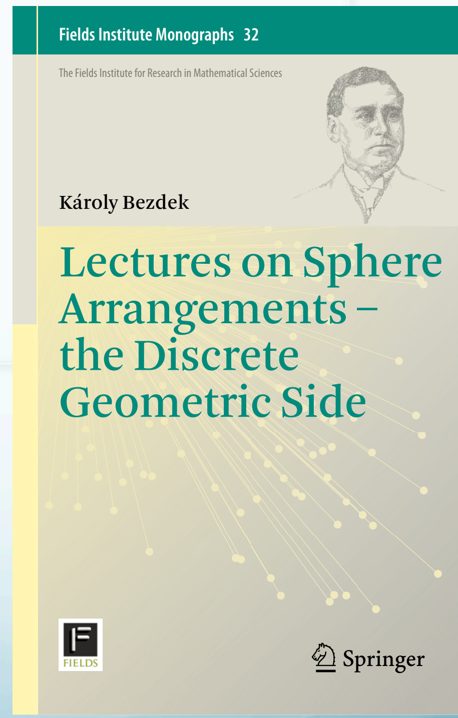
Plank theorems via successive inradii



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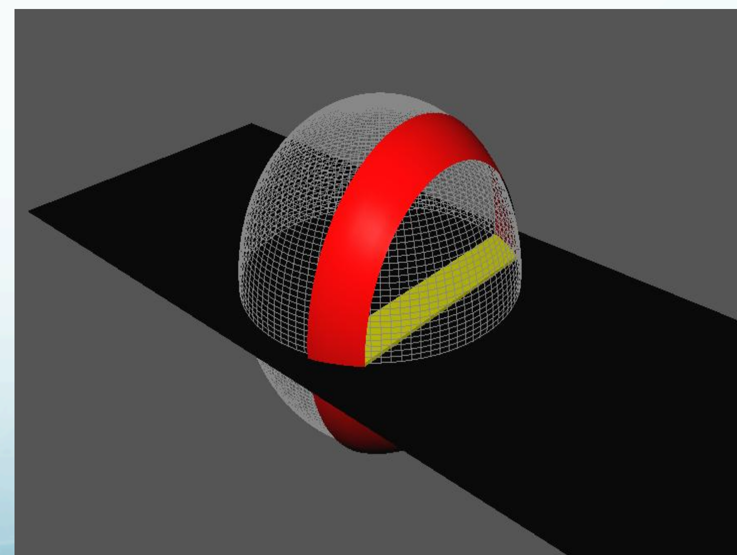
Alfred Tarski:**Alfred Tarski** (January 14, 1901, Warsaw, Russian-ruled Poland – October 26, 1983, Berkeley, California) was a Polish logician and mathematician. Educated in the Warsaw School of Mathematics and philosophy, he emigrated to the USA in 1939, and taught and did research in mathematics at the University of California, Berkeley, from 1942 until his death.[1]

The plank problem of Tarski (1932)

Recall that in the 1930's, Tarski posed what came to be known as the plank problem. A *plank* P in $\mathbb{E}^d$ is the (closed) set of points between two distinct parallel hyperplanes. The *width* w(P) of P is simply the distance between the two boundary hyperplanes of P. Tarski conjectured that if a convex body of minimal width w is covered by a collection of planks in $\mathbb{E}^d$, then the sum of the widths of these planks is at least w. This conjecture was



Recall that in the 1930's, Tarski posed what came to be known as the plank problem. A *plank* $\mathbf{P}$ in $\mathbb{E}^d$ is the (closed) set of points between two distinct parallel hyperplanes. The *width* $w(\mathbf{P})$ of $\mathbf{P}$ is simply the distance between the two boundary hyperplanes of $\mathbf{P}$. Tarski conjectured that if a convex body of minimal width w is covered by a collection of planks in $\mathbb{E}^d$, then the sum of the widths of these planks is at least w. This conjecture was proved by Bang in his memorable paper [**5**]. (In fact, the proof presented in that paper is a simplification and generalization of the proof published by Bang somewhat earlier in [**4**].) Thus, we call the following statement Bang's plank theorem.

THEOREM 1.1. *If the convex body* $\mathbf{C}$ *is covered by the planks* $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ *in* $\mathbb{E}^d, d \geq 2$ *(i.e.,* $\mathbf{C} \subset \mathbf{P}_1 \cup \mathbf{P}_2 \cup \cdots \cup \mathbf{P}_n \subset \mathbb{E}^d$*), then* $\sum_{i=1}^{n} w(\mathbf{P}_i) \geq w(\mathbf{C})$.

