Quantum Groups and Free Araki-Woods Factors

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Motivation: Orthogonal Groups and Gaussian Random Variables

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- The goal of this talk is discuss some interesting connections between quantum groups and free probability theory.
- As a motivation, consider the $N \times N$ orthogonal group O_N with its Haar measure dg.
- Denote by

$$v_{ij} : g \in O_N \mapsto g_{ij} \in \mathbb{R}$$

  be the (i,j)-th coordinate function on O_N. Then

$$L_\infty(O_N) = \{v_{ij}\}''_{1 \le i,j \le N} \subset \mathcal{B}(L_2(O_N)).$$

  We will simultaneously think of $\{v_{ij}\}_{i,j}$ as functions and as random variables over (O_N, dg).
- There are two interesting ways in which O_N appears in in connection to independent Gaussian random variables.

Motivation: Orthogonal Groups and Gaussian RVs

1. Rotational Symmetry: Consider a real, i.i.d. $N(0,1)$ Gaussian vector $\mathbf{x} = (x_1, x_2, \ldots, x_N) \subset L_{\infty-}(\Omega, \mu)$. Then the joint distribution of $\mathbf{x}$ and the "randomly rotated vector"

$$\mathbf{y} = (y_1, \ldots, y_N); \quad y_i = \sum_{j=1}^{N} v_{ij} \otimes x_j \in L_\infty(O_N) \otimes L_{\infty-}(\Omega, \mu)$$

are the same:

$$(\iota \otimes \mathbb{E}_\mu)(P(\mathbf{y})) = \mathbb{E}_\mu(P(\mathbf{x})) 1_{L_\infty(O_N)} \quad \forall P \in \mathbb{C}\langle X_1, \ldots, X_N \rangle.$$

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2. Asymptotic Gaussianity in (O_N, dg): Let $\mathbf{X} = \{x_{ij}\}_{i,j \in \mathbb{N}} \subset L_{\infty-}(\Omega, \mu)$ be a real, i.i.d. $N(0,1)$ Gaussian array. Then the rescaled random variables $\sqrt{N} v_{ij} \in L_\infty(O_N)$ satisfy the following convergence result:

$$\{\sqrt{N} v_{ij}\}_{1 \le i,j \le N} \longrightarrow \mathbf{X} \quad \text{in distribution as } N \to \infty. \text{ I.e,}$$

$$\lim_{N \to \infty} \int_{O_N} P(\{\sqrt{N} v_{ij}\}) dg = \mathbb{E}_\mu(P(\mathbf{X})) \qquad (P \in \mathbb{C}\langle X_{ij} : i,j \in \mathbb{N} \rangle).$$

From Classical to Free Probability

- Replace $L_{\infty-}(\Omega, \mu)$ with a vN algebra (M, φ) equipped with a n.f. state φ (a non-commutative probability space).
- Replace classical independence with Voiculescu's free independence with respect to φ. ($\longrightarrow$ free probability theory).
- The free probability analogue of a Gaussian vector $\mathbf{x}$ is a free semicircular system $\mathbf{s} = (s_1, s_2, \ldots, s_N) \subset (M, \varphi)$, determined by $s_i = s_i^*$ and joint distribution

$$\varphi(s_{i(1)} s_{i(2)} \ldots s_{i(k)}) := |NC_2^{i(1),\ldots,i(k)}(k)| \qquad (1 \leq i(r) \leq N).$$

- **Important Fact**: $(W^*(s_1, \ldots, s_N), \varphi) \cong (L(\mathbb{F}_N), \text{trace})$, the free group factor on N generators.

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Question

What do distributional symmetries of free semicircular systems $\mathbf{s} = (s_1, \ldots, s_N)$ look like?

The Distributional Symmetries of $\mathbf{s} = (s_1, \ldots, s_N)$

- Consider a generic "random rotation" of $\mathbf{s} = (s_1, \ldots, s_N)$ given by

$$s_i \mapsto y_i := \sum_{j=1}^{N} u_{ij} \otimes s_j \qquad (1 \leq i \leq N)$$

where $\{u_{ij}\}_{1 \leq i,j \leq N} \subseteq \mathcal{A}$ are the coordinate functions implementing the symmetry and $\mathcal{A}$ is the unital $*$-algebra they generate.

