Boundaries of reduced C*-algebras of discrete groups

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Definition

A discrete group G is *amenable* if there is a left-invariant mean

$$\lambda : \ell^{\infty}(G) \to \mathbb{C},$$

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In this case, λ is a unital positive G-equivariant projection.

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Therefore, G is non-amenable if $\mathbb{C}$ is "too small" to be the range of a unital positive G-equivariant projection on $\ell^\infty(G)$.

Idea

Consider the minimal C*-subalgebra $\mathcal{A}_G$ of $\ell^\infty(G)$ such that there is a unital positive G-equivariant projection

$$P : \ell^\infty(G) \to \mathcal{A}_G.$$

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The size of $\mathcal{A}_G$ should somehow "measure" the non-amenability of G.

Theorem (Kalantar-K 2014)

There is a unique minimal C-algebra $\mathcal{A}_G$ arising as the range of a unital positive G-equivariant projection*

$$P : \ell^\infty(G) \to \mathcal{A}_G.$$

The algebra $\mathcal{A}_G$ is isomorphic to the algebra $C(\partial_F G)$ of continuous functions on the Furstenberg boundary $\partial_F G$ of G.

Motivation

Kirchberg proved that every exact C*-algebra can be embedded into a nuclear C*-algebra.

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In the separable case, Kirchberg and Phillips proved the nuclear C*-algebra can be taken to be the Cuntz algebra on two generators.

Ozawa conjectured the existence of what he calls a “tight” nuclear embedding.

Conjecture (Ozawa 2007)

Let $\mathcal{A}$ be an exact C*-algebra. There is a canonical nuclear C*-algebra $N(\mathcal{A})$ such that

$$\mathcal{A} \subset N(\mathcal{A}) \subset I(\mathcal{A}),$$

where $I(\mathcal{A})$ denotes the injective envelope of $\mathcal{A}$.

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The algebra $\mathcal{N}(\mathcal{A})$ will inherit many properties from $\mathcal{A}$, for example simplicity and primality.

Ozawa proved this conjecture for the reduced C*-algebra of the free group $\mathbb{F}_n$ on $n \geq 2$ generators.

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Theorem (Ozawa 2007)

Let $\mathrm{C}^*_r(\mathbb{F}_n)$ denote the reduced C-algebra of $\mathbb{F}_n$ for $n \geq 2$. There is a canonical nuclear C*-algebra $N(\mathrm{C}^*_r(\mathbb{F}_n))$ such that*

$$\mathrm{C}^*_r(\mathbb{F}_n) \subset N(\mathrm{C}^*_r(\mathbb{F}_n)) \subset I(\mathrm{C}^*_r(\mathbb{F}_n)),$$

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where $I(\mathrm{C}^*_r(\mathbb{F}_n))$ denotes the injective envelope of $\mathrm{C}^*_r(\mathbb{F}_n)$.

Note that $\mathrm{C}^*_r(\mathbb{F}_n)$ is exact since $\mathbb{F}_n$ is an exact group.

Ozawa takes $N(\mathrm{C}_r^*(\mathbb{F}_n)) = C(\partial\mathbb{F}_n) \rtimes_r \mathbb{F}_n$, where $\partial\mathbb{F}_n$ denotes the hyperbolic boundary of $\mathbb{F}_n$.

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Key Proposition (Ozawa 2007)

Let μ be a quasi-invariant doubly ergodic measure on ∂G. If

$$\varphi : C(\partial\mathbb{F}_n) \to L^\infty(\partial G, \mu)$$

is a unital positive $\mathbb{F}_n$-equivariant map, then $\varphi = \mathrm{id}$.

Equivariant Injective Envelopes

An *operator system* is a unital self-adjoint subspace of a C*-algebra.

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A *G-operator system* is an operator system equipped with the action of a group G, i.e. a unital homomorphism from G into the group of order isomorphisms on $\mathcal{S}$.

Let $\mathcal{C}$ be a category consisting of objects and morphisms. An object I is *injective* in $\mathcal{C}$ if, for every pair of objects $E \subset F$ and and every morphism $\varphi : E \to I$, there is an extension $\tilde{\varphi} : F \to I$.

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Theorem (Hamana 1985)

If $\mathcal{S}$ is a G-operator system, then $\mathcal{S}$ has a unique G-injective envelope $I_G(\mathcal{S})$. Every unital completely isometric G-equivariant embedding

$$\varphi : \mathcal{S} \to \mathcal{T},$$

extends to a unital completely isometric G-equivariant embedding

$$\tilde{\varphi} : I_G(\mathcal{S}) \to \mathcal{T}.$$

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Since there is a unital completely isometric G-equivariant embedding of $\mathcal{S}$ into $\ell^\infty(G, \mathcal{S})$ there are unital completely isometric G-equivariant embeddings

$$\mathcal{S} \subset I_G(\mathcal{S}) \subset \ell^\infty(G, \mathcal{S}).$$

Upshot

If $\mathcal{S}$ is an operator system equipped with a G-action, then there are unital completely isometric G-equivariant embeddings

$$\mathcal{S} \subset I_G(\mathcal{S}) \subset \ell^\infty(G, \mathcal{S}),$$

and a unital positive G-equivariant projection $P : \ell^\infty(G, \mathcal{S}) \to I_G(\mathcal{S})$.

