Smale spaces, their C^*-algebras and a homology theory for them

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- to describe certain hyperbolic dynamical systems called Smale spaces

- to describe C^*-algebras constructed from them

- to find algebraic invariants for them, and show how the C^*-algebras provided key ideas in their construction

Smale spaces (D. Ruelle)

(X, d) compact metric space,

$\varphi : X \to X$ homeomorphism $0 < \lambda < 1$,

For x in X and $\epsilon > 0$ and small, there is a local stable set $X^s(x, \epsilon)$ and a local unstable set $X^u(x, \epsilon)$ which satisfy:

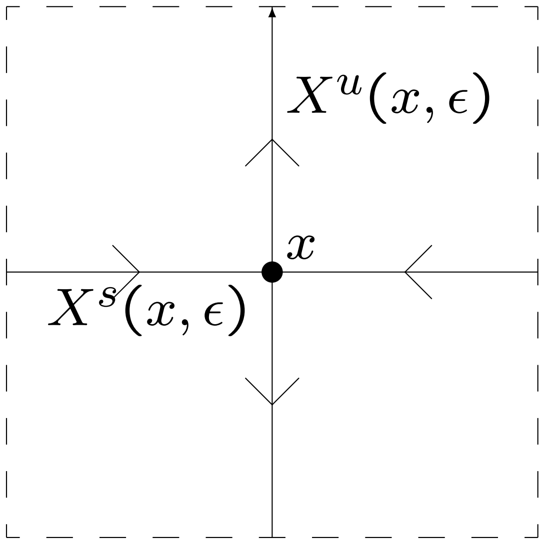
1. $X^s(x, \epsilon) \times X^u(x, \epsilon)$ is homeomorphic to a neighbourhood of x,

2. φ-invariance,

3.

$$\begin{aligned} d(\varphi(y), \varphi(z)) &\leq \lambda d(y, z), \quad y, z \in X^s(x, \epsilon), \\ d(\varphi^{-1}(y), \varphi^{-1}(z)) &\leq \lambda d(y, z), \quad y, z \in X^u(x, \epsilon), \end{aligned}$$

That is, we have a local picture:



Global stable and unstable sets:

$$X^s(x) = \{y \mid \lim_{n \to +\infty} d(\varphi^n(x), \varphi^n(y)) = 0\}$$
$$X^u(x) = \{y \mid \lim_{n \to +\infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0\}$$

These are equivalence relations and

$X^s(x,\epsilon) \subset X^s(x)$,
$X^u(x,\epsilon) \subset X^u(x)$.

Example 1(from linear algebra)

The linear map

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$$

is hyperbolic. Let $\gamma > 1$ be the golden mean,

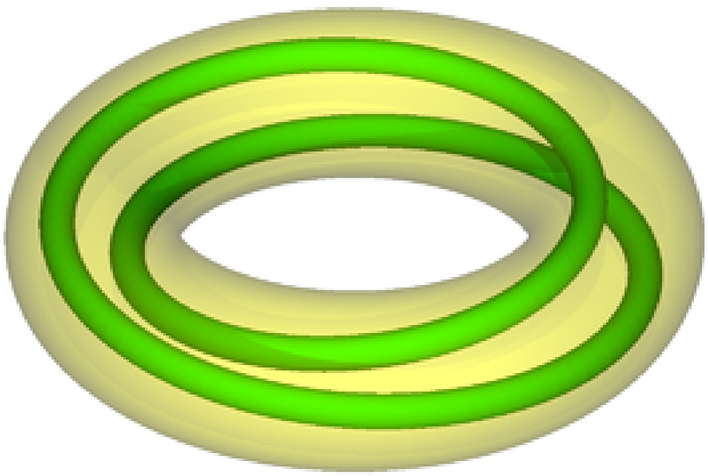
$$\begin{aligned} (\gamma, 1)A &= \gamma(\gamma, 1) \\ (-1, \gamma)A &= -\gamma^{-1}(-1, \gamma) \end{aligned}$$

Of course, $\mathbb{R}^2$ is not compact, but letting $X = \mathbb{R}^2/\mathbb{Z}^2$, as $det(A) = -1$, A induces a map with the same local structure, but is a Smale space.

X^s and X^u are Kronecker foliations with lines of slope $-\gamma^{-1}$ and γ.

Example 2 (from topology)

Let $X_0 = \overline{\mathbb{D}} \times \mathbb{S}^1$, be the solid torus and define $\varphi_0 : X_0 \to X_0$ with image as shown:



It is not onto, but if we let

$$X = \cap_{n \geq 1} \varphi_0^n(X_0) \quad \varphi = \varphi_0|_X,$$

then (X, φ) is a Smale space. The unstable set is the $\mathbb{S}^1$ coordinate, while the stable set is a totally disconnected subset of $\overline{\mathbb{D}}$.

