Avoiding Three Consecutive Blocks of the Same Length and Sum

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Pattern avoidance

Problem

Find an infinite word **w** over a finite alphabet Σ such that no factor matches a given pattern.

- A *kth power* is a word of the form x^k for some $x \in \Sigma^*$.
  - `murmur` is a *square*.
- An *abelian kth power* is a word of the form $x_1 \cdots x_k$, where each x_i is a permutation of x_1.
  - `reappear` is an abelian square.
- Let $\Sigma \subseteq \mathbb{Z}$. An *additive kth power* is a word of the form $x_1 \cdots x_k$ such that

$$|x_1| = \cdots = |x_k|$$
$$\sum x_1 = \cdots = \sum x_k.$$

  - `facade` is an additive square if we let $\mathtt{a} = 1$, $\mathtt{b} = 2$, ...

Additive Powers

Pirillo and Varricchio (1994) discuss avoiding additive kth powers. Independently, Halbeisen and Hungerbühler (2000) considered additive squares.

Theorem (Dekking, 1979)

Abelian 4th powers are avoidable over a binary alphabet.

Corollary

Additive 4th powers are avoidable over a binary alphabet.

Questions

Are additive squares/cubes avoidable? How many symbols are required?

Our Result

Theorem

Suppose $\Sigma = \{0, 1, 3, 4\}$ *and* $\varphi\colon \Sigma^* \to \Sigma^*$ *is the morphism*

$$\begin{aligned}\varphi(0) &= 03\\ \varphi(1) &= 43\\ \varphi(3) &= 1\\ \varphi(4) &= 01.\end{aligned}$$

Then the fixed point

$$\mathbf{w} := \varphi^{\omega}(0) = 031430110343430\cdots$$

avoids additive cubes.

The morphism was found by brute force search (Shallit).

Proof Outline

- Start with an infinite tree $\mathcal{T}$ representing all prefixes of **w**.
  - Based on recursive structure of **w**.
- Construct a tree $\mathcal{T}^4$ representing all triples of consecutive blocks.
- Store information (state) at each node such that we can
  - compute the state of a child from its parent and the edge label, and
  - determine whether the node represents an additive cube given the state.

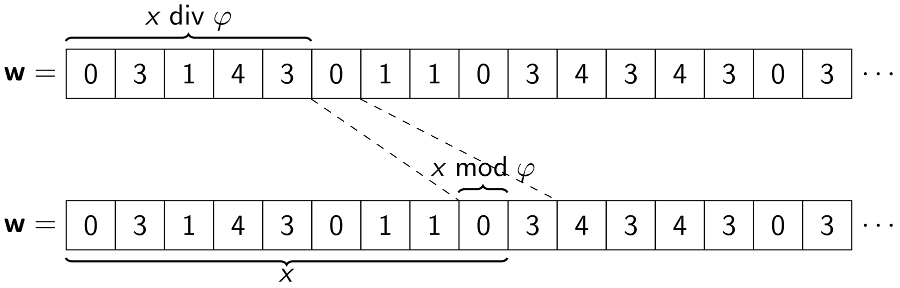
  We use two vectors in $\mathbb{N}^4$.
- Use linear algebra to show that, along a (hypothetical) path to an additive cube, the vectors are bounded.
- Exhaustively check the remaining (finite) search space for additive cubes.

