACBBCBBABCAB$, variable order : (A, B, C)

    $w_{25} = 0010$

    $$R_{25} = \begin{cases} D = & 0^{24}01^{21} \\ L = & [\{2,3\}] \\ X = & [011100] \end{cases} \quad |A| = 2 \quad |B| = 1 \quad |C| = \frac{21 - 3\times 2 - 6 \times 1}{3} = 3$$

    By induction, (w_{i-1}, R_{i-1}) gives V_{i-1}.

    $V_i = V_{i-1} \cdot V[i]$

Let us show that $|\mathcal{V}| \leq |\mathscr{O}|$

Lemme

After i steps, V_i can be recovered from the couple (w_i, R_i).

Proof

- Step 0 : $w_0 = \varepsilon$, $R_0 = (\varepsilon, [\,], [\,])$, $V_0 = \varepsilon$
- Step i :
  - If D ends with 0
  - If D ends with 01^ℓ

$p = ACBBCBBABCAB$, variable order : (A, B, C)

$w_{25} = 0010$

$$R_{25} = \begin{cases} D = & 0^{24}01^{21} \\ L = & [\{2,3\}] \quad |A| = 2 \quad |B| = 1 \quad |C| = \frac{21 - 3\times 2 - 6\times 1}{3} = 3 \\ X = & [011100] \end{cases}$$

By induction, (w_{i-1}, R_{i-1}) gives V_{i-1}.

$V_i = V_{i-1} \cdot V[i]$

Let us show that $|\mathcal{V}| \leq |\mathscr{O}|$

Lemme

After i steps, V_i can be recovered from the couple (w_i, R_i).

Proof

- Step 0 : $w_0 = \varepsilon$, $R_0 = (\varepsilon, [\,], [\,])$, $V_0 = \varepsilon$
- Step i :
  - If D ends with 0
  - If D ends with 01^ℓ

$p = ACBBCBBABCAB$, variable order : (A, B, C)

$w_{24} = 001001100111001101110001$1

$$R_{24} = \begin{cases} D = & 0^{24}01^{21} \\ L = & [\{2,3\}] \quad |A|=2 \quad |B|=1 \quad |C| = \frac{21-3\times2-6\times1}{3} = 3 \\ X = & [011100] \end{cases}$$

By induction, (w_{i-1}, R_{i-1}) gives V_{i-1}.

$V_i = V_{i-1} \cdot V[i]$

Let us show that $|\mathcal{V}| \leq |\mathscr{O}|$

Lemme

After i steps, V_i can be recovered from the couple (w_i, R_i).

Proof

- Step 0 : $w_0 = \varepsilon$, $R_0 = (\varepsilon, [\,], [\,])$, $V_0 = \varepsilon$
- Step i :
  - If D ends with 0
  - If D ends with 01^ℓ

$p = ACBBCBBABCAB$, variable order : (A, B, C)

$w_{24} = 0010011001110011011100011\!\!\!/$

$$R_{24} = \begin{cases} D = & 0^{24}\not{0} \\ L = & [\{2,3\}] \quad |A|=2 \quad |B|=1 \quad |C| = \frac{21-3\times2-6\times1}{3} = 3 \\ X = & [011100] \end{cases}$$

By induction, (w_{i-1}, R_{i-1}) gives V_{i-1}.

$V_i = V_{i-1} \cdot V[i]$

Let us show that $|\mathcal{V}| \leq |\mathscr{O}|$

Lemme

After i steps, V_i can be recovered from the couple (w_i, R_i).

Proof

- Step 0 : $w_0 = \varepsilon$, $R_0 = (\varepsilon, [\,], [\,])$, $V_0 = \varepsilon$
- Step i :
  - If D ends with 0
  - If D ends with 01^ℓ

    $p = ACBBCBBABCAB$, variable order : (A, B, C)

    $w_{24} = 001001100111001101110001$

    $$R_{24} = \begin{cases} D = & 0^{24} \\ L = & [] \\ X = & [] \end{cases}$$

    By induction, (w_{i-1}, R_{i-1}) gives V_{i-1}.