In [**5**], Bang raised the following stronger version of Tarski's plank problem called the affine plank problem. We phrase it via the following definition. Let $\mathbf{C}$ be a convex body and let $\mathbf{P}$ be a plank with boundary hyperplanes parallel to the hyperplane H in $\mathbb{E}^d$. We define the $\mathbf{C}$-*width* of the plank $\mathbf{P}$ as $\frac{w(\mathbf{P})}{w(\mathbf{C},H)}$ and label it $w_{\mathbf{C}}(\mathbf{P})$. (This notion was introduced by Bang [**5**] under the name "relative width".)

CONJECTURE 1.2. *If the convex body* $\mathbf{C}$ *is covered by the planks* $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ *in* $\mathbb{E}^d, d \geq 2$*, then* $\sum_{i=1}^{n} w_{\mathbf{C}}(\mathbf{P}_i) \geq 1$.

4. T. Bang, On covering by parallel-strips, *Mat. Tidsskr. B.* **1950** (1950), 49–53.
5. T. Bang, A solution of the "Plank problem", *Proc. Am. Math. Soc.* **2** (1951), 990–993.

Thoger S. V. Bang (1917-1997) was a professor at the University of Copenhagen Mathematical Institute.

The special case of Conjecture 1.2, when the convex body to be covered is centrally symmetric, has been proved by Ball in [**3**]. Thus, the following is Ball's plank theorem.

Keith M. Ball

Theorem 1.3. *If the centrally symmetric convex body* $\mathbf{C}$ *is covered by the planks* $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ *in* $\mathbb{E}^d, d \geq 2$, *then* $\sum_{i=1}^{n} w_{\mathbf{C}}(\mathbf{P}_i) \geq 1$.

3. K. Ball, The plank problem for symmetric bodies, *Invent. Math.* **104** (1991), 535–543.

It was Alexander [**2**] who noticed that Conjecture 1.2 is equivalent to the following generalization of a problem of Davenport.

Conjecture 1.4. *If a convex body* $\mathbf{C}$ *in* $\mathbb{E}^d, d \geq 2$ *is sliced by* $n-1$ *hyperplane cuts, then there exists a piece that covers a translate of* $\frac{1}{n}\mathbf{C}$.

2. R. Alexander, A problem about lines and ovals, *Amer. Math. Monthly* **75** (1968) 482–487.

We note that the paper [**7**] of A. Bezdek and the author proves Conjecture 1.4 for successive hyperplane cuts (i.e., for hyperplane cuts when each cut divides one piece). Also, the same paper ([**7**]) introduced two additional equivalent versions of Conjecture 1.2. As they seem to be of independent interest we recall them following the terminology used in [**7**].

7. A. Bezdek and K. Bezdek, Conway's fried potato problem revisited, *Arch. Math.* **66/6** (1996), 522–528.

Let $\mathbf{C}$ and $\mathbf{K}$ be convex bodies in $\mathbb{E}^d$ and let H be a hyperplane of $\mathbb{E}^d$. The $\mathbf{C}$-*width of* $\mathbf{K}$ *parallel to* H is denoted by $w_{\mathbf{C}}(\mathbf{K}, H)$ and is defined as $\frac{w(\mathbf{K},H)}{w(\mathbf{C},H)}$. The *minimal* $\mathbf{C}$-*width of* $\mathbf{K}$ is denoted by $w_{\mathbf{C}}(\mathbf{K})$ and is defined as the minimum of $w_{\mathbf{C}}(\mathbf{K}, H)$, where the minimum is taken over all possible hyperplanes H of $\mathbb{E}^d$. Recall that the inradius of $\mathbf{K}$ is the radius of the largest ball contained in $\mathbf{K}$. It is quite natural then to introduce the $\mathbf{C}$-*inradius of* $\mathbf{K}$ as the factor of the largest positive homothetic copy of $\mathbf{C}$, a translate of which is contained in $\mathbf{K}$. We need to do one more step to introduce the so-called successive $\mathbf{C}$-inradii of $\mathbf{K}$ as follows.

