The Spectral Geometry of Curved Noncommutative Tori

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Spectral Triples (Connes)

- Noncommutative geometric spaces are described by *spectral triples* (first order elliptic PDE's on NC spaces), $(\mathcal{A}, \mathcal{H}, D)$, where

$$\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H}) \qquad (\ast\text{-representation}),$$
$$D = D^* : Dom(D) \subset \mathcal{H} \to \mathcal{H}, \quad \text{s.a.}$$
$$D\,\pi(a) - \pi(a)\,D \in \mathcal{L}(\mathcal{H}), \quad \text{bounded commutators},$$
$$D \text{ has compact reseolvant.}$$

- Example: The Dirac spectral triple $\big(C^\infty(M), L^2(M,S), D\big)$, e.g. $D = \frac{1}{i}\frac{d}{dx}$, or the Cauchy-Riemann operator $\frac{\partial}{\partial \bar{z}}$.

The scalar curvature of a spectral triple

- *Connes' distance formula* recovers the metric from D, but a more difficult issue is how to define and compute the scalar curvature using D.
- A spectral triple is a NC Riemannian manifold. It is tempting to think that one might be able to define a Levi-Civita type connection for a spectral triple and then define the curvature of this connection. For many reasons this algebraic approach does not work in NCG in general.
- Instead one needs to import ideas of spectral geometry to NCG.

Spectral geometry: can one hear the shape of a drum?

- Weyl's law: for a compact Riemannian manifold M

$$N(\lambda) \sim \frac{\omega_n \mathsf{Vol}(M)}{(2\pi)^n} \lambda^{\frac{n}{2}} \qquad \lambda \to \infty,$$

where $N(\lambda) = \#\{\lambda_i \leq \lambda\}$ is the eigenvalue counting function for the Laplacian Δ on M.

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- A better way to think of Weyl's law: quantize the classical Hamiltonian $h(x,p) = \frac{p^2}{2m} + V(x)$, to the quantum Hamiltonian $H = -\frac{\hbar^2}{2m}\Delta + V(x)$. Then

$$N(a \leq \lambda \leq b) = \frac{1}{(2\pi\hbar)^d} \mathsf{Vol}\left\{a \leq h \leq b\right\} + o(\hbar^{-d})$$

(Physics proof: by Heisenberg unceratinly relation, each quantum state occupies a volume of $\sim (2\pi\hbar)^d$ in phase space. quantized energy levels are approximated by phase space volumes; Bohr's correspondence principle; semiclassical approximation)

- Weyl's law: One can hear the volume and dimension of a manifold. We shall see one can hear the volume and scalar curvature of curved noncommutative tori too.

Beyond Weyl's law

- (M,g) = closed Riemannian manifold. Laplacian on forms

$$\triangle = (d + d^*)^2 : \Omega^p(M) \to \Omega^p(M),$$

has pure point spectrum:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

- Fact: Dimension, volume, total scalar curvature, Betti numbers, and hence the Euler characteristic of M are fully determined by the spectrum of Δ (on all p-forms).

Heat trace asymptotics

- $N(\lambda) = \operatorname{Tr} P_\lambda$ is too brutal. Mollify it by a smoothing operator like $\operatorname{Tr}(e^{-t\Delta})$ and use Tauberian theorems to obtain information about $N(\lambda)$.

- $k(t, x, y) =$ kernel of $e^{-t\triangle}$. Asymptotic expansion near $t = 0$:

$$k(t, x, x) \sim \frac{1}{(4\pi t)^{m/2}}(a_0(x, \triangle)+a_1(x, \triangle)t+a_2(x, \Delta)t^2+\cdots)$$

- $a_i(x, \triangle)$, Seeley-De Witt-Gilkey coefficients.