    $V_i = V_{i-1} \cdot V[i]$

Let us show that $|\mathcal{V}| \leq |\mathscr{O}|$

Lemme

After i steps, V_i can be recovered from the couple (w_i, R_i).

Proof

- Step 0 : $w_0 = \varepsilon$, $R_0 = (\varepsilon, [\,], [\,])$, $V_0 = \varepsilon$
- Step i :
  - If D ends with 0
  - If D ends with 01^ℓ

    $p = ACBBCBBABCAB$, variable order : (A, B, C)

    $w_{24} = 00100110011100110111000 1$

    $$R_{24} = \begin{cases} D = & 0^{24} \\ L = & [] \\ X = & [] \end{cases}$$

    By induction, (w_{i-1}, R_{i-1}) gives V_{i-1}.

    $V_i = V_{i-1} \cdot V[i]$

Distinct entry vectors produce distinct outputs (w, R).

We show that $|\mathcal{R}| = o(2^t)$

Keep in mind :

- $R = R_t = (D, L, X)$
- $|\mathcal{R}| \leq |\mathcal{D}| \times |\mathcal{L}| \times |\mathcal{X}|$
- t letters are added, $t - |w_t|$ letters are erased
- Let m be the number erased factors
- $(f_i)_{1 \leq i \leq m}$ is the set of m erased factors
- $|f_i| \geq 3 \times 2^{k-1}$
- $\sum_{1 \leq i \leq m} |f_i| = t - |w_t| \leq t$

Analysis of $\mathcal{D}$

- $|D| = t + t - |w_t| = 2t - n$
- D is a partial Dyck word.
- The length of a descent (consecutive 1's) is $\geq 3 \times 2^{k-1}$.
- $C_{t,d}$: number of Dyck words of length $2t$ with descents of length $\geq d$.

Let $\phi_d(x) = 1 + \sum_{i \geq d} x^i = 1 + \frac{x^d}{1-x}$.

Lemme (Esperet & Parreau, 2013)

Let d be an integer such that the equation $\phi_d(x) - x\phi_d'(x) = 0$ has a solution τ with $0 < \tau < r$, where r is the radius of convergence of ϕ_d. Then τ is the unique solution of the equation in the open interval $(0, r)$. Moreover, there exists a constant c_d such that $C_{t,d} \leq c_d \gamma_d^t t^{-\frac{3}{2}}$ where $\gamma_d = \phi_d'(\tau) = \frac{\phi_d(\tau)}{\tau}$.

Analysis of $\mathcal{D}$

- $|D| = t + t - |w_t| = 2t - n$
- D is a partial Dyck word.
- The length of a descent (consecutive 1's) is $\geq 3 \times 2^{k-1}$.
- $C_{t,d}$: number of Dyck words of length $2t$ with descents of length $\geq d$.

- $|\mathcal{D}| \leq 1.27575^t$ if $d \geq 24$
- $|\mathcal{D}| \leq 1.15685^t$ if $d \geq 48$
- $|\mathcal{D}| \leq 1.08603^t$ if $d \geq 100$

Analysis of $\mathcal{X}$

- For every erased factor f_i, we add at most $\left\lfloor \frac{|f_i|}{2} \right\rfloor$ letters to X
- $|X| \leq \left\lfloor \frac{|f_1|}{2} \right\rfloor + \left\lfloor \frac{|f_2|}{2} \right\rfloor + \ldots + \left\lfloor \frac{|f_m|}{2} \right\rfloor \leq \frac{t}{2}$
- $|\mathcal{X}| \leq 2^{\frac{t}{2}}$