Let r be the $\mathbf{C}$-inradius of $\mathbf{K}$. For any $0 < \rho \leq r$ let the $\rho\mathbf{C}$-*rounded body of* $\mathbf{K}$ be denoted by $\mathbf{K}^{\rho\mathbf{C}}$ and be defined as the union of all translates of $\rho\mathbf{C}$ that are covered by $\mathbf{K}$. (See Fig. 7.1.)

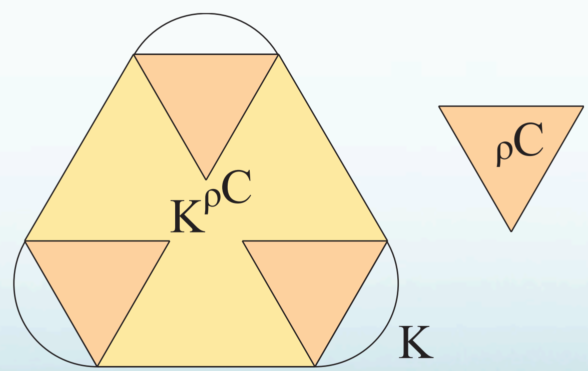


Fig. 7.1 The $\rho\mathbf{C}$-*rounded body of* $\mathbf{K}$: $\mathbf{K}^{\rho\mathbf{C}}$.

40. A. Bezdek and K. Bezdek, Conway's fried potato problem revisited, *Arch. Math.* **66/6** (1996), 522–528.

Now, take a fixed integer $m \geq 1$. On the one hand, if $\rho > 0$ is sufficiently small, then $w_{\mathbf{C}}(\mathbf{K}^{\rho\mathbf{C}}) > m\rho$. On the other hand, $w_{\mathbf{C}}(\mathbf{K}^{r\mathbf{C}}) = r \leq mr$. As $w_{\mathbf{C}}(\mathbf{K}^{\rho\mathbf{C}})$ is a decreasing continuous function of $\rho > 0$ and $m\rho$ is a strictly increasing continuous function of ρ, there exists a uniquely determined $\rho > 0$ such that

$$w_{\mathbf{C}}(\mathbf{K}^{\rho\mathbf{C}}) = m\rho.$$

This uniquely determined ρ is called the *mth successive* **C**-*inradius of* **K** and is denoted by $r_{\mathbf{C}}(\mathbf{K}, m)$. (See Fig. 7.2.) Notice that $r_{\mathbf{C}}(\mathbf{K}, 1) = r$.

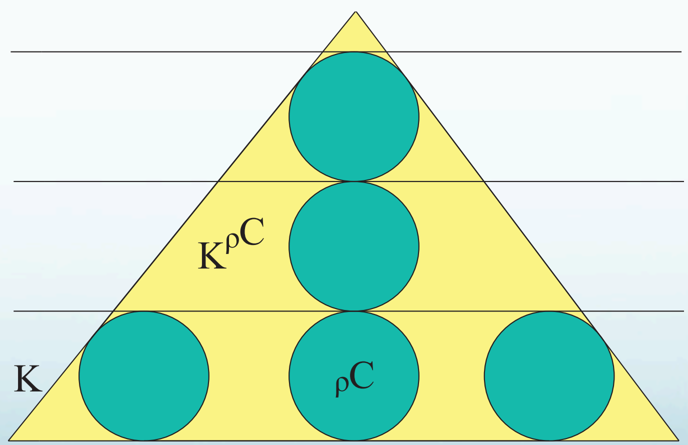


Fig. 7.2 The 3rd successive **C**-inradius of **K**: $r_{\mathbf{C}}(\mathbf{K}, 3) = \rho$ with the characteristic property $w_{\mathbf{C}}(\mathbf{K}^{\rho\mathbf{C}}) = 3\rho$.

Now, the two equivalent versions of Conjecture 1.2 and Conjecture 1.4 introduced in [**7**] can be phrased as follows.

CONJECTURE 1.5. *If a convex body* $\mathbf{K}$ *in* $\mathbb{E}^d, d \geq 2$ *is covered by the planks* $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, *then* $\sum_{i=1}^{n} w_{\mathbf{C}}(\mathbf{P}_i) \geq w_{\mathbf{C}}(\mathbf{K})$ *for any convex body* $\mathbf{C}$ *in* $\mathbb{E}^d$.

