What is the minimal critical exponent of quasiperiodic words?

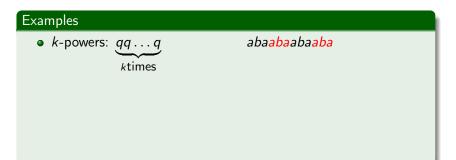
Gwenaël Richomme

Montpellier, France

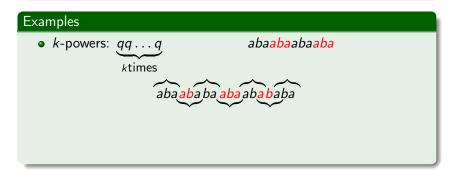
April 22th, 2013 Workshop "Challenges in Combinatorics on Words"

Definition

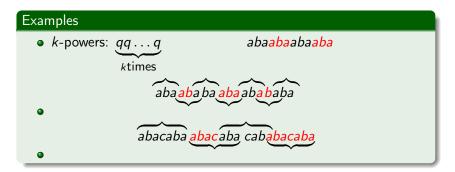
Definition



Definition



Definition



Critical exponent?

Fractional power

 $x^{\frac{p}{q}} = x^n y$ with $n = \lfloor \frac{p}{q} \rfloor$, q = |x| and y prefix of x of length p - nq

$$ababa = (ab)^{5/2}$$
 $abaabaab = (aba)^{8/3}$

Critical exponent?

Fractional power

$$x^{\frac{p}{q}} = x^n y$$
 with $n = \lfloor \frac{p}{q} \rfloor$, $q = |x|$ and y prefix of x of length $p - nq$

Critical exponent of w

 $E(w) = sup\{k \in \mathbf{Q} \mid w \text{ contains a } k\text{th power}\}$

E(Thue-Morse) = 2 E(Fibonacci) = 2 + ϕ

Reformulation of the question

Question

$\min\{E(w) \mid w \text{ quasiperiodic}\}$?

Question

$$\min\{E(w) \mid w \text{ quasiperiodic}\}$$
?

Observation

w quasiperiodic
$$\Rightarrow E(w) > 2$$
.

Indeed w contains an overlap of q or q^2 .

Result to be verified

For all $\epsilon>$ 0, over a 3-letter alphabet, there exists an infinite word with critical exponent less than 2 + ϵ

So the question holds only on binary alphabets: Is the smallest exponent $\frac{7}{3}$? $\frac{5}{2}$? $\frac{8}{3}$? other ?

Result to be verified

For all $\epsilon>$ 0, over a 3-letter alphabet, there exists an infinite word with critical exponent less than 2 + ϵ

So the question holds only on binary alphabets: Is the smallest exponent $\frac{7}{3}$?

Recent idea (friday)

to use Karhumäki, Shallit 1994 and their 21-uniform morphism: $\Rightarrow \frac{7}{3}$

Ideas for the 7-letter alphabet

Step 1

$$f \left\{ \begin{array}{l} a \mapsto xyxzxyx \\ b \mapsto xyxzxy \\ c \mapsto xyxz \end{array} \right.$$

for all infinite word w, f(w) is xyxzxyx-quasiperiodic

Ideas for the 7-letter alphabet

Step 1

$$\left\{ \begin{array}{l} a \mapsto xyxzxyx \\ b \mapsto xyxzxy \\ c \mapsto xyxz \end{array} \right.$$

for all infinite word w, f(w) is xyxzxyx-quasiperiodic

Step 2

Choose:

- w, y and z square-free
- x letter, $x \notin alph(yz)$, $alph(y) \cap alph(z) = \emptyset$

Maximal runs of exponent > 2 are:

Ideas for the 7-letter alphabet (continue)

Consequence of Step 2

$$E(w) = \max(2 + \frac{1}{1 + |y|}, 2 + \frac{1}{1 + \frac{|z|}{2 + |y|}})$$

Final step

y and z can be chosen on disjoint 3-letter alphabets such that $E(w) \leq 2 + \epsilon$

Use following square-free Brandenburg's morphism (1983) twice:

 $a_1 \mapsto aba \ cab \ cac \ bab \ cba \ cbc$ $a_2 \mapsto aba \ cab \ cac \ bac \ aba \ cbc$ $a_3 \mapsto aba \ cab \ cac \ bca \ bcb \ abc$ $a_4 \mapsto aba \ cab \ cba \ cba \ cab \ acb \ abc$ $a_5 \mapsto aba \ cab \ cba \ cba \ cbc \ acb \ abc$

Use following square-free Brandenburg's morphism (1983) twice:

 $a_1 \mapsto aba \ cab \ cac \ bab \ cba \ cbc$ $a_2 \mapsto aba \ cab \ cac \ bac \ aba \ cbc$ $a_3 \mapsto aba \ cab \ cac \ bca \ bcb \ abc$ $a_4 \mapsto aba \ cab \ cba \ cba \ cab \ acb \ abc$ $a_5 \mapsto aba \ cab \ cab \ cba \ cbc \ acb \ abc$

with following extensions for the first time: $a_6 \mapsto dbd \ cdb \ cdc \ bdb \ cbd \ cbc$, $a_7 \mapsto ebe \ ceb \ cec \ beb \ cbc$

