Projections in Eberlein compactifications

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A classical decomposition

G – locally compact group

$\pi : G \to \mathcal{U}(\mathcal{H})$ continuous unitary representation

Theorem [Jacobs–de Leeuw–Glicksberg]

$$\pi = \pi_{\mathrm{wm}} \oplus \pi_{\mathrm{ret}} \text{ on } p_{\mathrm{wm}}\mathcal{H} \oplus^2 p_{\mathrm{ret}}\mathcal{H}$$

where

$$\mathcal{H}_{\mathrm{wm}} = \left\{ \xi \in \mathcal{H} : 0 \in \overline{\pi(G)\xi}^{\mathrm{w}} \right\}$$

$$\mathcal{H}_{\mathrm{ret}} = \left\{ \xi \in \mathcal{H} : \xi \in \overline{\pi(G)\eta}^{\mathrm{w}} \text{ whenever } \eta \in \overline{\pi(G)\xi}^{\mathrm{w}} \right\}.$$

Semigroup perspective

$(\mathrm{ball}(\mathcal{B}(\mathcal{H})), \mathrm{w.o.t.})$ – semitopological semigroup
i.e. $x \mapsto xy, yx$ each continuous for each fixed y

$G^\pi = \overline{\pi(G)}^{\mathrm{w.o.t.}}$ – compact semitopological semigroup
E.g.: $\lambda : G \to \mathcal{U}(\mathrm{L}^2(G))$ left reg. rep'n, $G^\lambda = G_\infty$

Theorem [de Leeuw–Glicksberg, Troallic]

- p_{ret} minimal projection (idempotent) in G^π
- $G^\pi_{\mathrm{ret}} = p_{\mathrm{ret}} G^\pi$ compact group & ideal in G^π

Eberlein compactification

S – compact semitop'l semigroup
called Eberlein if $S \hookrightarrow (\mathrm{ball}(\mathcal{B}(\mathcal{H})), \mathrm{w.o.t.})$ homeo'lly

$\varpi : G \to \mathcal{U}(\mathcal{H})$ – universal representation

Theorem [Megrelishvili, S.–Stokke]

$G^{\mathcal{E}} := G^{\varpi}$ universal Eberlein compactification of G
S Eberlein semigroup, $\eta : G \to S$ homo'm w. dense range
(i.e. (η, S) is an Eberlein compactification of G)
$\Rightarrow \exists$ extension $\tilde{\eta} : G^{\mathcal{E}} \twoheadrightarrow S$

Can be done for non-locally compact G as well.

Eberlein groups & topologies

(G,τ_G) – (complete) topological group

$$\mathrm{B}(G) = \left\{ s \mapsto \langle \pi(s)\xi|\eta\rangle : \begin{array}{c} \pi : G \to \mathcal{U}(\mathcal{H})\ \tau_G\text{-w.o.t.-cts.} \\ \xi,\eta \in \mathcal{H},\ \mathcal{H}\text{ Hil. space} \end{array} \right\}$$

(G,τ_G) is <u>Eberlein</u> if $\tau_G = \sigma(G,\mathrm{B}(G))$.
Equivalently, $\varpi : G \hookrightarrow \varpi(G) \subset G^{\mathcal{E}}$ is a homeomorphism.
E.g. (G,τ_G) locally compact, or discrete.

Coarser Eberlein topologies:

$$\widetilde{\mathcal{T}}(G) = \{\tau \subseteq \tau_G : (G,\tau)\text{ top'l group},\ \tau = \sigma(G,\mathrm{B}_\tau(G))\}$$

where $\mathrm{B}_\tau(G) = \mathrm{B}(G) \cap \mathcal{C}(G,\tau)$.

... Eberlein topologies

$\tau \in \widetilde{\mathcal{T}}(G)$
$N_\tau = \bigcap\{U : U\ \tau\text{-nbhd. of } e\}$ is a τ-closed normal subgroup
$\bar{\tau}$ – (Hausdorff) toplogy induced on G/N_τ
$\mathcal{U}_{\bar{\tau}}$ – two-sided uniformity on G/N_τ generated by $\bar{\tau}$.

Facts

- $G_\tau = \overline{(G/N_\tau, \bar{\tau})}^{\mathcal{U}_{\bar{\tau}}}$ is an Eberlein group
- $\exists$ cts. homo'm $\eta_\tau : G \to G_\tau$ w. dense range
- $\exists$ unique cts. ext'n $\tilde{\eta}_\tau : G^{\mathcal{E}} \twoheadrightarrow G_\tau^{\mathcal{E}}$

Relations to central projections

$$\mathrm{ZE}(G^{\mathcal{E}}) = \{z \in G^{\mathcal{E}} : z^2 = z \;\&\; tz = zt \,\forall t \in G^{\mathcal{E}}\}$$

Theorem (after [Ruppert] for abelian G)

(i) $\exists$ map $T : \mathrm{ZE}(G^{\mathcal{E}}) \to \widetilde{\mathcal{T}}(G)$:

- define for z, $\eta_z : G \to G^{\mathcal{E}}$ by $\eta_z(s) = z\varpi(s)$
- let $T(z) = \sigma(G, \{\eta_z\})$

(ii) $\exists$ map $E : \widetilde{\mathcal{T}}(G) \to \mathrm{ZE}(G^{\mathcal{E}})$:

- given τ, the compact semigroup $\tilde{\eta}_\tau^{-1}(\{e_\tau\}) \subset G^{\mathcal{E}}$ admits a unique min'l idempotent, $z = E(\tau)$ [Ruppert, Troallic]
- $E(\tau)$ is central in $G^{\mathcal{E}}$

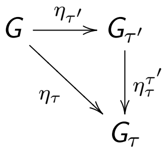
Notes.
- $E \circ T = \mathrm{id}_{\mathrm{ZE}(G^{\mathcal{E}})}$, $T \circ E(\tau) \supseteq \tau$.
- $G_{T(z)} \cong G^{\mathcal{E}}(z) := \{t \in G^{\mathcal{E}} : tz = t \,\&\, tt^* = z = t^*t\}$
- $\tau \subseteq \tau' \Rightarrow E(\tau) \leq E(\tau')$, $\quad z \leq z' \Rightarrow T(z) \subseteq T(z')$

When is $T \circ E(\tau) = \tau$?

$\tau \subseteq \tau'$ in $\widetilde{\mathcal{T}}(G)$

get cts. homo'ms w. dense range

$\eta_\tau^{\tau'} \circ \eta_{\tau'} = \eta_\tau$



Co-compact/Cauchy containment

$\tau \subseteq_c \tau'$ in $\widetilde{\mathcal{T}}(G)$ if $\tau \subseteq \tau'$ &

- $\ker \eta_\tau^{\tau'}$ compact & $\eta_\tau^{\tau'}$ open.
- Eq'ly, each τ-Cauchy net in G admits τ'-Cauchy refinement.

Theorem

$\tau \subseteq \tau'$ in $\widetilde{\mathcal{T}}(G)$: $\tau \subseteq_c \tau' \Leftrightarrow E(\tau) = E(\tau')$ & $\tau \subseteq_c T \circ E(\tau)$

"Reasonable" Eberlein topologies: $\mathcal{T}(G) = T(\mathrm{ZE}(G^{\mathcal{E}})) \subseteq \widetilde{\mathcal{T}}(G)$

Jacobs–de Leeuw–Glicksberg revisted

$\mathrm{B}(G) \cong (\varpi(G)'')_*$, Banach algebra of functions on G:

$$\langle \pi(\cdot)\xi|\eta\rangle + \langle \pi'(\cdot)\xi'|\eta'\rangle = \langle \pi \oplus \pi'(\cdot)\xi \oplus \xi'|\eta \oplus \eta'\rangle$$
$$\langle \pi(\cdot)\xi|\eta\rangle\langle \pi'(\cdot)\xi'|\eta'\rangle = \langle \pi \otimes \pi'(\cdot)\xi \otimes \xi'|\eta \otimes \eta'\rangle$$

Almost periodic (Bohr) topology $\tau_{\mathrm{ap}} = T(p_{\mathrm{ret}})$ satisfies

- $\pi : G \to \mathcal{U}(\mathcal{H})$ rep'n, $\pi = \pi_{\tau_{\mathrm{ap}}^{\perp}} \oplus \pi_{\tau_{\mathrm{ap}}}$, $p_{\mathrm{ret}} = \pi''(E(\tau_{\mathrm{ap}}))$
- $\mathrm{B}(G) = \mathrm{I}_{\tau_{\mathrm{ap}}}(G) \oplus \mathrm{B}_{\tau_{\mathrm{ap}}}(G)$, $\mathrm{B}_{\tau_{\mathrm{ap}}}(G) = E(\tau_{\mathrm{ap}}) \cdot \mathrm{B}(G)$,

Theorem

Let $\tau \in \mathcal{T}(G)$. Then

- $\pi : G \to \mathcal{U}(\mathcal{H})$ rep'n, $\pi = \pi_{\tau^{\perp}} \oplus \pi_{\tau}$, $\pi_{\tau} = \pi''(E(\tau))\pi$
- $\mathrm{B}(G) = \mathrm{I}_{\tau}(G) \oplus \mathrm{B}_{\tau}(G)$ where
  $\mathrm{B}_{\tau}(G) = E(\tau) \cdot \mathrm{B}(G), \quad \mathrm{I}_{\tau}(G) \lhd \mathrm{B}(G)$

Operator amenability of $\mathrm{B}(G)$

G locally compact

Theorem [Dales–Ghahramani–Helemskiĭ, Brown–Moran]

Measure algebra $\mathrm{M}(G)$ (op.) amenable $\Leftrightarrow$ G discrete & amenable.
G abelian: $\mathrm{B}(G) \cong \mathrm{M}(\widehat{G})$ (op.) amenable $\Leftrightarrow$ G compact.

False conjecture: $\mathrm{B}(G)$ op. amenable $\Leftrightarrow$ G compact.

Theorem [Runde-S.] (after [Ilie-S.])

$G_{n,p} = \mathbb{Q}_p^n \rtimes \mathrm{GL}_n(\mathbb{O}_p)$ has $\mathrm{B}(G_{n,p})$ op. amenable.

Proposition

$\mathrm{B}(G)$ op. amenable $\Rightarrow |\mathrm{ZE}(G^{\mathcal{E}})| = |\mathcal{T}(G)| < \infty$.

[Elgün] G abelian non-compact, $|\mathrm{ZE}(G^{\mathcal{E}})| \geq \mathfrak{c}$

Thank you for your attention!

Thank you

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