Recursive Structure

$$\mathbf{w} = 0\ 3\ 1\ 4\ 3\ 0\ 1\ 1\ 0\ 3\ 4\ 3\ 4\ 3\ 0\ 3 \cdots$$

$$\mathbf{w} = 0\ 3\ 1\ 4\ 3\ 0\ 1\ 1\ 0\ 3\ 4\ 3\ 4\ 3\ 0\ 3 \cdots$$

Quotients and Remainders



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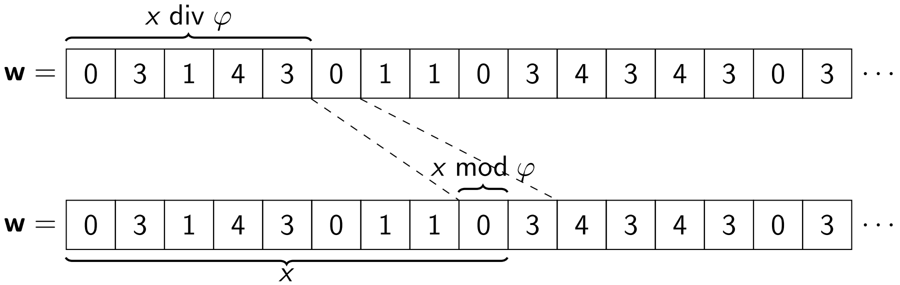
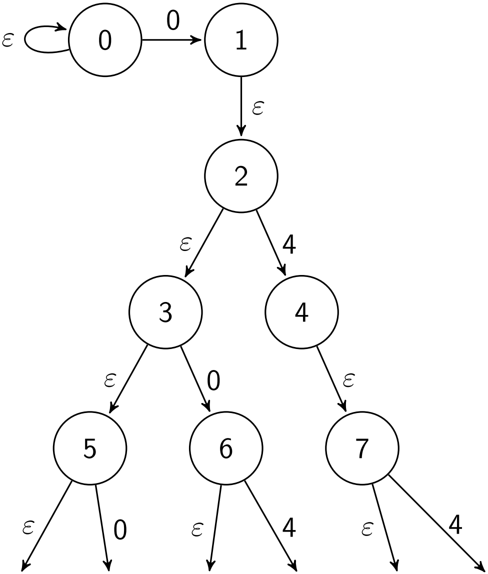


Idea

Build a tree with a node for each prefix. For each x, draw an edge from x div φ to x labelled x mod φ.

$\mathcal{T}$



$\mathcal{T}^4$

Three consecutive blocks are delimited by four positions: the start of each block, and the end of the last block.

Definition

Suppose V is the set of nodes in $\mathcal{T}$. We define a tree $\mathcal{T}^4$ on nodes V^4 such that there is an edge from $(x_1, x_2, x_3, x_4) \in V^4$ to $(y_1, y_2, y_3, y_4) \in V^4$ labelled $(a_1, a_2, a_3, a_4) \in \{\varepsilon, 0, 4\}^*$ if and only if there is an edge from x_i to y_i labelled a_i for $i = 1, 2, 3, 4$.

Any triple of blocks $b_1 b_2 b_3$ corresponds to a node in $\mathcal{T}^4$.

Next Step

Annotate each node with information to identify additive cubes.

Add some "state" to each node such that we can

- compute the state of the child given the state of the parent and edge label, and
- additive cubes can be identified.

Example

Associate a word with each node in $\mathcal{T}$. Let ε be the word for the root node, and compute the word for a child as follows:

$$x \xrightarrow{y} \varphi(x)y.$$

Then (by induction) node i is associated with $w[0..i-1]$.

- We can recursively compute $w[0..i_1-1], w[0..i_2-1], w[0..i_3-1], w[0..i_4-1]$ for a node (i_1, i_2, i_3, i_4) in $\mathcal{T}^4$.
- Given $w[0..i_1-1], w[0..i_2-1], w[0..i_3-1], w[0..i_4-1]$, we can check if $w[i_1..i_4-1]$ is an additive cube.

Parikh vectors

Definition

The Parikh map, $\psi \colon \Sigma^* \to \mathbb{N}^\Sigma$, maps a word x to a vector $\psi(x)$ that counts the number of occurrences of each symbol $a \in \Sigma$ in x.
For example, $\psi(034343) = (1, 0, 3, 2)$.

Idea

Store $\psi(x)$ instead of x.

Parikh Vector Operations

- We can compute $\psi(\varphi(x)y)$ given $\psi(x)$ and y.
  - $$\psi(\varphi(x)) = M\psi(x)$$

    where M is the *incidence matrix* of φ:

    $$M := \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$
  - $\psi(xy) = \psi(x) + \psi(y)$

$$\psi(x) \xrightarrow{y} \psi(\varphi(x)y) = M\psi(x) + \psi(y)$$

- We can detect additive cubes.

$$|x| = \psi(x) \cdot (1,1,1,1)$$

$$\sum x = \psi(x) \cdot (0,1,3,4)$$

Block Differences

Let $b_1 = w[i_1..i_2 - 1]$, $b_2 = w[i_2..i_3 - 1]$, $b_3 = w[i_3..i_4 - 1]$ be three consecutive blocks.

- Given $t_1 = \psi(w[0..i_1 - 1])$, $t_2 = \psi(w[0..i_2 - 1])$, $t_3 = \psi(w[0..i_3 - 1])$ and $t_4 = \psi(w[0..i_4 - 1])$, we can tell if $b_1 b_2 b_3$ is an additive cube.
- It suffices to have the Parikh vector for each block:

$$\psi(b_1) = t_2 - t_1$$
$$\psi(b_2) = t_3 - t_2$$
$$\psi(b_3) = t_4 - t_3$$

  or even just the *block differences*:

$$u := \psi(b_2) - \psi(b_1) = t_3 - 2t_2 + t_1$$
$$v := \psi(b_3) - \psi(b_2) = t_4 - 2t_3 + t_2$$

  to detect additive cubes.