- Theorem: $a_i(x, \triangle)$ are universal polynomials in the curvature tensor $R = R^1_{jkl}$ and its covariant derivatives:

$$\begin{aligned} a_0(x, \triangle) &= 1 && \text{Weyl's law} \\ a_1(x, \triangle) &= \frac{1}{6} S(x) && \text{scalar curvature} \\ a_2(x, \triangle) &= \frac{1}{360}(2|R(x)|^2 - 2|\mathrm{Ric}(x)|^2 + 5|S(x)|^2) \\ a_3(x, \triangle) &= \cdots\cdots \end{aligned}$$

Noncommutative Local Invariants

- Local geometric invariants such as scalar curvature of $(A, \mathcal{H}, D)$ are detected by the high frequency behavior of the spectrum of D and the action of A via heat kernel asymptotic expansions of the form

$$\text{Trace}\left(a\, e^{-tD^2}\right) \sim \sum_{j=0}^{\infty} a_j(a, D^2)\, t^{(-n+j)/2}, \quad t \searrow 0, \qquad a \in A.$$

Example: Gauss-Bonnet

- For surfaces

$$\chi(\Sigma) = \frac{1}{2\pi} \int_\Sigma K dA$$

- Spectral zeta function: Let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ be the eigenvalues of $\triangle$,
  and

$$\zeta_\triangle(s) = \sum \lambda_j^{-s}, \qquad \Re(s) > 1.$$

  It has a mermorphic extension to $\mathbb{C}$ with a simple pole at $s = \frac{1}{2}$. G-B is equivalent to

$$\zeta_\triangle(s) + 1 = 0$$

Curved noncommutative tori

- A_θ: universal C^*-algebra generated by unitaries U and V

$$VU = e^{2\pi i\theta} UV.$$

- Smooth structure:

$$A_\theta^\infty = \Big\{ \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n : \quad (a_{m,n}) \in \mathcal{S}(\mathbb{Z}^2) \Big\}.$$

- Derivations $\delta_1, \delta_2 : A_\theta^\infty \to A_\theta^\infty$

$$\delta_1(U) = U, \quad \delta_1(V) = 0, \quad \delta_2(U) = 0, \quad \delta_2(V) = V,$$

- Canonical trace $\varphi_0 : A_\theta \to \mathbb{C}$

Complex structure on A_θ

- Fix $\tau = \tau_1 + i\tau_2, \quad \tau_2 = \Im(\tau) > 0$, and define the Dolbeault operators

$$\partial := \delta_1 + \tau\delta_2, \qquad \partial^* := \delta_1 + \bar{\tau}\delta_2.$$

- Let $\mathcal{H}_0 = L^2(A_\theta)$= GNS completion of A_θ w.r.t. φ_0.

- $\mathcal{H}^{(1,0)}$ = Hilbert space of $(1,0)$-forms: completion of finite sums $\sum a\partial b$, $a, b \in A_\theta^\infty$, under

$$\langle a\partial b, a'\partial b'\rangle := \varphi_0((a'\partial b')^* a\partial b).$$

- ∂^* is the formal adjoint of $\partial : \mathcal{H}_0 \to \mathcal{H}^{(1,0)}$.

- Flat Dolbeault Laplacian:

$$\triangle = \partial^* \partial = \delta_1^2 + 2\tau_1 \delta_1 \delta_2 + |\tau|^2 \delta_2^2.$$

For $\tau = i$, we get

$$\triangle = \delta_1^2 + \delta_2^2.$$

Conformal perturbation of the metric

- Fix a Weyl factor: $h = h^* \in A_\theta^\infty$. Replace φ_0 by

$$\varphi(a) = \varphi_0(a\, e^{-h}).$$

- φ is a KMS state

$$\varphi(a\, b) = \varphi\big(b\, \Delta(a)\big),$$

with modular automorphism

$$\Delta(a) = \sigma_i(a) = e^{-h}\, a\, e^{h},$$

and modular group

$$\sigma_t(a) = e^{ith}\, a\, e^{-ith}.$$

- Warning: $\triangle$ and Δ are very different operators!