CONJECTURE 1.6. *Let* $\mathbf{K}$ *and* $\mathbf{C}$ *be convex bodies in* $\mathbb{E}^d, d \geq 2$. *If* $\mathbf{K}$ *is sliced by* $n-1$ *hyperplanes, then the minimum of the greatest* $\mathbf{C}$*-inradius of the pieces is equal to the* n*th successive* $\mathbf{C}$*-inradius of* $\mathbf{K}$, *i.e., it is* $r_{\mathbf{C}}(\mathbf{K}, n)$.

6. A. Bezdek and K. Bezdek, A solution of Conway's fried potato problem, *Bull. London Math. Soc.* **27/5** (1995), 492–496.
7. A. Bezdek and K. Bezdek, Conway's fried potato problem revisited, *Arch. Math.* **66/6** (1996), 522–528.

Recall that Theorem 1.3 gives a proof of (Conjecture 1.5 as well as) Conjecture 1.6 for centrally symmetric convex bodies $\mathbf{K}$ in $\mathbb{E}^d, d \geq 2$ (with $\mathbf{C}$ being an arbitrary convex body in $\mathbb{E}^d, d \geq 2$). Another approach that leads to a partial solution of Conjecture 1.6 was published in [**7**]. Namely, in that paper A. Bezdek and the author proved the following theorem that (under the condition that $\mathbf{C}$ is a ball) answers a question raised by Conway ([**6**]) as well as proves Conjecture 1.6 for successive hyperplane cuts.

Andras Bezdek

THEOREM 1.7. *Let* $\mathbf{K}$ *and* $\mathbf{C}$ *be convex bodies in* $\mathbb{E}^d$, $d \geq 2$. *If* $\mathbf{K}$ *is sliced into* $n \geq 1$ *pieces by* $n-1$ *successive hyperplane cuts (i.e., when each cut divides one piece), then the minimum of the greatest* $\mathbf{C}$*-inradius of the pieces is the* n*th successive* $\mathbf{C}$*-inradius of* $\mathbf{K}$ *(i.e.,* $r_{\mathbf{C}}(\mathbf{K}, n)$*). An optimal partition is achieved by* $n-1$ *parallel hyperplane cuts equally spaced along the minimal* $\mathbf{C}$*-width of the* $r_{\mathbf{C}}(\mathbf{K}, n)\mathbf{C}$*-rounded body of* $\mathbf{K}$.

Akopyan and Karasev ([**1**]) just very recently have proved a related partial result on Conjecture 1.5. Their theorem is based on a nice generalization of successive hyperplane cuts. The more exact details are as follows. Under the *convex partition* $\mathbf{V}_1 \cup \mathbf{V}_2 \cup \cdots \cup \mathbf{V}_n$ of $\mathbb{E}^d$ we understand the family $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n$ of closed convex sets having pairwise disjoint non-empty interiors in $\mathbb{E}^d$ with $\mathbf{V}_1 \cup \mathbf{V}_2 \cup \cdots \cup \mathbf{V}_n = \mathbb{E}^d$. Then we say that the convex partition $\mathbf{V}_1 \cup \mathbf{V}_2 \cup \cdots \cup \mathbf{V}_n$ of $\mathbb{E}^d$ is an *inductive partition* of $\mathbb{E}^d$ if for any $1 \leq i \leq n$, there exists an inductive partition $\mathbf{W}_1 \cup \cdots \cup \mathbf{W}_{i-1} \cup \mathbf{W}_{i+1} \cup \cdots \cup \mathbf{W}_n$ of $\mathbb{E}^d$ such that $\mathbf{V}_j \subset \mathbf{W}_j$ for all $j \neq i$. A partition into one part $\mathbf{V}_1 = \mathbb{E}^d$ is assumed to be inductive. We note that if $\mathbb{E}^d$ is sliced into n pieces by $n-1$ successive hyperplane cuts (i.e., when each cut divides one piece), then the pieces generate an inductive partition of $\mathbb{E}^d$. Also, the Voronoi cells of finitely many points of $\mathbb{E}^d$ generate an inductive partition of $\mathbb{E}^d$. Now, the main theorem of [**1**] can be phrased as follows.

THEOREM 1.8. *Let* $\mathbf{K}$ *and* $\mathbf{C}$ *be convex bodies in* $\mathbb{E}^d, d \geq 2$ *and let* $\mathbf{V}_1 \cup \mathbf{V}_2 \cup \cdots \cup \mathbf{V}_n$ *be an inductive partition of* $\mathbb{E}^d$ *such that* $\operatorname{int}(\mathbf{V}_i \cap \mathbf{K}) \neq \emptyset$ *for all* $1 \leq i \leq n$. *Then* $\sum_{i=1}^{n} r_{\mathbf{C}}(\mathbf{V}_i \cap \mathbf{K}, 1) \geq r_{\mathbf{C}}(\mathbf{K}, 1)$.