Use following square-free Brandenburg's morphism (1983) twice:

 $\begin{cases} a_1 \mapsto aba \ cab \ cac \ bab \ cba \ cbc \\ a_2 \mapsto aba \ cab \ cac \ bac \ aba \ cbc \\ a_3 \mapsto aba \ cab \ cac \ bca \ bcb \ abc \\ a_4 \mapsto aba \ cab \ cba \ cba \ cab \ acb \ abc \\ a_5 \mapsto aba \ cab \ cba \ cba \ cbc \ abc \\ abc \ ab$

with following extensions for the first time: $a_6 \mapsto dbd \ cdb \ cdc \ bdb \ cbd \ cbc$, $a_7 \mapsto ebe \ ceb \ cec \ beb \ cbc$

If w has a run of period p and exponent $2 + \epsilon$ with $\epsilon > 0$, then f(w) has a run of exponent $2 + \epsilon + 17/p$

Use following square-free Brandenburg's morphism (1983) twice:

 $\left\{\begin{array}{l} a_1 \mapsto aba \ cab \ cac \ bab \ cba \ cbc \\ a_2 \mapsto aba \ cab \ cac \ bac \ aba \ cbc \\ a_3 \mapsto aba \ cab \ cac \ bca \ bcb \ abc \\ a_4 \mapsto aba \ cab \ cba \ cba \ cab \ abc \\ a_5 \mapsto aba \ cab \ cba \ cbc \ abc \\ abc \ abc$

with following extensions for the first time: $a_6 \mapsto dbd \ cdb \ cdc \ bdb \ cbd \ cbc$, $a_7 \mapsto ebe \ ceb \ cec \ beb \ cbc$

If w has a run of period p and exponent $2 + \epsilon$ with $\epsilon > 0$, then f(w) has a run of exponent $2 + \epsilon + 17/p$

(In the construction on 7 letter alphabt, we can prove periods of repetitions of exponent at least 2 are > |xyz|.)

Use paper by Karhumäki and Shallit in 1994 and their morphism:

- $\left\{ \begin{array}{l} a \mapsto 011010011001001101001 \\ b \mapsto 10010110010011001010 \\ c \mapsto 10010110011011001010 \\ d \mapsto 011010011011001001 \end{array} \right.$

KS1994: If w is square-free:

- f(w) contains no square yy with |y| > 13;
- f(w) contains no $\frac{7}{2}^+$ -powers.

It seems that taking suitable w quasiperiodic over $\{a, b, c\}$ with exponent $2 < E(w) < \frac{7}{3}$, we can get $E(f(w)) = \frac{7}{3}$.

Theorem (Karhumäki, Shallit 1994)

Let x be a word avoiding α -powers, with $2 < \alpha \leq \frac{7}{3}$. Let μ be the Thue–Morse morphism. Then there exist u, v with u, $v \in \{\varepsilon, 01, 00, 11\}$ and a word y avoiding α -powers, such that $x = u\mu(y)v$.

Theorem (Karhumäki, Shallit 1994)

Let x be a word avoiding α -powers, with $2 < \alpha \leq \frac{7}{3}$. Let μ be the Thue–Morse morphism. Then there exist u, v with u, $v \in \{\varepsilon, 01, 00, 11\}$ and a word y avoiding α -powers, such that $x = u\mu(y)v$.

Consequence:

for w infinite avoiding such α -powers, $n \ge a$, $w = u\mu^n(w')$ with w'. w q-quasiperiodic + n such that $3|q| \le |\mu^n(a)|$: contradiction.

Theorem (Karhumäki, Shallit 1994)

Let x be a word avoiding α -powers, with $2 < \alpha \leq \frac{7}{3}$. Let μ be the Thue–Morse morphism. Then there exist u, v with u, $v \in \{\varepsilon, 01, 00, 11\}$ and a word y avoiding α -powers, such that $x = u\mu(y)v$.

Consequence:

for w infinite avoiding such α -powers, $n \ge a$, $w = u\mu^n(w')$ with w'. w q-quasiperiodic + n such that $3|q| \le |\mu^n(a)|$: contradiction.

 $E(w) \geq \frac{7}{3}$

Characterization of quasiperiodic-free morphism ? That is w non-quasiperiodic $\Rightarrow f(w)$ non-quasiperiodic.

Characterization of quasiperiodic-free morphism ? That is w non-quasiperiodic $\Rightarrow f(w)$ non-quasiperiodic.

They are prefix and suffix.

Characterization of quasiperiodic-free morphism ? That is w non-quasiperiodic $\Rightarrow f(w)$ non-quasiperiodic.

They are prefix and suffix.

?

If f does not preserve non-quasiperiodic words, then exists uv^{ω} non-quasiperiodic with $f(uv^{\omega})$ non-quasiperiodic?

Characterization of quasiperiodic-free morphism ? That is w non-quasiperiodic $\Rightarrow f(w)$ non-quasiperiodic.

They are prefix and suffix.

?

If f does not preserve non-quasiperiodic words, then exists uv^{ω} non-quasiperiodic with $f(uv^{\omega})$ non-quasiperiodic?

What about bounds on |u| and |v|?