Curved Laplacian

- Hilbert space $\mathcal{H}_\varphi = GNS$ completion of A_θ under
$$\varphi(a) = \varphi_0(a\, e^{-h}).$$
- Let $\partial_\varphi = \delta_1 + \tau\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}$. It has an adjoint
$$\partial_\varphi^* = R_{k^2}\partial^* : \mathcal{H}^{(1,0)} \to \mathcal{H}_\varphi.$$
- Curved Laplacian
$$\triangle' = \partial_\varphi^* \partial_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi.$$

A Spectral Triple $(A_\theta^\infty, \mathcal{H}, D)$

$$\mathcal{H} := \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)},$$

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : \mathcal{H} \to \mathcal{H},$$

$$D := \begin{pmatrix} 0 & \partial_\varphi^* \\ \partial_\varphi & 0 \end{pmatrix} : \mathcal{H} \to \mathcal{H},$$

$$\partial_\varphi := \partial = \delta_1 + \bar{\tau}\delta_2 : \mathcal{H}_\varphi \to \mathcal{H}^{(1,0)}.$$

Anti-Unitary Equivalence of the Laplacians

$$D^2 = \begin{pmatrix} \partial_\varphi^* \partial_\varphi & 0 \\ 0 & \partial_\varphi \partial_\varphi^* \end{pmatrix} : \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)} \to \mathcal{H}_\varphi \oplus \mathcal{H}^{(1,0)}.$$

Lemma: Let

$$k = e^{h/2}.$$

We have

$$\begin{aligned} \partial_\varphi^* \partial_\varphi : \mathcal{H}_\varphi \to \mathcal{H}_\varphi \qquad &\sim \qquad k \bar{\partial} \partial k : \mathcal{H}_0 \to \mathcal{H}_0, \\ \partial_\varphi \partial_\varphi^* : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)} \qquad &\sim \qquad \bar{\partial} k^2 \partial : \mathcal{H}^{(1,0)} \to \mathcal{H}^{(1,0)}. \end{aligned}$$

(The Tomita anti-unitary map J is used.)

Conformal Geometry of $\mathbb{T}^2_\theta$ with $\tau = i$ (Cohen-Connes, late 80's)

Let

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \qquad \text{be the eigenvalues of } \partial_\varphi^* \partial_\varphi,$$

and

$$\zeta(s) = \sum \lambda_j^{-s}, \qquad \Re(s) > 1.$$

Then

$$\zeta(0) + 1 =$$

$$\varphi\big(f(\Delta)(\delta_1(e^{h/2}))\,\delta_1(e^{h/2})\big) + \varphi\big(f(\Delta)(\delta_2(e^{h/2}))\,\delta_2(e^{h/2})\big),$$

where

$$f(u) = \frac{1}{6}u^{-1/2} - \frac{1}{3} + \mathcal{L}_1(u) - 2(1+u^{1/2})\mathcal{L}_2(u) + (1+u^{1/2})^2\mathcal{L}_3(u),$$

$$\mathcal{L}_m(u) = (-1)^m (u-1)^{-(m+1)}\Big(\log u - \sum_{j=1}^{m} (-1)^{j+1}\frac{(u-1)^j}{j}\Big).$$

The Gauss-Bonnet theorem for $\mathbb{T}_\theta^2$

Theorem. (Connes-Tretkoff; Fathizadeh-Kh.) For any $\theta \in \mathbb{R}$, complex parameter $\tau \in \mathbb{C} \setminus \mathbb{R}$ and Weyl conformal factor $e^h, h = h^* \in A_\theta^\infty$, we have

$$\zeta(0) + 1 = 0.$$

Final Part of the Proof

$$\zeta(0) + 1 =$$

$$\frac{2\pi}{\Im(\tau)}\varphi_0\Big(K(\nabla)(\delta_1(\frac{h}{2}))\,\delta_1(\frac{h}{2})\Big) + \frac{2\pi|\tau|^2}{\Im(\tau)}\varphi_0\Big(K(\nabla)(\delta_2(\frac{h}{2}))\,\delta_2(\frac{h}{2})\Big)$$