Arseniy Akopyan

Roman Karasev

1. A. Akopyan and R. Karasev, Kadets-type theorems for partitions of a convex body, *Discrete Comput. Geom.* **48** (2012), 766–776.

2. Extensions to Successive Inradii

First, we state the following stronger version of Theorem 1.7. Its proof is an extension of the proof of Theorem 1.7 published in [**7**].

Theorem 2.1. *Let* $\mathbf{K}$ *and* $\mathbf{C}$ *be convex bodies in* $\mathbb{E}^d$*,* $d \geq 2$ *and let* m *be a positive integer. If* $\mathbf{K}$ *is sliced into* $n \geq 1$ *pieces by* $n-1$ *successive hyperplane cuts (i.e., when each cut divides one piece), then the minimum of the greatest* m*th successive* $\mathbf{C}$*-inradius of the pieces is the* (mn)*th successive* $\mathbf{C}$*-inradius of* $\mathbf{K}$ *(i.e.,* $r_{\mathbf{C}}(\mathbf{K}, mn)$*). An optimal partition is achieved by* $n-1$ *parallel hyperplane cuts equally spaced along the minimal* $\mathbf{C}$*-width of the* $r_{\mathbf{C}}(\mathbf{K}, mn)\mathbf{C}$*-rounded body of* $\mathbf{K}$*.*

Second, the method of Akopyan and Karasev ([**1**]) can be extended to prove

Theorem 2.2. *Let* $\mathbf{K}$ *and* $\mathbf{C}$ *be convex bodies in* $\mathbb{E}^d, d \geq 2$ *and let* m *be a positive integer. If* $\mathbf{V}_1 \cup \mathbf{V}_2 \cup \cdots \cup \mathbf{V}_n$ *is an inductive partition of* $\mathbb{E}^d$ *such that* $\mathrm{int}(\mathbf{V}_i \cap \mathbf{K}) \neq \emptyset$ *for all* $1 \leq i \leq n$*, then* $\sum_{i=1}^{n} r_{\mathbf{C}}(\mathbf{V}_i \cap \mathbf{K}, m) \geq r_{\mathbf{C}}(\mathbf{K}, m)$*.*

Corollary 2.3. *Let* $\mathbf{K}$ *and* $\mathbf{C}$ *be convex bodies in* $\mathbb{E}^d, d \geq 2$*. If* $\mathbf{V}_1 \cup \mathbf{V}_2 \cup \cdots \cup \mathbf{V}_n$ *is an inductive partition of* $\mathbb{E}^d$ *such that* $\mathrm{int}(\mathbf{V}_i \cap \mathbf{K}) \neq \emptyset$ *for all* $1 \leq i \leq n$*, then* $\sum_{i=1}^{n} w_{\mathbf{C}}(\mathbf{V}_i \cap \mathbf{K}) \geq w_{\mathbf{C}}(\mathbf{K})$*.*

Finally, we close this section stating that Conjectures 1.2, 1.4, 1.5, and 1.6 are all equivalent to the following two conjectures:

Conjecture 2.4. *Let* $\mathbf{K}$ *and* $\mathbf{C}$ *be convex bodies in* $\mathbb{E}^d, d \geq 2$ *and let* m *be a positive integer. If* $\mathbf{K}$ *is covered by the planks* $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ *in* $\mathbb{E}^d$*, then* $\sum_{i=1}^{n} r_{\mathbf{C}}(\mathbf{P}_i, m) \geq r_{\mathbf{C}}(\mathbf{K}, m)$ *or equivalently,* $\sum_{i=1}^{n} w_{\mathbf{C}}(\mathbf{P}_i) \geq m r_{\mathbf{C}}(\mathbf{K}, m)$*.*

Conjecture 2.5. *Let* $\mathbf{K}$ *and* $\mathbf{C}$ *be convex bodies in* $\mathbb{E}^d, d \geq 2$ *and let the positive integer* m *be given. If* $\mathbf{K}$ *is sliced by* $n-1$ *hyperplanes, then the minimum of the greatest* m*th successive* $\mathbf{C}$*-inradius of the pieces is the* (mn)*th successive* $\mathbf{C}$*-inradius of* $\mathbf{K}$*, i.e., it is* $r_{\mathbf{C}}(\mathbf{K}, mn)$*.*