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and a unital positive G-equivariant projection $P : \ell^\infty(G, \mathcal{S}) \to I_G(\mathcal{S})$.

The G-injective envelope $I_G(\mathcal{S})$ has a natural C*-algebra structure (induced by the Choi-Effros product).

Corollary

Let G be a discrete group acting trivially on $\mathbb{C}$ and let $I_G(\mathbb{C})$ denote the G-injective envelope of $\mathbb{C}$. Then

$$\mathbb{C} \subset I_G(\mathbb{C}) \subset \ell^\infty(G),$$

and there is a unital positive G-equivariant projection

$$P : \ell^\infty(G) \to I_G(\mathbb{C}).$$

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The G-injective envelope $I_G(\mathbb{C})$ is a commutative C*-algebra equipped with a G-action, so there is a compact G-space space $\partial_H G$ such that $I_G(\mathbb{C}) \simeq C(\partial_H G)$.

We call $\partial_H G$ the *Hamana boundary of G*.

The Furstenberg Boundary

Definition

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2. The G-action on X is *strongly proximal* if, for every probability measure ν on X, the weak*-closure of the G-orbit

$$G\nu = \{s\nu \mid s \in G\}$$

   contains a point mass δ_x for some $x \in X$.

Definition (Furstenberg 1972)

A compact G-space X is a *boundary* if it is minimal and strongly proximal.

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Key Property

If X is a boundary, then for every probability measure ν on X, the weak*-closure of the G-orbit $G\nu$ contains all of X.

Here $x \in X$ is identified with the point mass δ_x on X.

Theorem (Kalantar-K 2014)

The Hamana boundary $\partial_H G$ is a boundary in the sense of Furstenberg.

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Every group G has a unique boundary $\partial_F G$ that is universal, in the sense that every boundary of G is a continuous G-equivariant image of $\partial_F G$.

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Theorem (Kalantar-K 2014)

For a discrete group G, the Hamana boundary $\partial_H G$ can be identified with the Furstenberg boundary $\partial_F G$.

Properties of injective envelopes (injectivity, rigidity and essentiality) imply corresponding results about the Furstenberg boundary.

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Let G be a discrete group and let $\partial_F G$ denote the Furstenberg boundary of G. Then the C-algebra $C(\partial_F G)$ is G-injective. Moreover, we have the following rigidity results:*

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1. *Every unital positive G-equivariant map from $C(\partial_F G)$ is completely isometric.*

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1. *Every unital positive G-equivariant map from $C(\partial_F G)$ is completely isometric.*
2. *The only positive G-equivariant map from $C(\partial_F G)$ to itself is the identity map.*

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2. *The only positive G-equivariant map from $C(\partial_F G)$ to itself is the identity map.*
3. *If M is a minimal G-space, then there is at most one unital G-equivariant map from $C(\partial_F G)$ to $C(M)$, and if such a map exists, then it is a unital injective *-homomorphism.*

Exactness and Nuclear Embeddings

Definition (Kirchberg-Wasserman 1999)

A discrete group G is *exact* if the reduced C*-algebra $\mathrm{C}^*_r(G)$ is exact.

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Applying a result of Anantharaman-Delaroche gives the following corollary.

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Let G be a discrete group. Then G is exact if and only if the G-action on on the Furstenberg boundary $\partial_F G$ is amenable.

Applying a result of Anantharaman-Delaroche gives the following corollary.

Corollary

If G is a discrete exact group, then the reduced crossed product $C(\partial_F G) \rtimes_r G$ is nuclear.

Theorem (Kalantar-K 2014)

Let G be a discrete exact group. Then there is a canonical nuclear C-algebra $N(\mathrm{C}^*_r(G))$ such that*

$$\mathrm{C}^*_r(G) \subset N(\mathrm{C}^*_r(G)) \subset I(\mathrm{C}^*_r(G)),$$

where $I(\mathrm{C}^*_r(G))$ denotes the injective envelope of $\mathrm{C}^*_r(G)$.

Theorem (Kalantar-K 2014)

Let G be a discrete exact group. Then there is a canonical nuclear C-algebra $N(\mathrm{C}_r^*(G))$ such that*

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where $I(\mathrm{C}_r^*(G))$ denotes the injective envelope of $\mathrm{C}_r^*(G)$.