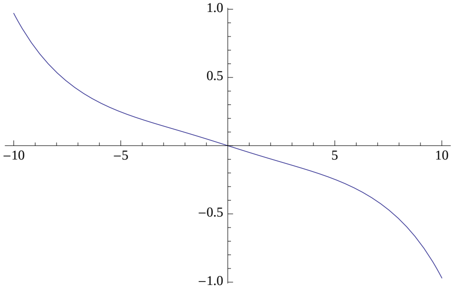
$$+\frac{2\pi\Re(\tau)}{\Im(\tau)}\varphi_0\Big(K(\nabla)(\delta_1(\frac{h}{2}))\,\delta_2(\frac{h}{2})\Big)+\frac{2\pi\Re(\tau)}{\Im(\tau)}\varphi_0\Big(K(\nabla)(\delta_2(\frac{h}{2}))\,\delta_1(\frac{h}{2})\Big),$$

where

$$K(x) = -\frac{\left(3x - 3\sinh\left(\frac{x}{2}\right) - 3\sinh(x) + \sinh\left(\frac{3x}{2}\right)\right)\mathsf{csch}^2\left(\frac{x}{2}\right)}{3x^2}$$

is an odd entire function, and $\nabla = \log\Delta$.

$$K(x) = -\frac{x}{20} + \frac{x^3}{2240} - \frac{23x^5}{806400} + O\left(x^6\right).$$



Scalar curvature for A_θ

- The scalar curvature of the curved nc torus $(\mathbb{T}^2_\theta, \tau, k)$ is the unique element $R \in A_\theta^\infty$ satisfying

$$\text{Trace}\,(a\triangle^{-s})_{|_{s=0}} + \text{Trace}\,(aP) = \mathfrak{t}\,(aR), \qquad \forall a \in A_\theta^\infty,$$

where P is the projection onto the kernel of $\triangle$.

- In practice this is done by finding an asymptotic expansin for the kernel of the operator $ae^{-t\triangle}$,

$$\text{Trace}(a\,e^{-tD^2}) \sim \sum_{n\geq 0} B_n(a, D^2)\,t^{\frac{n-2}{2}}, \qquad a \in A_\theta^\infty.$$

using Connes' pseudodifferential calculus for nc tori. A good pseudo diff calculus for general nc spaces is still illusive.

Final Formula for the Scalar Curvature of $\mathbb{T}_\theta^2$

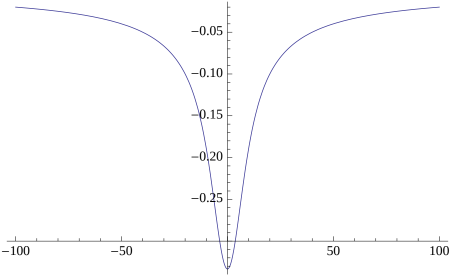
Theorem. (Connes-Moscovici; Fathizadeh-Kh.) Up to an overall factor of $\frac{-\pi}{\Im(\tau)}$, R is equal to

$$R_1(\nabla)\big(\delta_1^2(\frac{h}{2}) + 2\,\tau_1\,\delta_1\delta_2(\frac{h}{2}) + |\tau|^2\,\delta_2^2(\frac{h}{2})\big)$$

$$+R_2(\nabla,\nabla)\Big(\delta_1(\frac{h}{2})^2 + |\tau|^2\,\delta_2(\frac{h}{2})^2 + \Re(\tau)\,\big\{\delta_1(\frac{h}{2}),\delta_2(\frac{h}{2})\big\}\Big)$$

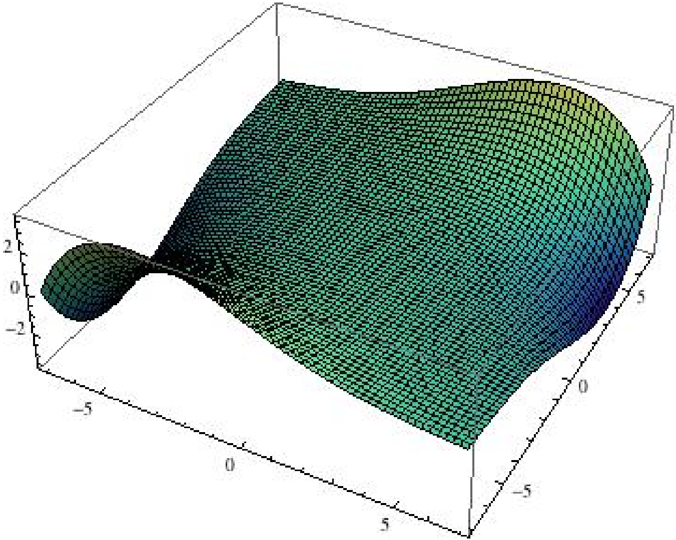
$$+i\,W(\nabla,\nabla)\Big(\Im(\tau)\,[\delta_1(\frac{h}{2}),\delta_2(\frac{h}{2})]\Big).$$