4.1. Successive Inradii Revisited. We give a somewhat different but still equivalent description of $r_{\mathbf{C}}(\mathbf{K}, m)$. If $\mathbf{C}$ is a convex body in $\mathbb{E}^d$, then

$$\mathbf{t} + \mathbf{C}, \mathbf{t} + \lambda_2 \mathbf{v} + \mathbf{C}, \ldots, \mathbf{t} + \lambda_m \mathbf{v} + \mathbf{C}$$

is called a *linear packing* of m translates of $\mathbf{C}$ positioned parallel to the line $\{\lambda \mathbf{v} \mid \lambda \in \mathbb{R}\}$ with direction vector $\mathbf{v} \neq \mathbf{o}$ if the m translates of $\mathbf{C}$ are pairwise non-overlapping, i.e., if

$$(\mathbf{t} + \lambda_i \mathbf{v} + \mathrm{int}\mathbf{C}) \cap (\mathbf{t} + \lambda_j \mathbf{v} + \mathrm{int}\mathbf{C}) = \emptyset$$

holds for all $1 \leq i \neq j \leq m$ (with $\lambda_1 = 0$). Furthermore, the line $l \subset \mathbb{E}^d$ passing through the origin $\mathbf{o}$ of $\mathbb{E}^d$ is called a *separating direction* for the linear packing

$$\mathbf{t} + \mathbf{C}, \mathbf{t} + \lambda_2 \mathbf{v} + \mathbf{C}, \ldots, \mathbf{t} + \lambda_m \mathbf{v} + \mathbf{C}$$

if

$$\mathrm{Pr}_l(\mathbf{t} + \mathbf{C}), \mathrm{Pr}_l(\mathbf{t} + \lambda_2 \mathbf{v} + \mathbf{C}), \ldots, \mathrm{Pr}_l(\mathbf{t} + \lambda_m \mathbf{v} + \mathbf{C})$$

are pairwise non-overlapping intervals on l, where $\mathrm{Pr}_l : \mathbb{E}^d \to l$ denotes the orthogonal projection of $\mathbb{E}^d$ onto l. It is easy to see that every linear packing

$$\mathbf{t} + \mathbf{C}, \mathbf{t} + \lambda_2 \mathbf{v} + \mathbf{C}, \ldots, \mathbf{t} + \lambda_m \mathbf{v} + \mathbf{C}$$

possesses at least one separating direction in $\mathbb{E}^d$. Finally, let $\mathbf{K}$ be a convex body in $\mathbb{E}^d$ and let $m \geq 1$ be a positive integer. Then let $\overline{\rho} > 0$ be the largest positive real with the following property: for every line l passing through the origin $\mathbf{o}$ in $\mathbb{E}^d$ there exists a linear packing of m translates of $\overline{\rho}\mathbf{C}$ lying in $\mathbf{K}$ and having l as a separating direction. It is straightforward to show that

$$\overline{\rho} = r_{\mathbf{C}}(\mathbf{K}, m).$$

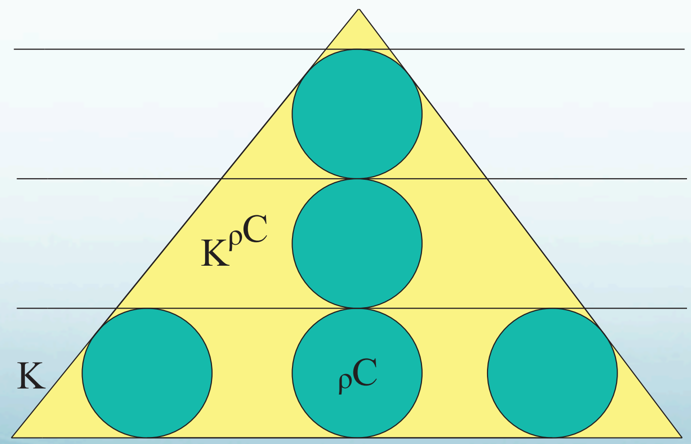


Fig. 7.2 The 3rd successive **C**-inradius of **K**: $r_{\mathbf{C}}(\mathbf{K}, 3) = \rho$ with the characteristic property $w_{\mathbf{C}}(\mathbf{K}^{\rho\mathbf{C}}) = 3\rho$.