Example 3 (from number theory)

For a prime p, $\mathbb{Q}_p$ is the p-adic numbers. It is a field and a metric space which is the completion of $\mathbb{Q}$. It is totally disconnected. Multiplication by p contracts by a factor p^{-1}, while multiplication by any integer relatively prime to p is an isometry.

Let $p < q$ be primes. On $\mathbb{Q}_p \times \mathbb{R} \times \mathbb{Q}_q$, define

$$\varphi(x, y, z) = (p^{-1}qx, p^{-1}qy, p^{-1}qz).$$

It expands the first factor and the second ($p < q$), but contracts the third.

But the space is not compact. However,

$$X = \mathbb{Q}_p \times \mathbb{R} \times \mathbb{Q}_q / \mathbb{Z}[1/pq]$$

is, φ induces a homeomorphism which has the same local structure.

Example 4: Shifts of finite type (SFTs)

Let $G = (G^0, G^1, i, t)$ be a finite directed graph. Then we have the shift space of bi-infinite paths and shift map:

$$\begin{aligned} \Sigma_G &= \{(e^k)_{k=-\infty}^{\infty} \mid e^k \in G^1, \\ & \qquad i(e^{k+1}) = t(e^k), \text{ for all } n\} \\ \sigma(e)^k &= e^{k+1}, \text{ "left shift"} \end{aligned}$$

The metric $d(e, f) = 2^{-k}$, where $k \geq 0$ is the least integer where $(e^{-k}, e^k) \neq (f^{-k}, f^k)$.

The local stable and unstable sets at some point e are:

$$\Sigma^s(e, 1) = \{(\ldots, *, *, *, e^0, e^1, e^2, \ldots)\}$$
$$\Sigma^u(e, 1) = \{(\ldots, e^{-2}, e^{-1}, e^0, *, *, *, \ldots)\}$$

Note that Σ_G is totally disconnected; if fact, these are precisely the totally disconnected Smale spaces.

C^*-algebra: $C^*(X^s)$

For C^*-algebras of equivalence relations, it is nice if we can find an abstract transversal, as in Muhly, Renault, Williams.

Space:

$$X^u(\mathcal{O}(x)) = \cup_{n\in\mathbb{Z}} X^u(\varphi^n(x)),$$

(Caution: in a new topology, not the relative topology!)

Equivalence relation:

$$X^u(\mathcal{O}(x))^s = X^s \cap (X^u(\mathcal{O}(x)) \times X^u(\mathcal{O}(x)))$$

This is an étale equivalence relation and we consider $S(X,\varphi,x) = C^*(X^u(\mathcal{O}(x))^s)$.

Alternately, we could study
$U(X,\varphi,x) = C^*(X^s(\mathcal{O}(x))^u)$.

Up to Morita equivalence these are independent of the choice of x.

Our original map φ induces a homeomorphism of the space $X^u(\mathcal{O}(x))$ and an automorphism of $X^u(\mathcal{O}(x))^s$ and hence automorphisms of $S(X, \varphi, x)$, as well as $U(X, \varphi, x)$. We can also look at

$$S(X, \varphi, x) \times_\varphi \mathbb{Z}, \qquad U(X, \varphi, x) \times_\varphi \mathbb{Z}.$$

Case 1: Shifts of finite type (Krieger)

$S(X, \varphi, x)$ is an AF-algebra.

$$K_0(S(\Sigma_G, \sigma, x)) \cong \lim \mathbb{Z}^N \xrightarrow{A_G^T} \mathbb{Z}^N \xrightarrow{A_G^T} \cdots$$

where A_G is the adjacency matrix of the graph G.

The same for $U(X, \varphi, x)$ (change A_G^T to A_G).

Moreover, we have

$$S(\Sigma_G, \sigma, x) \times_\varphi \mathbb{Z} \cong \mathcal{O}_{A_G^T} \otimes \mathcal{K}.$$

Case 2: One-dimensional solenoids

Klaus Thomsen : C^*-algebras fall under Elliott's classification program. (Torsion can occur in K-theory!)

Case 3: General properties

P-Spielberg : amenability, simplicity, purely infinite, etc.

Back to dynamics...

Smale spaces have a large supply of periodic points and it is interesting to count them.

Theorem 1. *Let A_G be the adjacency matrix of the graph G. For any $p \geq 1$, we have*

$$\#\{e \in \Sigma_G \mid \sigma^p(e) = e\} = Tr(A_G^p).$$

This is reminiscent of the Lefschetz fixed-point formula for smooth maps of compact manifolds.