$$R_1(x) = \frac{\frac{1}{2} - \frac{\sinh(x/2)}{x}}{\sinh^2(x/4)}.$$



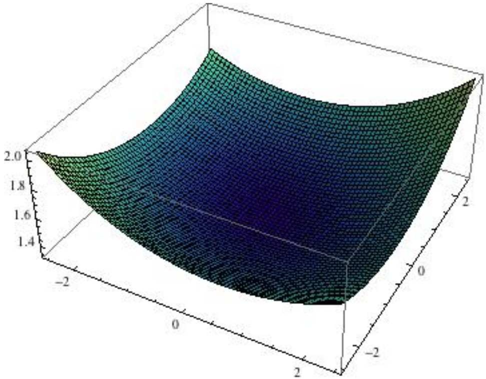
$R_2(s,t) =$

$$-\frac{(1+\cosh((s+t)/2))(-t(s+t)\cosh s+s(s+t)\cosh t-(s-t)(s+t+\sinh s+\sinh t-\sinh(s+t)}{st(s+t)\sinh(s/2)\sinh(t/2)\sinh^2((s+t)/2)}$$



$$W(s,t) =$$

$$\frac{(-s - t + t\cosh s + s\cosh t + \sinh s + \sinh t - \sinh(s+t))}{st\sinh(s/2)\sinh(t/2)\sinh((s+t)/2)}.$$



Noncommutative 4-Torus $\mathbb{T}^4_\theta$

- Complex Structure on $\mathbb{T}^4_\theta$

$$\begin{aligned}
\partial = \partial_1 \oplus \partial_2, &\qquad \bar{\partial} = \bar{\partial}_1 \oplus \bar{\partial}_2, \\
\partial_1 = \frac{1}{2}\,(\delta_1 - i\delta_3), &\qquad \partial_2 = \frac{1}{2}\,(\delta_2 - i\delta_4), \\
\bar{\partial}_1 = \frac{1}{2}\,(\delta_1 + i\delta_3), &\qquad \bar{\partial}_2 = \frac{1}{2}\,(\delta_2 + i\delta_4).
\end{aligned}$$

Conformal perturbation of the metric

Let $h = h^* \in C^\infty(\mathbb{T}^4_\theta)$ and replace the trace φ_0 by

$$\varphi : C(\mathbb{T}^4_\theta) \to \mathbb{C},$$

$$\varphi(a) := \varphi_0(a\, e^{-2h}), \qquad a \in C(\mathbb{T}^4_\theta).$$

φ is a KMS state with the modular group

$$\sigma_t(a) = e^{2ith}\, a\, e^{-2ith}, \qquad a \in C(\mathbb{T}^4_\theta),$$

and the modular automorphism

$$\Delta(a) := \sigma_i(a) = e^{-2h}\, a\, e^{2h}, \qquad a \in C(\mathbb{T}^4_\theta).$$

$$\varphi(a\, b) = \varphi\big(b\, \Delta(a)\big), \qquad a, b \in C(\mathbb{T}^4_\theta).$$

Perturbed Laplacian on $\mathbb{T}_\theta^4$

$$d = \partial \oplus \bar{\partial} : \mathcal{H}_\varphi \to \mathcal{H}_\varphi^{(1,0)} \oplus \mathcal{H}_\varphi^{(0,1)},$$

$$\triangle_\varphi := d^* d.$$

Remark. If $h = 0$ then $\varphi = \varphi_0$ and

$$\triangle_{\varphi_0} = \delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2 = \partial^* \partial$$

(the underlying manifold is Kähler).