Proposal

Keep two *block difference* vectors,

$$u = \psi(x_2) - \psi(x_1)$$
$$v = \psi(x_3) - \psi(x_2).$$

On transition (a_1, a_2, a_3, a_4), we compute u', v' where

$$u' = Mu - f(a_1, a_2, a_3)$$
$$v' = Mv - f(a_2, a_3, a_4)$$

with $f(a, b, c) = \psi(a) - 2\psi(b) + \psi(c)$.

Eigenbasis

Idea

Change basis so the matrix is in Jordan canonical form.

Entries are complex numbers, not integers.

Eigencoordinates are decoupled for individual analysis.

Suppose $M = P^{-1}DP$, where D is a diagonal matrix with diagonal elements $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, the eigenvalues of M.

$$\lambda_1 \doteq 1.69028 \qquad \lambda_2 = -1.50507$$
$$\lambda_3 \doteq 0.40739 + 0.47657i \qquad \lambda_4 = 0.40739 - 0.47657i$$

Note that $|\lambda_3| = |\lambda_4| \doteq 0.62696$.

Coordinates along a path

Recall the equation

$$u' = Mu - f(a_1, a_2, a_3)$$

For each coordinate $i = 1, 2, 3, 4$ in the eigenbasis, we have

$$u_i' = \lambda_i u_i - f_i(a_1, a_2, a_3).$$

Note that $u_i' - \lambda_i u_i = f_i(a_1, a_2, a_3)$ is bounded.

Question

Suppose $\lambda \in \mathbb{C}$ and $(z_j)_{j=0}^{\infty}$ is a sequence of complex numbers with $z_0 = 0$ and

$$|z_{j+1} - \lambda z_j| \leq B$$

for all j. What can we say about the asymptotic behaviour of such sequences?

Inside the Unit Circle ($|\lambda_i| < 1$)

Theorem

Let $\lambda \in \mathbb{C}$ be a complex number such that $|\lambda| < 1$. Suppose $(z_j)_{j=0}^{\infty}$ is a complex sequence such that $z_0 = 0$ and

$$|z_{j+1} - \lambda z_j| \leq B$$

for all j. Then $|z_j| \leq \frac{B}{1-\lambda}$ for all j.

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for all j. Then $|z_j| \leq \frac{B}{1-\lambda}$ for all j.

Since $\lambda_3 \doteq 0.40739 + 0.47657i$ and $\lambda_4 = 0.40739 - 0.47657i$ are inside the unit circle,

Corollary

For any node in the tree, the third and fourth eigencoordinates of u and v are bounded.

Inside the Unit Circle - Corollaries

- For three consecutive blocks $x_1 x_2 x_3$, the block difference vectors

$$u = \psi(b_2) - \psi(b_1)$$

$$v = \psi(b_3) - \psi(b_2)$$

  are close to a plane (2-dimensional subspace).

- If $b_1 b_2 b_3$ is an additive cube then we have two linear equations per vector:

$$\big(\psi(b_2) - \psi(b_1)\big) \cdot (1,1,1,1) = 0 \quad \big(\psi(b_3) - \psi(b_2)\big) \cdot (1,1,1,1) = 0$$
$$\big(\psi(b_2) - \psi(b_1)\big) \cdot (0,1,3,4) = 0 \quad \big(\psi(b_3) - \psi(b_2)\big) \cdot (0,1,3,4) = 0$$

  So $u = \psi(b_2) - \psi(b_1)$ and $v = \psi(b_3) - \psi(b_2)$ are bounded.

Bounded endpoints

Theorem

Let $b_1 b_2 b_3$ be an additive cube. Then $\psi(b_2) - \psi(b_1)$ and $\psi(b_3) - \psi(b_2)$ are bounded.

In a path to an additive cube, the first and last nodes have bounded u and v. What happens in the middle of the path?

Outside the Unit Circle ($|\lambda_i| > 1$)

Theorem

Let $\lambda \in \mathbb{C}$ be a complex number such that $|\lambda| > 1$. Suppose $(z_j)_{j=0}^{\infty}$ is a complex sequence such that $z_0 = 0$ and

$$|z_{j+1} - \lambda z_j| \leq B$$

for all j. Then either $|z_j| \leq \frac{B}{\lambda - 1}$ for all j, or the sequence grows exponentially.

Corollary

Suppose x is a node along a path to a (hypothetical) additive cube in $\mathbf{w}$. Then the first and second eigencoordinates of u and v are bounded.

- Along a path to a hypothetical additive cube, all eigencoordinates of u and v are bounded.
- Hence, $u = \psi(b_2) - \psi(b_1)$ and $v = \psi(b_3) - \psi(b_2)$ are bounded, integer vectors.
- The search space is finite. A computer-assisted search for additive cubes finishes the proof.