Analysis of $\mathcal{L}$

- $\{A, B, C, \ldots\}$

Analysis of $\mathcal{L}$

- ~~$\{A, B, C, \ldots\}$~~ $\{A_1, A_2, \ldots, A_k\}$

Analysis of $\mathcal{L}$

- ~~$\{A, B, C, \ldots\}$~~ $\{A_1, A_2, \ldots, A_k\}$
- $L = \{L_1, L_2, \ldots, L_m\}$
- Every L_i in L corresponds to an erased factor f_i
- $L_i = \{|A_1|, |A_1 \cdot A_2|, \ldots, |A_1 \cdot A_2 \cdot \ldots \cdot A_{k-1}|\}$
- $h_k(\ell)$: number of $(k-1)$-sets corresponding to a factor of length ℓ
- $|\mathcal{L}| \leq h_k(|f_1|) \times h_k(|f_2|) \times \ldots \times h_k(|f_m|)$
- $g_k(\ell) = h_k(\ell)^{\frac{1}{\ell}}$
- $|\mathcal{L}| \leq g_k(|f_1|)^{|f_1|} \times g_k(|f_2|)^{|f_2|} \times \ldots \times g_k(|f_m|)^{|f_m|}$

- If we show that $g_k(\ell) \leq c$, then
  $|\mathcal{L}| \leq c^{|f_1|} \times c^{|f_2|} \times \ldots \times c^{|f_m|} \leq c^t$

Bound on $g_k(\ell)$ for $k = 4, \ell \geq 100$ or $k \geq 5, \ell \geq 48$

- $L_i = \{\underbrace{|A_1|}_{\geq 1}, |A_1 \cdot A_2|, |A_1 \cdot A_2 \cdot A_3|, \ldots, \underbrace{|A_1 \cdot A_2 \cdot \ldots \cdot A_{k-1}|}_{\leq \left\lfloor \frac{|f_i|}{2} \right\rfloor}\}$
- L_i is a $(k-1)$-set of distinct integers between 1 and $\left\lfloor \frac{|f_i|}{2} \right\rfloor$
- $h_k(\ell) \leq \binom{\lfloor \frac{\ell}{2} \rfloor}{k-1} \quad \Rightarrow \quad g_k(\ell) \leq \binom{\lfloor \frac{\ell}{2} \rfloor}{k-1}^{\frac{1}{\ell}}$
- $g_k(\ell) \leq \overline{g_k}(l) = \left(\frac{\left(\lfloor \frac{\ell}{2} \rfloor\right)^{k-1}}{(k-1)!} \right)^{\frac{1}{\ell}}$ (decreasing for $\ell \geq 3 \times 2^{k-1}$)

- $\forall \ell \geq 100,\ \overline{g_4}(\ell) \leq \overline{g_4}(100) \leq 1.10456$
- $\forall k \geq 5,\ \forall \ell \geq 48,\ \overline{g_k}(\ell) \leq \overline{g_5}(48) \leq 1.21973$

Bound on $g_4(\ell)$ for $24 \leq \ell \leq 99$

- $k = 4$. Variables : A_1, A_2, A_3, A_4.
- a_i : # appearance of A_i in p. $a_i \geq 2$.
- $\sum a_i = |p|$
- $L_i = \{|A_1|, |A_1 \cdot A_2|, |A_1 \cdot A_2 \cdot A_3|\}$. Gives $\{\ell_1, \ell_2, \ell_3, \ell_4\}$
- $\mathcal{A}_{|p|} = \sum_{i \geq |p|} b_i \, x^i$ (generating function)

  b_i : # 4-uplets $(\ell_1, \ell_2, \ell_3, \ell_4)$ with $\ell_i \geq 1$
  such that $a_1 \times \ell_1 + a_2 \times \ell_2 + a_3 \times \ell_3 + a_4 \times \ell_4 = i$
  By definition : $h_4(\ell) = b_\ell$, and then $g_4(\ell) = (b_\ell)^{\frac{1}{\ell}}$.

Pennies, nickels, dimes, quarters, and half dollars

- $\mathcal{C} = \sum_{i \geq 1} c_i x^i$ (c_i : number of ways to change i cents)
- $\mathcal{C} = \frac{1}{1-x} \times \frac{1}{1-x^2} \times \frac{1}{1-x^5} \times \frac{1}{1-x^{10}} \times \frac{1}{1-x^{20}} \times \frac{1}{1-x^{50}}$
- In our case :
  - Four coins with (possibly the same) values a_i
  - Every coin appears at least once.
  - $\mathcal{A}_{|p|} = \sum_{i \geq |p|} b_i \, x^i = \frac{x^{a_1}}{1-x^{a_1}} \times \frac{x^{a_2}}{1-x^{a_2}} \times \frac{x^{a_3}}{1-x^{a_3}} \times \frac{x^{a_4}}{1-x^{a_4}}$

