Traveling wave solutions for a chemotaxis system

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Reaction-Diffusion-Chemotaxis System

$$\begin{cases} \dfrac{\partial U}{\partial T} = \nabla \cdot (D_U \nabla U) - \nabla \cdot (U \cdot \chi(V) \cdot \nabla V) + f(U) \\ \dfrac{\partial V}{\partial T} = \nabla \cdot (D_V \nabla V) + g(U, V) \end{cases} \tag{1}$$

$U(T, X)$ = density of bacteria at time T and location X
$V(T, X)$ = density of attractant substrate

D_U, D_V = diffusion coefficients
$f(U)$, $g(U, V)$ = reaction/kinetic terms
$\chi(V)$ = chemotactic coefficient

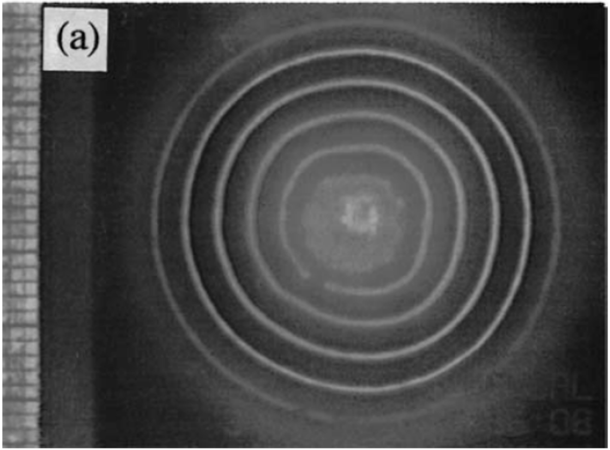

Figure: *Salmonella typhimurium* pattern (from "Mathematical Biology" by J. D. Murray, experiments by Howard Berg and Elena Budrene (1991)).

One Space Dimension

$$\begin{cases} \dfrac{\partial U}{\partial T} = D_U \cdot \dfrac{\partial^2 U}{\partial X^2} - \dfrac{\partial}{\partial X}\left(U \cdot \chi(V) \cdot \dfrac{\partial V}{\partial X} \right) \\ \dfrac{\partial V}{\partial T} = D_V \cdot \dfrac{\partial^2 V}{\partial X^2} - k \cdot U \end{cases} \quad (2)$$

$$\chi(V) = D_U \frac{m - pV^{p-1}}{mV - V^p} \quad (3)$$

for some $p \geq 1$ and $m > p$, $k > 0$

$(T, X) \in (0, \infty) \times \mathbb{R}$. Want U, V positive, and for any $T > 0$ fixed

$$\begin{aligned} &U(T, X) \to 0 \text{ as } |X| \to \infty, \\ &V(T, X) \to 0 \text{ as } X \to -\infty, V(T, X) \to 1 \text{ as } X \to \infty. \end{aligned} \quad (4)$$

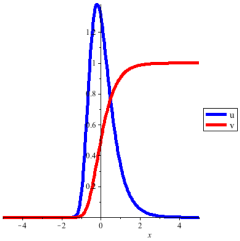

Figure: Typical solution at time T.

With these asymptotic conditions, the mass of bacteria is conserved; i.e. for any $T > 0$,

$$\int_{-\infty}^{\infty} U(T,X)\, dX = \int_{-\infty}^{\infty} U(0,X)\, dX$$

Traveling Wave Solutions of (2)

Solutions of the form

$$U(X,T) = u(\xi) \text{ and } V(X,T) = v(\xi), \text{ where } \xi = X - cT$$

The system (2) becomes

$$\begin{cases} -cu' = \left(D_U \cdot u' - u \cdot \chi(v) \cdot v'\right)' \\ -cv' = D_V \cdot v'' - k \cdot u, \end{cases} \tag{5}$$

with asymptotic conditions

$$\begin{aligned} u(\xi) &\to 0 \text{ as } |\xi| \to \infty, \\ v(\xi) &\to 0 \text{ as } \xi \to -\infty, \\ v(\xi) &\to 1 \text{ as } \xi \to \infty. \end{aligned} \tag{6}$$

Assumptions

$D_U > 0$, $D_V \geq 0$ and $k > 0$ are constants.