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  where $\{u_{ij}\}_{1 \leq i,j \leq N} \subseteq \mathcal{A}$ are the coordinate functions implementing the symmetry and $\mathcal{A}$ is the unital $*$-algebra they generate.
- If $\mathbf{s} = (s_1, \dots, s_N)$ is invariant under this transformation, then the invariance condition

$$(\iota \otimes \varphi)(P(\mathbf{y})) = \varphi(P(\mathbf{s}))1_{\mathcal{A}} \qquad (P \in \mathbb{C}\langle X_1, \dots, X_N \rangle)$$

  imposes certain relations on the generators u_{ij} of $\mathcal{A}$.
- It turns out that the only relations imposed are
  1. $U := [u_{ij}] \in M_N(\mathcal{A})$ is unitary (R1)
  2. $\overline{U} = U$, where $\overline{U} = [u_{ij}^*]$. (R2).
- These are the same relations as for $\{v_{ij}\} \subset L_\infty(O_N)$ BUT $\{u_{ij}\}$ are not required to commute!

The Quantum Group O_N^+

- This leads us to define a universal (non-commutative) unital C*-algebra
  $C(O_N^+) = C^*\big(\{u_{ij}\}_{1 \le i,j \le N} \;\big|\; U = [u_{ij}] \text{ unitary } \& \overline{U} = U\big)$.
- The algebra $C(O_N^+)$ encodes all the symmetries of a free semicircular system $\mathbf{s} = (s_1, \ldots, s_N)$.

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Theorem (Wang '93)

$C(O_N^+)$ is the C^*-algebra of a compact quantum group - the free orthogonal quantum group O_N^+.

In particular, we have a coproduct

$$\Delta : C(O_N^+) \to C(O_N^+) \otimes C(O_N^+); \quad \Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$$

and a Δ-bi-invariant Haar state

$$h_N : C(O_N^+) \to \mathbb{C}; \quad (\iota \otimes h_N)\Delta = (h_N \otimes \iota)\Delta = h_N(\cdot)1.$$

Note: O_N is a quantum subgroup of O_N^+.

O_N^+ and Free Semicircular Systems

Let $L_\infty(O_N^+) = \{u_{ij}\}''_{1\leq i,j\leq N} \subset \mathcal{B}(L_2(h_N))$. In summary, we obtain the following:

Theorem (Curran '09)

Free semicircular systems are invariant under quantum rotations. In particular, there is a trace-preserving quantum group action $O_N^+ \curvearrowright^\alpha L(\mathbb{F}_N)$ given by a unital injective normal $*$-homomorphism

$$\alpha : L(\mathbb{F}_N) = W^*(s_1, \ldots, s_N) \to L_\infty(O_N^+)\overline{\otimes} L(\mathbb{F}_N); \quad \alpha(s_i) = \sum_j u_{ij} \otimes s_j$$

satisfying $(\iota \otimes \alpha) \circ \alpha = (\Delta \otimes \iota) \circ \alpha$ *and* $(\iota \otimes \varphi) \circ \alpha = \varphi(\cdot)1$.

O_N^+ and Free Semicircular Systems

By replacing O_N with O_N^+, we also obtain a free analogue of the asymptotic Gaussianity result for O_N.

Theorem (Banica-Collins '07, B. '13)

The normalized generators $\{\sqrt{N}u_{ij}\}_{1\leq ij\leq N} \subset (L_\infty(O_N^+), h_N)$ *are (strongly) asymptotically free and semicircular: Let* $\mathbf{S} = \{s_{ij}\}_{i,j\in\mathbb{N}}$ *be a free semicircular array, then for any NC polynomial* P*,*

$$\lim_N h_N\Big(P(\{\sqrt{N}u_{ij}\})\Big) = \varphi(P(\mathbf{S}))$$

$$\textit{and } \lim_N \|P(\{\sqrt{N}u_{ij}\})\|_{L_\infty(O_N^+)} = \|P(\mathbf{S})\|_{L(\mathbb{F}_N)}.$$

Type III Deformations of O_N^+

The QG O_N^+ can be "deformed" to get many more interesting QGs.

Theorem (Van Daele-Wang '95)

For any $F \in GL(N, \mathbb{C})$ such that $F\overline{F} \in \mathbb{C}1$, there exists a compact quantum group O_F^+ with

$$C(O_F^+) = C^*\big(u_{ij}, 1 \leq i, j \leq N \mid U = [u_{ij}] \text{ unitary and } U = F\overline{U}F^{-1}\big),$$

$$\text{and } \Delta(u_{ij}) = \sum_{j=1}^{N} u_{ik} \otimes u_{kj}.$$

Note: In most cases (i.e., $F \notin \mathbb{C}U_N$), the Haar state h_F on O_F^+ is non-tracial ($L_\infty(O_F^+)$ is a type III vN algebra).