4.2. On an Extension of a Helly-type Result of Klee. Recall the following Helly-type result of Klee [**9**]. Let $\mathcal{F} := \{\mathbf{A}_i \mid i \in I\}$ be a family of compact convex sets in $\mathbb{E}^d, d \geq 2$ containing at least $d+1$ members. Suppose $\mathbf{C}$ is a compact convex set in $\mathbb{E}^d$ such that the following holds: For each subfamily of $d+1$ sets in $\mathcal{F}$, there exists a translate of $\mathbf{C}$ that is contained in all $d+1$ of them. Then there exists a translate of $\mathbf{C}$ that is contained in all the members of $\mathcal{F}$. In what follows we give a proof of the following extension of Klee's theorem to linear packings.

THEOREM 4.1. *Let $\mathcal{F} := \{\mathbf{A}_i \mid i \in I\}$ be a family of convex bodies in $\mathbb{E}^d, d \geq 2$ containing at least $d+1$ members. Suppose $\mathbf{C}$ is a convex body in $\mathbb{E}^d$ and $m \geq 1$ is a positive integer moreover, l is a line passing through the origin $\mathbf{o}$ in $\mathbb{E}^d$ such that the following holds: For each subfamily of $d+1$ convex bodies in $\mathcal{F}$, there exists a linear packing of m translates of $\mathbf{C}$ with separating direction l that is contained in all $d+1$ of them. Then there exists a linear packing of m translates of $\mathbf{C}$ with separating direction l that is contained in all the members of $\mathcal{F}$.*

9. V. Klee, The critical set of a convex body, *Amer. J. Math.* **75** (1953), 178-188.

4.3. On Some Concave Functions of Successive Inradii. A rather straightforward extension of the method of Akopyan and Karasev ([**1**]) combined with Theorem 4.1 gives the following statement. For the statement below as well as its proof we extend the definition of the mth successive $\mathbf{C}$-inradius of convex bodies $\mathbf{K} \subset \mathbb{E}^d$ via including all non-empty compact convex sets $\mathbf{K} \subset \mathbb{E}^d$ having $\mathrm{int}\mathbf{K} = \emptyset$ with the definition $r_{\mathbf{C}}(\mathbf{K}, m) := 0$ and via including the empty set $\emptyset$ with the definition $r_{\mathbf{C}}(\emptyset, m) := -\infty$.

THEOREM 4.2. *Let $\mathbf{K}$ and $\mathbf{C}$ be convex bodies in $\mathbb{E}^d, d \geq 2$ and let m be a positive integer. Moreover, let $\mathbf{V}_1 \cup \mathbf{V}_2 \cup \cdots \cup \mathbf{V}_n$ be an inductive partition of $\mathbb{E}^d$ and let $\mathbf{K}_i(\mathbf{x}) := \mathbf{K} \cap (\mathbf{x} + \mathbf{V}_i)$ for all $\mathbf{x} \in \mathbb{E}^d$ and $1 \leq i \leq n$. Then the function*

$$r(\mathbf{x}, m) := \sum_{i=1}^{n} r_{\mathbf{C}}\left(\mathbf{K}_i(\mathbf{x}), m\right)$$

is a concave function of $\mathbf{x} \in \mathbb{E}^d$.

COROLLARY 4.4. *Let $\mathbf{K}_1, \ldots, \mathbf{K}_N$, and $\mathbf{C}$ be convex bodies in $\mathbb{E}^d$ and let m be a positive integer. Then*

$$r_{\mathbf{C}}\left((\mathbf{y}_1 + \mathbf{K}_1) \cap \cdots \cap (\mathbf{y}_N + \mathbf{K}_N), m\right)$$

is a concave function of $(\mathbf{y}_1, \ldots, \mathbf{y}_N) \in \mathbb{E}^{Nd}$.

