Irreducible holomorphic symplectic manifolds: geometry

(K. Hulek: Lecture I)

0. Introduction

X: compact Kähler manifold

$$c_1(X)_{\mathbb{R}} = c_1(T_X)_{\mathbb{R}} \in H^2(X, \mathbb{R})$$

<u>Theorem</u> (Bogomolov 1974) Assume that X is a compact Ricci flat, i.e. $c_1(X)_{\mathbb{R}} = 0$ manifold. Then there is a finite étale cover $\tilde{X} \to X$ such that

$$\tilde{X} \cong T \times \prod Y_j \times \prod X_h$$

torus — T

CY manifolds ($h^i(\mathcal{O}_Y) = 0$, $1 \le i \le \dim Y - 1$), $SU(n)$-holonomy — $\prod Y_j$

IHSM manifolds, $Sp(n)$-holonomy — $\prod X_h$

<u>Definition</u>: A compact complex Kähler manifold X is called an <u>irreducible holomorphic symplectic manifold</u> (<u>IHSM</u>) (or <u>complex hyperkähler manifold</u>) if

(i) $\pi_1(X) = 0$

(ii) $H^0(X, \Omega^2_X) \cong \mathbb{C}\sigma$ where σ is a nowhere degenerate holomorphic 2-form.

<u>Remarks</u>: (i) $\dim_{\mathbb{C}} X = 2n$, $\dim_{\mathbb{C}} X = 2$: IHSM $=$ K3

(ii) $\omega_X = \Omega^{2n}_X = \mathcal{O}_X$ (trivialized by $\sigma \wedge \ldots \wedge \sigma$)

(iii) $H^0(X, \Omega^{2i}_X) = \mathbb{C}\sigma \wedge \ldots \wedge \sigma$, $H^0(X, \Omega^{2i+1}_X) = 0$.

(iv) Deformations of X are unobstructed (Bogomolov) and

$$T_{[0]} \mathrm{Def}(X) \cong H^1(X, T_X) \cong H^1(X, \Omega^1_X)$$

Examples

① Let S = K3 surface.

$$X = \mathrm{Hilb}^n S \quad \text{(degree } n \text{ Hilbert scheme).}$$

Note: The deformation space of X is 21-dimensional. Hence there are deformations of X which are not of type $\mathrm{Hilb}^n S$.

② Let A be a 2-dimensional torus. Let

$$p : \mathrm{Hilb}^{n+1} A \to A$$

be given by addition.

$$K^{[n]} A = p^{-1}(0)$$

This is called the generalized Kummer variety. If $n \geq 2$ then the deformation space has dimension $5 > 4$ = def. of tori.

③ O'Grady's example of dimension 10 (derived from a moduli space of sheaves on a K3 surface), 22 parameters

④ O'Grady's example of dimension 6 (derived from a moduli space of sheaves on an abelian surface), 6 parameters.

Question: Are there other examples of IHSM ?

I. The Beauville - Bogomolov lattice

$$S = \text{K3 surface.}$$

The cup-product defines a non-degenerate pairing

$$(\ ,\ ) : H^2(S,\mathbb{Z}) \times H^2(S,\mathbb{Z}) \to H^4(S,\mathbb{Z}) = \mathbb{Z}$$

This makes S a lattice, in fact an even lattice ($x^2 \in 2\mathbb{Z}$). In fact

$$H^2(S,\mathbb{Z}) = 3U \oplus 2E_8(-1) \qquad (\text{sign} = (3,19))$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \text{hyperbolic plane}$$

$E_8(-1)$: rank 8, even, unimodular

For an IHSM X one can still equip $H^2(X,\mathbb{Z})$ with a lattice structure. This is the Beauville - Bogomolov lattice:

① Let $\sigma \in H^0(X, \Omega^2_X)$, $\int_X (\sigma\bar{\sigma})^n = 1$. We can decompose $\alpha \in H^2(X,\mathbb{C})$

$$\alpha = \lambda\sigma + \beta + \mu\bar{\sigma}, \qquad \beta \in H^{1,1}(X).$$

Then we set

$$q_X'(\alpha) := \lambda\mu + \frac{n}{2}\int_X \beta^2(\sigma\bar{\sigma})^{n-1}.$$

This defines a quadratic form on $H^2(X,\mathbb{C})$. There is a $\gamma > 0$ such that the quadratic form $q_X = \gamma q_X'$ restricted to $H^2(X,\mathbb{Z})$ defines a lattice

$$(\ ,\ )_X : H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to \mathbb{Z},$$

$$\text{sign}\ (\ ,\ )_X = (3, b_2(X) - 3).$$

② For $\alpha \in H^2(X,\mathbb{Z})$ let

$$V_X(\alpha) := \alpha^{2n} \in H^{4n}(X,\mathbb{Z}) \cong \mathbb{Z}$$

Then Fujiki has shown that there is some $c \in \mathbb{Q}_{>0}$ such that

$$\boxed{v_X(\alpha) = c \cdot q_X(\alpha)^n} \qquad (c = \text{Fujiki constant}).$$

<u>Examples</u> ① $X \sim \mathrm{Hilb}^n S =: S^{[n]}$

$$L = 3U \oplus 2E_8(-1) \oplus \langle -2(n-1) \rangle \quad (n \geq 2), \quad c = \frac{(2n)!}{n!\,2^n}$$