Wave Speed

Integrating the second equation in (5)

$$cv' = -D_V \cdot v'' + k \cdot u$$

from $-\infty$ to ∞, because $v(\xi) \to 0$ as $\xi \to -\infty$ and $v(\xi) \to 1$ as $\xi \to \infty$, we get the speed of the wave in terms of the total mass of bacteria

$$c = k \int_{-\infty}^{\infty} u(\xi)\ d\xi$$

Chemotactic Coefficient

$$\chi(v) = D_U \frac{m - pv^{p-1}}{mv - v^p} \quad \text{with} \quad m > p \geq 1 \tag{7}$$

Table: Cases considered

$D_V = 0$	$p = 1$	linear first order explicit solutions
$D_V = 0$	$p > 1$	nonlinear first order explicit solutions
$D_V > 0$	$p = 1$	linear second order explicit solutions
$D_V = 2D_U > 0$	$p > 1$	nonlinear second order explicit solutions
$D_V > 0$	$p > 1$	nonlinear second order fixed point argument

First equation in (5) can be integrated to

$$\frac{u'}{u} = -\frac{c}{D_U} + \frac{\chi(v)}{D_U} \cdot v'$$

Integrating again from 0 to ξ we get

$$\ln(u(\xi)) = \int_0^{\xi} -\frac{c}{D_U} + \frac{\chi(v(\zeta))}{D_U} \cdot v'(\zeta)\ d\zeta + \ln(u(0))$$

$$u(\xi) = u(0) \cdot \exp\left(-\frac{c}{D_U}\xi\right) \cdot \exp\left(\int_0^{\xi} \frac{\chi(v(\zeta))}{D_U} \cdot v'(\zeta)\ d\zeta\right)$$

$$u(\xi) = u(0) \cdot \exp\left(-\frac{c}{D_U}\xi\right) \cdot \exp\left(\int_0^{\xi} \frac{\chi(v(\zeta))}{D_U} \cdot v'(\zeta)\, d\zeta\right) \quad (8)$$

If χ has no singularity at $v = 0$ then

$$\lim_{\xi \to -\infty} \int_0^{\xi} \chi(v(\zeta)) \cdot v'(\zeta)\, d\zeta = \lim_{v(\xi) \to 0} \int_{v(0)}^{v(\xi)} \chi(v)\, dv, \quad \text{finite.}$$

Equation (8) implies $\lim_{\xi \to -\infty} u(\xi) = \infty$ and this violates the requirement $\lim_{\xi \to -\infty} u(\xi) = 0$.

In the Keller and Segel paper the choice of a singular $\chi(v) = \frac{\delta}{v}$ (with $\delta > D_U$) is attributed to the "pervasiveness of the Weber-Fechner Law".

$$\chi(v) = D_U \frac{m - pv^{p-1}}{mv - v^p} \text{ for } p = 1 \text{ becomes } \chi(v) = \frac{D_U}{v}$$

Denote the values of the unknowns at $\xi = 0$ by $u(0) = a > 0$ and $v(0) = b \in (0, 1)$

$$u(\xi) = \frac{a}{mb - b^p}\left(mv(\xi) - v^p(\xi)\right) \cdot \exp\left(-\frac{c\xi}{D_U}\right) \quad (9)$$

$$\text{for } p = 1 \text{ it simplifies to } u(\xi) = \frac{a}{b} v(\xi) \cdot \exp\left(-\frac{c\xi}{D_U}\right) \quad (10)$$

where v is solution of the second equation in (5)

$$D_V v''(\xi) + cv'(\xi) - ka\frac{mv(\xi) - v^p(\xi)}{mb - b^p} \cdot \exp\left(-\frac{c\xi}{D_U}\right) = 0 \quad (11)$$

Under the change of variables

$$t = \exp\left(-\frac{c\xi}{2D_U}\right) \in (0, \infty) \quad \text{and} \quad v(\xi) = f(t) > 0 \qquad (12)$$

Equation (11) becomes

$$D_V f_{tt} + \frac{D_V - 2D_U}{t} f_t - \frac{4akD_U^2}{c^2(mb - b^p)}(mf - f^p) = 0 \qquad (13)$$

with boundary conditions $f(0) = 1$ and $f(\infty) = 0$, and the extra-requirement that $f(1) = b \in (0, 1)$

$$\frac{-2D_U}{t} f_t - \frac{4akD_U^2}{c^2(mb - b^p)}(mf - f^p) = 0 \qquad (14)$$

If $p = 1$, then $v(\xi) = b^{\exp\left(-\frac{c\xi}{D_U}\right)}$, $c = \sqrt{-\frac{akD_U}{b \ln b}}$