Question 2 (Bowen)**.** *Is the right hand side actually the result of σ acting on some homology theory of (Σ_G, σ)? Is there a more general version of the theory for Smale spaces?*

Krieger: $K_0(S(\Sigma, \sigma, x))$ or $K_0(U(\Sigma, \sigma, x))$, which we will now denote by $D^u(\Sigma, \sigma)$ and $D^s(\Sigma, \sigma)$, respectively.

Bowen's Theorem

Theorem 3 (Bowen)**.** *For a non-wandering Smale space, (X, φ), there exists a SFT (Σ, σ) and*

$$\pi : (\Sigma, \sigma) \to (X, \varphi),$$

with $\pi \circ \sigma = \varphi \circ \pi$, continuous, surjective and finite-to-one.

Problem Does a map $\pi : (Y, \psi) \to (X, \varphi)$ induce a $*$-homomorphism between the C^*-algebras?

A map $\pi : (Y, \psi) \to (X, \varphi)$ map between Smale spaces is π is s-*bijective* if, for all y in Y

$$\pi : Y^s(y, \epsilon) \to X^s(\pi(y), \epsilon')$$

is a local homeomorphism.

Theorem 4. *Let $\pi : (Y, \psi) \to (X, \varphi)$ be a factor map between Smale spaces and y in Y be periodic and such that $\pi|\mathcal{O}(y)$ is injective.*

If π is u-bijective, then there is a $*$-homomorphism

$$\pi^s : S(Y, \psi, y) \to S(X, \varphi, \pi(y)).$$

If π is s-bijective, then there is a $*$-homomorphism

$$\pi^{u*} : U(X, \varphi, \pi(y)) \to U(Y, \psi, y).$$

If π is u-bijective

$$\pi : Y^u(\mathcal{O}(y)) \to X^u(\mathcal{O}(\pi(y)))$$

is a homeomorphism and

$$\pi \times \pi(Y^u(\mathcal{O}(y)))^s \subseteq X^u(\mathcal{O}(\pi(y)))^s$$

is an open subgroupoid.

If π is s-bijective

$$\pi \times \pi : Y^u(\mathcal{O}(y))^s \to X^u(\mathcal{O}(\pi(y)))^s$$

is a proper morphism of groupoids.

A better Bowen's Theorem

Let (X, φ) be a Smale space. We look for a Smale space (Y, ψ) and a factor map

$$\pi_s : (Y, \psi) \to (X, \varphi)$$

satisfying:

1. π_s is s-bijective,

2. $dim(Y^u(y, \epsilon)) = 0$.

That is, $Y^u(y, \epsilon)$ is totally disconnected, while $Y^s(y, \epsilon)$ is homeomorphic to $X^s(\pi_s(y), \epsilon)$.

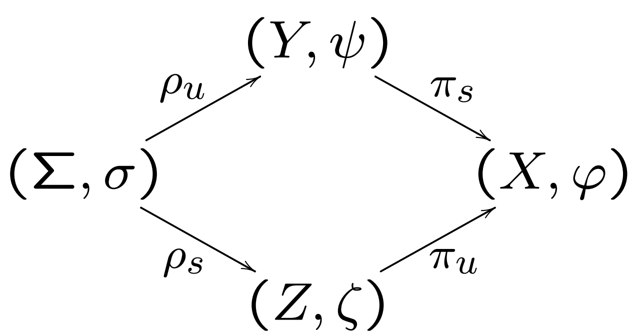
This is a "one-coordinate" version of Bowen's Theorem.

Similarly, we look for a Smale space (Z, ζ) and a factor map $\pi_u : (Z, \zeta) \to (X, \varphi)$ satisfying $dim(Z^s(z, \epsilon)) = 0$, and π_u is u-bijective.

We call $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ a *s/u-bijective pair* for (X, φ).

Theorem 5 (Better Bowen)**.** *If (X, φ) is a non-wandering Smale space, then there exists an s/u-bijective pair.*

Fibred product recovers Bowen's (Σ, σ):



with

$$\pi = \rho_s \circ \pi_u = \rho_u \circ \pi_s.$$

A homology theory

For $L, M \geq 0$, we define

$$\begin{aligned}\Sigma_{L,M}(\pi) \;=\; & \{(y_0, \ldots, y_L, z_0, \ldots, z_M) \mid \\ & y_l \in Y, z_m \in Z, \\ & \pi_s(y_l) = \pi_u(z_m)\}.\end{aligned}$$

Each of these is a SFT.

Moreover, the maps

$$\begin{aligned}\delta_{l,} : \; \Sigma_{L,M} \rightarrow \; & \Sigma_{L-1,M}, \\ \delta_{,m} : \; \Sigma_{L,M} \rightarrow \; & \Sigma_{L,M-1}\end{aligned}$$

which delete y_l and z_m are s-bijective and u-bijective, respectively.