② X = 10-dim O'Grady example

$$L = 3U \oplus 2E_8(-1) \oplus A_2(-1), \qquad c = 945$$

③ $X \sim K^{[n]}(A)$

$$L = 3U \oplus \langle -2(n+1) \rangle, \qquad c = (n+1)\frac{(2n)!}{n!\,2^n}$$

④ X = 6-dim O' Grady example

$$L = 3U \oplus \langle -2 \rangle \oplus \langle -2 \rangle, \qquad c = 60.$$

II. <u>Periods</u>

$$X = \text{IHSM}, \quad \sigma \in H^0(X, \Omega_X^2) \subset H^2(X,\mathbb{C}) = H^{20} \oplus H^{11} \oplus H^{02}$$

Then

$$(\sigma,\sigma) = 0, \; (\sigma, \bar{\sigma}) > 0.$$

This leads to the definition of <u>period domains</u>.

<u>Definition:</u> A <u>marking</u> of X is an isometry

$$\varphi : H^2(X, \mathbb{Z}) \xrightarrow{\cong} L$$

(L an abstract Beauville lattice).

We define

$$\Omega_L = \{ [x] \in \mathbb{P}(L \otimes \mathbb{C}),\ (x,x) = 0,\ (x,\bar{x}) > 0 \}$$

If φ is a marking, then we have the period point

$$\omega(X,\varphi) := [\varphi(\sigma)] = [\varphi(H^{20})] \in \Omega_L . \quad \text{"Hodge structure"}$$

Question (Torelli): In how far does $\omega(X,\varphi)$ determine X?

III. Polarizations

$$X = \text{IHSM}, \quad \mathcal{L} \in \operatorname{Pic} X \text{ ample, primitive.}$$

Let

$$c_1(\mathcal{L}) \in H^2(X,\mathbb{Z}).$$

Then by ampleness $c_1(\mathcal{L})^2 > 0$, and

$$(\sigma, c_1(\mathcal{L})) = 0. \qquad (*)$$

Now let $\varphi : H^2(X,\mathbb{Z}) \to L$ be a marking and set

$$h := \varphi(c_1(\mathcal{L})) \in L, \quad h^2 > 0.$$

By $(*)$ we are lead to consider the lattice

$$L_h := h^{\perp} \subset L, \quad \operatorname{sign} L_h = (2, b_2(X) - 3).$$

We also have the period domain

$$\Omega_{L_h} = \{ [x] \in \mathbb{P}(L_h \otimes \mathbb{C})\ ;\ (x,x) = 0,\ (x,\bar{x}) > 0 \}.$$

In this case

$$\omega(X,\varphi) = [\varphi(H^{20}(X))] \in \Omega_{L_h}.$$

We have

$$\Omega_{L_h} = D_{L_h} \amalg D'_{L_h}$$

In this case D_{L_h} is a type IV domain. This is a hermitian symmetric domain, more precisely

$$D_{L_h} = \frac{SO_0(2,19)}{SO_0(2) \times SO_0(19)}.$$

Examples: ① $S = K3$, $h^2 = 2d > 0$. Then up to $O(L_{K3})$ we can assume

$$h = e + df \in U \subset 3U \oplus 2E_8(-1) = L_{K3} \qquad (e^2 = f^2 = 0,\ e.f = 1)$$

Then

$$L_h \cong 2U \oplus 2E_8(-1) \oplus \underset{\langle e-df \rangle}{\overset{\|}{\langle -2d \rangle}}$$

② $X \sim Hilb^2 S$; $L = 3U \oplus 2E_8(-1) \oplus \langle -2 \rangle$; $h \in L$ primitive, $h^2 = 2d > 0$

Let

$$\operatorname{div}(h) = \text{positive of } (L, h) \subset \mathbb{Z}.$$

Then we have two cases

ⓐ $h^2 = 2d > 0$, $\operatorname{div} h = 1$:

$$L_h \cong 2U \oplus 2E_8(-1) \oplus \langle -2 \rangle \oplus \langle -2d \rangle \quad \text{"split case"}$$

ⓑ $h^2 = 2d > 0$, $d \equiv -1 \bmod 4$, $\operatorname{div} h = 2$

$$L_h \cong 2U \oplus 2E_8(-1) \oplus \begin{pmatrix} -2 & 1 \\ 1 & -\frac{d+1}{2} \end{pmatrix} \quad \text{"non-split case"}.$$

IV Torelli theorem for K3 surfaces

Torelli theorems come in different forms.

S, S': K3 surfaces.

<u>Definition</u>: A <u>Hodge isometry</u> $\varphi: H^2(S', \mathbb{Z}) \to H^2(S, \mathbb{Z})$ is an isometry of lattices with $\varphi_{\mathbb{C}}(H^{20}(S')) = H^{20}(S, \mathbb{C})$.

<u>Weak Torelli theorem</u>: Two K3 surfaces S, S' are isomorphic if and only if there exists a Hodge isometry $H^2(S', \mathbb{Z}) \cong H^2(S, \mathbb{Z})$.

<u>Strong Torelli theorem</u>: Let (S, h) and (S', h') be polarised K3-surfaces. Assume that there is a Hodge isometry $\Phi: H^2(S', \mathbb{Z}) \cong H^2(S, \mathbb{Z})$ with $\Phi(h') = h$. Then there exists a unique isomorphism $f: S \to S'$ with $\Phi = f^*$.