Scalar Curvature for $\mathbb{T}_\theta^4$

It is the unique element $R \in C^\infty(\mathbb{T}_\theta^4)$ such that

$$\mathsf{Res}_{s=1}\, \zeta_a(s) = \varphi_0(a\, R), \qquad a \in C^\infty(\mathbb{T}_\theta^4),$$

where

$$\zeta_a(s) := \mathsf{Trace}(a\, \triangle_\varphi^{-s}), \qquad \Re(s) \gg 0.$$

Final Formula for the Scalar Curvature of $\mathbb{T}_\theta^4$

Theorem. (Fathizadeh-Kh.) We have

$$R = e^{-h}\, k(\nabla)\Big(\sum_{i=1}^{4} \delta_i^2(h)\Big) + e^{-h}\, H(\nabla,\nabla)\Big(\sum_{i=1}^{4} \delta_i(h)^2\Big),$$

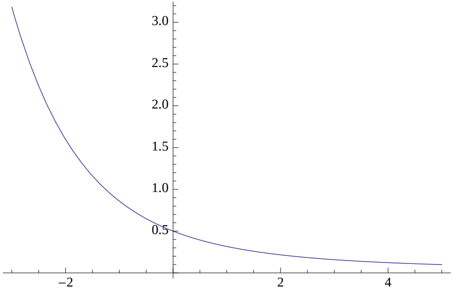
where

$$\nabla(a) = [-h, a], \qquad a \in C(\mathbb{T}_\theta^4),$$

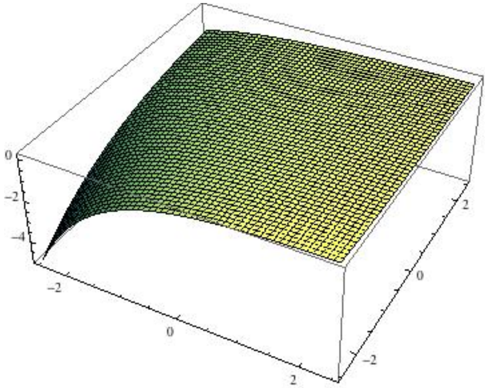
$$k(s) = \frac{1-e^{-s}}{2s},$$

$$H(s,t) = -\frac{e^{-s-t}\left(\left(-e^{s}-3\right) s\left(e^{t}-1\right)+\left(e^{s}-1\right)\left(3 e^{t}+1\right) t\right)}{4\, s\, t\,(s+t)}.$$

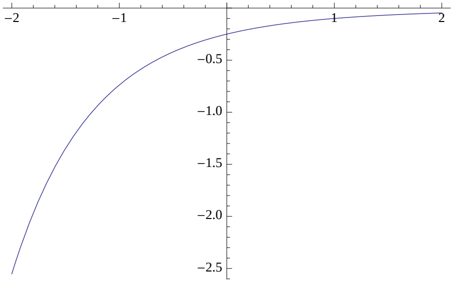
$$k(s) = \frac{1}{2} - \frac{s}{4} + \frac{s^2}{12} - \frac{s^3}{48} + \frac{s^4}{240} - \frac{s^5}{1440} + O\left(s^6\right).$$



$$
\begin{aligned}
H(s,t) \quad = \quad & \left(-\frac{1}{4}+\frac{t}{24}+O\left(t^3\right)\right)+s\left(\frac{5}{24}-\frac{t}{16}+\frac{t^2}{80}+O\left(t^3\right)\right) \\
& +s^2\left(-\frac{1}{12}+\frac{7t}{240}-\frac{t^2}{144}+O\left(t^3\right)\right)+O\left(s^3\right).
\end{aligned}
$$



$$\begin{aligned} H(s,s) &= -\frac{e^{-2s}\left(e^s-1\right)^2}{4s^2} \\ &= -\frac{1}{4}+\frac{s}{4}-\frac{7s^2}{48}+\frac{s^3}{16}-\frac{31s^4}{1440}+\frac{s^5}{160}+O\left(s^6\right). \end{aligned}$$



$$
\begin{aligned}
G(s) \quad := \quad H(s,-s) \quad &= \quad \frac{-4s - 3e^{-s} + e^{s} + 2}{4s^2} \\
&= \quad -\frac{1}{4} + \frac{s}{6} - \frac{s^2}{48} + \frac{s^3}{120} - \frac{s^4}{1440} + \frac{s^5}{5040} + O\left(s^6\right).
\end{aligned}
$$