Bound on $g_4(\ell)$ for $24 \leq \ell \leq 100$

- $k = 4$. Variables : A_1, A_2, A_3, A_4.
- a_i : # of appearance of A_i in p. $a_i \geq 2$. $\sum a_i = |p|$
- $L_i = \{|A_1|, |A_1 \cdot A_2|, |A_1 \cdot A_2 \cdot A_3|\}$. Gives $\{\ell_1, \ell_2, \ell_3, \ell_4\}$
- $\mathcal{A}_{|p|} = \sum_{i \geq |p|} b_i\, x^i$ (generating function)

  b_i : # 4-uplets $(\ell_1, \ell_2, \ell_3, \ell_4)$ with $\ell_i \geq 1$
  such that $a_1 \times \ell_1 + a_2 \times \ell_2 + a_3 \times \ell_3 + a_4 \times \ell_4 = i$
  By definition : $h_4(\ell) = b_\ell$ and so $g_4(\ell) = (b_\ell)^{\frac{1}{\ell}}$.
- $\mathcal{A}_{|p|} = \frac{x^{a_1}}{1 - x^{a_1}} \times \frac{x^{a_2}}{1 - x^{a_2}} \times \frac{x^{a_3}}{1 - x^{a_3}} \times \frac{x^{a_4}}{1 - x^{a_4}}$
- For all $24 \leq |p| \leq 99$ and all (a_1, a_2, a_3, a_4) such that $\sum a_i = |p|$, Maple computes
  $\mathcal{A}_{|p|} = b_{24}x^{24} + b_{25}x^{25} + \ldots + b_{99}x^{99} + O(x^{100})$.
- $(b_i)^{\frac{1}{i}}$ max for $|p| = 24$, $(a_1, a_2, a_3, a_4) = (2, 2, 2, 18)$, $i = 46$: $b_{46} = 84$

  there are 84 4-uplets which correspond to an occurrence f of a pattern p, such that $|f| = 46$ and $|p| = 24$
- $g_4(\ell) \leq 84^{\frac{1}{46}} < 1.10112$

Analyse of $\mathcal{L}$

- $g_4(\ell) < 1.10112$ for $24 \leq \ell \leq 99$
- $\forall \ell \geq 100,\ g_4(\ell) \leq 1.10456$
- $\forall k \geq 5,\ \forall \ell \geq 48,\ g_k(\ell) \leq 1.21973$

- If we show that $g_k(\ell) \leq c$ then $|\mathcal{L}| \leq c^{|f_1|} \times c^{|f_2|} \times \ldots \times c^{|f_m|} \leq c^t$
- $|\mathcal{L}| \leq (1.10456)^t$ if $k = 4$
- $|\mathcal{L}| \leq (1.21973)^t$ if $k \geq 5$

We show that $|\mathcal{R}| = o(2^t)$

- $|\mathcal{R}| \leq |\mathcal{D}| \times |\mathcal{L}| \times |\mathcal{X}|$
- If $k \geq 5 : |\mathcal{R}| \leq (1.15685 \times 1.21973 \times \sqrt{2})^t = o(2^t)$
- If $k = 4 : |\mathcal{R}| \leq (1.27575 \times 1.10456 \times \sqrt{2})^t = o(2^t)$

$2^t = |\mathcal{V}| \leq |\mathscr{O}| \leq 2^n \times |\mathcal{R}| = o(2^t)$

We have shown that :

- $|\mathcal{V}| \leq |\mathscr{O}|$
- $|\mathcal{R}| = o(2^t)$

And so :

- $2^t = |\mathcal{V}| \leq |\mathscr{O}| \leq 2^n \times |\mathcal{R}| = o(2^t)$

Questions

- Is every doubled pattern 3-avoidable ?
  remaining cases : $k = 4$ and $k = 5$.
- Is there a k such that every doubled pattern on at least k variables is 2-avoidable ?
  Such a k is at least 5 since $\lambda(ABCCBADD) = 3$.