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Note: In most cases (i.e., $F \notin \mathbb{C}U_N$), the Haar state h_F on O_F^+ is non-tracial ($L_\infty(O_F^+)$ is a type III vN algebra).

Questions

Do these deformed O_F^+ have any connections with free probability?

- Does free independence appear in the large rank limit?
- Does O_F^+ act on interesting NC probability spaces (M, φ)?

Shlyakhtenko's Free Araki-Woods Factors

The answer is yes to both of these questions! The relevant NC probabilistic objects (M, φ) are given by certain free Araki-Woods factors:

Shlyakhtenko's Free Araki-Woods Factors

The answer is yes to both of these questions! The relevant NC probabilistic objects (M, φ) are given by certain free Araki-Woods factors:

- Fix an orthogonal representation $(U_t)_{t\in\mathbb{R}}$ of $\mathbb{R}$ on a real Hilbert space $H_\mathbb{R}$ ($\dim H_\mathbb{R} \geq 2$).
- Extend U_t to the complexified Hilbert space $H_\mathbb{C}$, and write $U_t = A^{it} \in \mathcal{U}(H_\mathbb{C})$ for some (unbounded) $A > 0$.
- The generator A induces a new inner product

  $$\langle \xi|\eta\rangle_U = \left\langle \frac{2}{1+A^{-1}}\xi \middle| \eta \right\rangle \text{ on } H_\mathbb{C} \text{ with } \|\xi\|_U = \|\xi\| \ \forall \xi \in H_\mathbb{R}.$$

  This yields an isometric embedding $H_\mathbb{R} \hookrightarrow H = \overline{H_\mathbb{C}}^{\|\cdot\|_U}$.
- Consider the full Fock space

  $$\mathcal{F}(H) = \mathbb{C}\Omega \oplus \bigoplus_{n\geq 1} H^{\otimes n}$$

  and the canonical left creation operators

  $$\ell(\xi) \in \mathcal{B}(\mathcal{F}(H)) \qquad (\xi \in H).$$

Shlyakhtenko's Free Araki-Woods Factors

- The free Araki-Woods factor is the von Neumann algebra

$$\Gamma(H_{\mathbb{R}}, U_t)'' = \{\ell(\xi) + \ell(\xi)^* : \xi \in H_{\mathbb{R}}\}'' \subseteq \mathcal{B}(\mathcal{F}(H)).$$

- $\Gamma(H_{\mathbb{R}}, U_t)''$ has a n.f. state $\varphi_\Omega(\cdot) = \langle \cdot\Omega | \Omega \rangle$ - the free quasi-free state.
- φ_Ω is tracial iff $U_t = \mathrm{id}$ for all t.
- $\Gamma(H_{\mathbb{R}}, \mathrm{id}) = L(\mathbb{F}_{\dim H_{\mathbb{R}}})$. In fact, if $(e_i)_i$ is an ONS for $H_{\mathbb{R}}$, then $\mathbf{s} = (\ell(e_i) + \ell(e_i)^*)_{i=1}^N$ is a free semicircular system wrt. φ_Ω.
- For non-trivial U_t, $\Gamma(H_{\mathbb{R}}, U_t)''$ is a full type III_λ factor for some $0 \leq \lambda \leq 1$ (Shlyakhtenko).

O_F^+ and Free Araki-Woods Factors

Theorem (B.-Kirkpatrick '14)

Given any O_F^+ with $\dim F = N$, there exists a free Araki-Woods factor $(\Gamma(\mathbb{R}^N, U_t^F)'', \varphi_\Omega)$ with canonical generators $(c_1, \ldots, c_N)$ and a faithful φ_Ω-preserving action

$$O_F^+ \curvearrowright^{\alpha} \Gamma(\mathbb{R}^N, U_t^F)'' \quad \text{given by} \quad \alpha(c_i) = \sum_j u_{ij} \otimes c_j \quad (1 \leq i \leq N).$$

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Theorem (B.-Kirkpatrick '14)

For any almost periodic representation U_t on $H_\mathbb{R}$, there exists a sequence of quantum groups $\{O_{F(n)}^+\}_{n\geq 1}$ s.t. $((\Gamma(H_\mathbb{R}, U_t)'', \varphi_\Omega)$ arises as the Haar distributional limit of normalized generators of $(L_\infty(O_{F(n)}^+), h_{F(n)})$.

Bonus: When $\dim H_\mathbb{R} < \infty$, we can even take $\dim F(n) =$ constant for all n! (A purely non-unimodular phenomenon).