We take

$$N(\mathrm{C}_r^*(G)) = C(\partial_F G) \rtimes_r G.$$

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We take

$$N(\mathrm{C}^*_r(G)) = C(\partial_F G) \rtimes_r G.$$

Note: This is non-separable in general, but can be replaced by a separable nuclear C*-algebra at the expense of no longer being canonical.

C*-Simplicity

Open Problem

Let G be a discrete group. When is G C*-simple, i.e. when is the reduced group C*-algebra $\mathrm{C}^*_r(G)$ simple?

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Day showed in 1957 that every discrete group G has a largest amenable normal subgroup $R_a(G)$ called the *amenable radical of G*. If G is C*-simple, then $R_a(G)$ is necessarily trivial.

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Conjecture (de la Harpe, ?)

The reduced group C*-algebra $\mathrm{C}^*_r(G)$ is simple if and only if the amenable radical $R_a(G)$ is trivial.

Definition

Let G be a discrete group with identity element e. The G-action on a compact G-space X is *topologically free* if, for every $s \in G$, the set

$$X \backslash X^s = \{x \in X \mid sx \neq x\}$$

is dense in X.

The property of the G-action on the Furstenberg boundary $\partial_F G$ being topologically free is an intermediate property between C*-simplicity and triviality of the amenable radical $R_a(G)$.

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Theorem (Kalantar-K 2014)

Let G be a discrete group.

1. *If the G-action on $\partial_F G$ is topologically free, then $R_a(G)$ is trivial.*
2. *If G is exact, and the reduced C*-algebra $\mathrm{C}_r^*(G)$ is simple, then the G-action on $\partial_F G$ is topologically simple.*

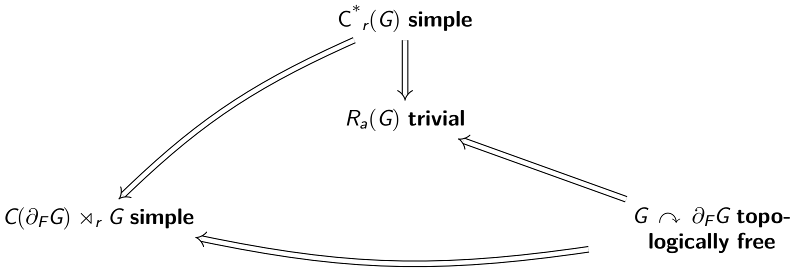

Figure: Implications for an arbitrary discrete group G.

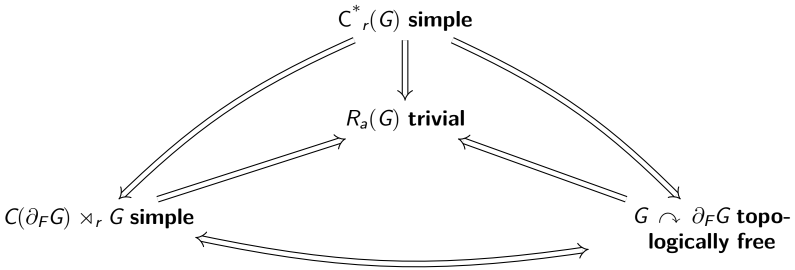

Figure: Implications for a discrete exact group G.

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This answered a question of von Neumann about the existence of non-amenable groups which do not contain non-abelian free groups.

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Theorem (Kalantar-K 2014)

If G is a Tarski monster group, then the G-action on the Furstenberg boundary $\partial_F G$ is topologically free.

Rigidity of Maps

Theorem (Kalantar-K 2014)

Let G be a non-amenable hyperbolic group, and let μ be an irreducible probability measure on G with finite first moment. Let ν be a μ-stationary probability measure on the hyperbolic boundary ∂G. If

$$\varphi : C(\partial G) \to L^\infty(\partial G, \nu)$$

is a unital positive G-equivariant map, then $\varphi = \mathrm{id}$.

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is a unital positive G-equivariant map, then $\varphi = \mathrm{id}$.

We apply Jaworski's theory of strongly approximately transitive measures, combined with a uniqueness result of Kaimanovich for stationary measures.

Corollary

Let G be as above, and let $\partial_F G$ denote the Furstenberg boundary of G. Then

$$I_G(C(\partial G)) = C(\partial_F G),$$

where $I_G(C(\partial G))$ denotes the G-injective envelope of $C(\partial G)$.

Corollary

Let G be as above, and let $\partial_F G$ denote the Furstenberg boundary of G. Then

$$I_G(C(\partial G)) = C(\partial_F G),$$

where $I_G(C(\partial G))$ denotes the G-injective envelope of $C(\partial G)$.

The Furstenberg boundary $\partial_F G$ can be thought of as a "projective cover" of the hyperbolic boundary ∂G.

Quantum Groups

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Many of our results hold in this setting. We intend to pursue this further...

Thanks!