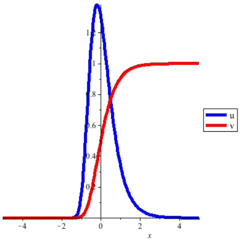

Figure: $D_U = 1,\ D_V = 0,\ p = 1,\ a = 1.3,\ b = 0.5,\ k = 1;\ c = 1.936751689$

If $p > 1$, then $v(\xi) = \left(\dfrac{m}{1+(m-1)\left(\frac{m-b^{p-1}}{b^{p-1}(m-1)}\right)^{\exp\left(-\frac{c\xi}{D_U}\right)}} \right)^{\frac{1}{p-1}}$,

$$c = \left(\frac{akD_U(p-1)}{b\left(1-\frac{b^{p-1}}{m}\right)\ln\left(\frac{m-b^{p-1}}{(m-1)b^{p-1}}\right)} \right)^{\frac{1}{2}}$$

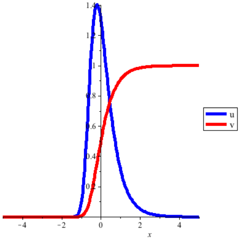

Figure: $D_U = 1,\ D_V = 0,\ p = 2,\ a = 1.3,\ b = 0.5,\ k = 1;\ c = 1.845273097$

If $p = 1$, let $\nu = \frac{D_U}{D_V}$, then

$$v(\xi) = \frac{2}{\Gamma(\nu)} \left(\frac{1}{c} \sqrt{\frac{ak\nu D_U}{b}} \right)^{\nu} \cdot \exp\left(-\frac{c\nu\xi}{2D_U} \right) \cdot K_{\nu}\left(\frac{2}{c} \sqrt{\frac{ak\nu D_U}{b}} \exp\left(-\frac{c\xi}{2D_U} \right) \right)$$

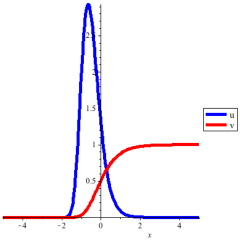

Figure: $D_U = 1,\ D_V = 2,\ p = 1,\ a = 1.3,\ b = 0.5,\ k = 1;\ c = 3.289850865$

If $p > 1$ and $\nu = \frac{D_U}{D_V} = \frac{1}{2}$, then for suitable t_0 and c,

$$v(\xi) = \frac{\left(\frac{m(p+1)}{2}\right)^{\frac{1}{p-1}}}{\cosh^{\frac{2}{p-1}}\left(\frac{p-1}{c}\sqrt{\frac{makD_U}{2(mb-b^p)}}\left(e^{-\frac{c}{2D_U}\xi} - t_0\right)\right)}$$

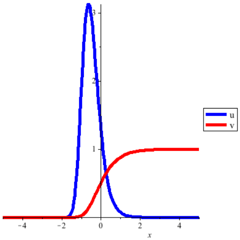

Figure: $D_U = 1$, $D_V = 2$, $p = 2$, $a = 1.3$, $b = 0.5$, $k = 1$, $m = 3$; $t_0 = -3.658207544$, $c = 3.300586405$

Theorem

For any $p > 1$, $D_U > 0$, $D_V > 0$ *let* $\nu = \frac{D_U}{D_V}$. *There exists* $m_0 = m_0(\nu, p) \geq p$, *such that for any* $m > m_0$ *in*

$$\chi(v) = \frac{m - pv^{p-1}}{mv - v^p},$$

the system (5) *with asymptotic conditions* (6), *admits solutions.*

Solutions are obtained by a Fixed Point argument.

Relation to Previous Work

Keller, Evelyn F. ; Segel, Lee A.; *Traveling Bands of Chemotactic Bacteria: A Theoretical Analysis.* Journal of Theoretical Biology (1971) Vol. **30**, 235-248. considered $D_V = 0$ and $\chi(v) = \frac{\delta}{v}$ with $\delta > D_U$.

Nagai, Toshitaka; Ikeda, Tsutomu; *Traveling waves in a chemotactic model.* J. Math. Biol. 30 (1991), no. 2, 169-184. considered $D_V \geq 0$ and $\chi(v) = \frac{\delta}{v}$ with $\delta > D_U$.

Horstmann, D.; Stevens, A.; *A constructive Approach to Traveling Waves in Chemotaxis.* J. Nonlinear Sci. Vol. 14. 1–25 (2005). considered general coefficients so that system (1) admits traveling wave solutions.

THANK YOU FOR YOUR ATTENTION!