Recap

- Construct an infinite search tree, $\mathcal{T}^4$, representing all triples of consecutive blocks
- Store a pair of vectors at each node.
- $|\lambda_3|, |\lambda_4| < 1 \implies$ two coordinates of u and v are bounded everywhere.
- At additive cube nodes, two additional equations make u and v bounded.
- $|\lambda_1|, |\lambda_2| > 1 \implies$ the other two coordinates u and v to be bounded on the path.
- Finite computer search.

Open Problems

1. Can we avoid additive squares?
2. Is it possible to avoid additive cubes over a 3 symbol alphabet?
3. Are there "nicer" words avoiding additive cubes?
4. Which subsets of the integers allow us to avoid additive cubes?
5. Suppose we have a coding $h(0) = a$, $h(1) = b$, $h(3) = c$ and $h(4) = d$ to $\mathbf{w}$. For which tuples $(a, b, c, d) \in \mathbb{Z}^4$ does $h(\mathbf{w})$ avoid additive cubes?

Recoding $\mathbf{w}$

Suppose $h\colon \Sigma^* \to \mathbb{Z}^*$ is a morphism where

$$\begin{aligned} h(0) &= a \\ h(1) &= b \\ h(3) &= c \\ h(4) &= d. \end{aligned}$$

Suppose $x_1 x_2$ is a factor in $\mathbf{w}$ with $|x_1| = |x_2|$. Then $\sum h(x_1) = \sum h(x_2)$ if and only if

$$\begin{aligned} \psi(x_1) \cdot (a, b, c, d) &= \psi(x_2) \cdot (a, b, c, d) \\ 0 &= (\psi(x_2) - \psi(x_1)) \cdot (a, b, c, d) \end{aligned}$$

We do not want $\sum h(x_1) = \sum h(x_2)$, so look for (a, b, c, d) *not* orthogonal to $\psi(x_2) - \psi(x_1)$.

Theorem

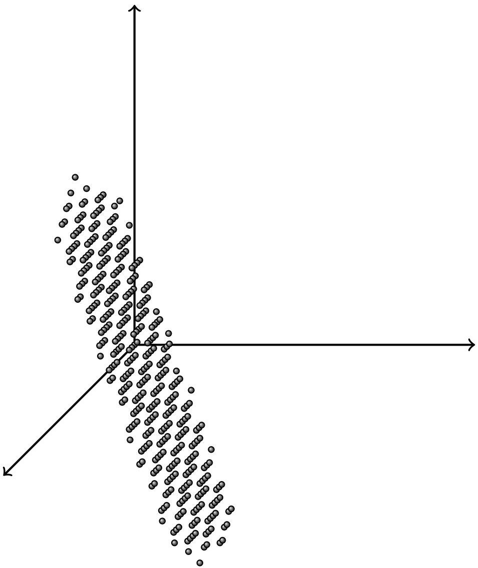
Suppose we have $(a, b, c, d) \in \mathbb{Z}^4$ *such that if* $x_1 x_2$ *is a factor in* $\mathbf{w}$ *with* $|x_1| = |x_2|$, *then* $(\psi(x_2) - \psi(x_1)) \cdot (a, b, c, d) = 0$ *if and only if* $\psi(x_1) = \psi(x_2)$. *Then* $h(\mathbf{w})$ *avoids additive cubes.*

Idea

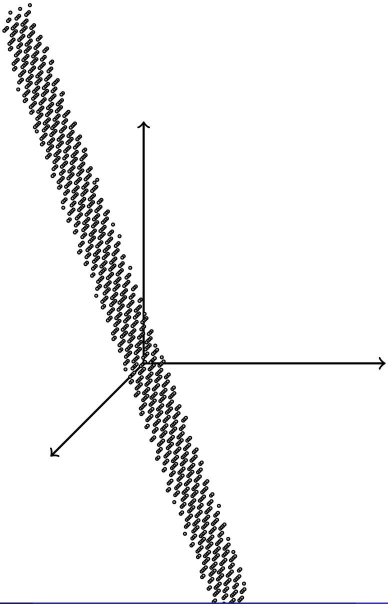
Plot $\psi(x_2) - \psi(x_1)$ for all $x_1 x_2$ in $\mathbf{w}$ such that $|x_1| = |x_2|$.

Note that $|x_1| = |x_2|$ implies $(\psi(x_2) - \psi(x_1)) \cdot (1, 1, 1, 1) = 0$, so there are only three degrees of freedom for us to plot.

Points



More Points



Points and Vector