We get a double complex:

$$
\begin{array}{ccccccc}
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,2})^{alt} & \longleftarrow & D^s(\Sigma_{1,2})^{alt} & \longleftarrow & D^s(\Sigma_{2,2})^{alt} & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,1})^{alt} & \longleftarrow & D^s(\Sigma_{1,1})^{alt} & \longleftarrow & D^s(\Sigma_{2,1})^{alt} & \longleftarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
D^s(\Sigma_{0,0})^{alt} & \longleftarrow & D^s(\Sigma_{1,0})^{alt} & \longleftarrow & D^s(\Sigma_{2,0})^{alt} & \longleftarrow &
\end{array}
$$

$$
\begin{aligned}
\partial_N^s : & \qquad \oplus_{L-M=N} D^s(\Sigma_{L,M})^{alt} \\
\rightarrow & \qquad \oplus_{L-M=N-1} D^s(\Sigma_{L,M})^{alt}
\end{aligned}
$$

$$
\partial_N^s = \sum_{l=0}^{L} (-1)^l \delta_{l,}^s + \sum_{m=0}^{M+1} (-1)^{m+M} \delta_{,m}^{s*}
$$

$$
H_N^s(\pi) = \ker(\partial_N^s)/Im(\partial_{N+1}^s).
$$

Topology	Dynamics
open cover $U_1, \ldots, U_I$	Bowen's Theorem $\pi_s, \pi_u : Y, Z \to X$
multiplicities $U_{i_0} \cap \cdots \cap U_{i_N} \neq \emptyset$	multiplicities $\Sigma_{L,M}(\pi)$
groups C^N	groups $D^s(\Sigma_N(\pi))^{alt}$

Theorem 6. *The groups* $H^s_N(\pi)$ *depend on* (X, φ)*, but not the choice of* s/u*-bijective pair* $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$.

From now on, we write $H^s_N(X, \varphi)$.

Theorem 7. *The functor* $H^s_*(X, \varphi)$ *is covariant for* s*-bijective factor maps, contravariant for* u*-bijective factor maps.*

Theorem 8. *The groups* $H^s_N(X, \varphi)$ *are all finite rank and non-zero for only finitely many* $N \in \mathbb{Z}$.

Theorem 9 (Lefschetz Formula). *Let* (X, φ) *be any non-wandering Smale space and let* $p \geq 1$*.*

$$\begin{aligned} \sum_{N \in \mathbb{Z}} (-1)^N \quad Tr[(\varphi^s)^{-p} : & \quad H_N^s(X, \varphi) \otimes \mathbb{Q} \\ \to & \quad H_N^s(X, \varphi) \otimes \mathbb{Q}] \\ = & \quad \#\{x \in X \mid \varphi^p(x) = x\} \end{aligned}$$

Example 4: Shifts of finite type

If $(X, \varphi) = (\Sigma, \sigma)$, then $Y = \Sigma = Z$ is an s/u-bijective pair.

The only non-zero group in the double complex occurs at $(0, 0)$.

$$\begin{aligned} H_0^s(\Sigma, \sigma) &= D^s(\Sigma), \\ H_N^s(\Sigma, \sigma) &= 0, N \neq 0. \end{aligned}$$

Example 3: $\frac{q}{p}$-solenoid[N. Burke-P.]

Let $p < q$ be primes and (X, φ) the $\frac{q}{p}$-solenoid.

$Z = X$, Y is the full q-shift and it maps down so that it is two-to-one on a full p-shift.

$$\begin{aligned} H_0^s(X, \varphi) &\cong \mathbb{Z}[1/q] \\ H_1^s(X, \varphi) &\cong \mathbb{Z}[1/p] \\ H_N^s(X, \varphi) &= 0, N \neq 0, 1. \end{aligned}$$

Example 2: 2^∞-**solenoid** [Bazett-P.]

$$\begin{aligned} H^s_0(X,\varphi) &\cong \mathbb{Z}[1/2], \\ H^s_1(X,\varphi) &\cong \mathbb{Z}, \\ H^s_N(X,\varphi) &= 0, N \neq 0,1 \end{aligned}$$

Generalized 1-solenoids (Williams, Yi, Thomsen): done by Amini, P, Saeidi Gholikandi and you can hear more at 4:00 PM.

Example 1: 2-**torus**[Bazett-P.]:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$$

N	$H^s_N(X,\varphi)$	φ^s
-1	$\mathbb{Z}$	1
0	$\mathbb{Z}^2$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
1	$\mathbb{Z}$	$-1.$