4.4. Estimating Sums of Successive Inradii. Now, we are set for an inductive proof of Theorem 2.2 on the number n of tiles in the relevant inductive partition. The details are as follows. By Theorem 4.2 the function $r(\mathbf{x}, m) = \sum_{i=1}^{n} r_{\mathbf{C}}(\mathbf{K}_i(\mathbf{x}), m)$ is a concave function of $\mathbf{x}$ and so, $\mathbf{X}_r := \{\mathbf{x} \in \mathbb{E}^d \mid r(\mathbf{x}, m) > -\infty\}$ is a closed convex set in $\mathbb{E}^d$. If $\mathbf{x}_0$ is a boundary point of $\mathbf{X}_r$, then at least one $\mathbf{K}_i(\mathbf{x}_0) = \mathbf{K} \cap (\mathbf{x}_0 + \mathbf{V}_i)$ must have an empty interior in $\mathbb{E}^d$ say, $\mathrm{int}\mathbf{K}_{i_0}(\mathbf{x}_0) = \emptyset$ for some $1 \leq i_0 \leq n$. Then take the inductive partition $\mathbf{W}_1 \cup \cdots \cup \mathbf{W}_{i_0-1} \cup \mathbf{W}_{i_0+1} \cup \cdots \cup \mathbf{W}_n$ of $\mathbb{E}^d$ such that $\mathbf{V}_j \subset \mathbf{W}_j$ for all $j \neq i_0$. Now, it is easy to see that if $\mathrm{int}\,(\mathbf{K} \cap (\mathbf{x}_0 + \mathbf{V}_j)) \neq \emptyset$ for some $j \neq i_0$, then $\mathbf{K} \cap (\mathbf{x}_0 + \mathbf{V}_j) = \mathbf{K} \cap (\mathbf{x}_0 + \mathbf{W}_j)$. Thus,

$$\sum_{i=1}^{n} r_{\mathbf{C}}(\mathbf{K} \cap (\mathbf{x}_0 + \mathbf{V}_i), m) = \sum_{j \neq i_0} r_{\mathbf{C}}(\mathbf{K} \cap (\mathbf{x}_0 + \mathbf{W}_j), m)$$

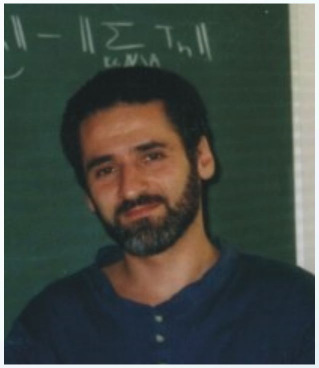
and therefore by induction we get that the inequality $r(\mathbf{x}_0, m) \geq r_{\mathbf{C}}(\mathbf{K}, m)$ holds for all boundary points $\mathbf{x}_0$ of $\mathbf{X}_r$. Then this fact and the concavity of $r(\mathbf{x}, m)$ imply in a straightforward way that the inequality $r(\mathbf{x}, m) \geq r_{\mathbf{C}}(\mathbf{K}, m)$ holds for all $\mathbf{x} \in \mathbf{X}_r$ unless $\mathbf{X}_r$ is a closed halfspace of $\mathbb{E}^d$. However, the latter case can happen only when (each $\mathbf{V}_i$, $1 \leq i \leq n$ contains the same halfspace and therefore) $n = 1$. As Theorem 2.2 clearly holds for $n = 1$, our inductive proof of Theorem 2.2 is complete.

PROBLEM 7.1. *Let* $\mathbf{K}$ *and* $\mathbf{C}$ *be convex bodies in* $\mathbb{E}^d$, $d \geq 2$ *and let* m *be a positive integer. Prove or disprove that if* $\mathbf{V}_1 \cup \mathbf{V}_2 \cup \ldots \cup \mathbf{V}_n$ *is a convex partition (resp., covering) of* $\mathbb{E}^d$ *such that* $\mathrm{int}(\mathbf{V}_i \cap \mathbf{K}) \neq \emptyset$ *for all* $1 \leq i \leq n$, *then* $\sum_{i=1}^{n} r_{\mathbf{C}}(\mathbf{V}_i \cap \mathbf{K}, m) \geq r_{\mathbf{C}}(\mathbf{K}, m)$.

PROBLEM 7.2. *Let* $\mathbf{K}$ *and* $\mathbf{C}$ *be convex bodies in* $\mathbb{E}^d$, $d \geq 2$ *and let* m *be a positive integer. Prove or disprove that if* $\mathbf{V}_1 \cup \mathbf{V}_2 \cup \ldots \cup \mathbf{V}_n$ *is a convex partition (resp., covering) of* $\mathbb{E}^d$ *such that* $\mathrm{int}(\mathbf{V}_i \cap \mathbf{K}) \neq \emptyset$ *for all* $1 \leq i \leq n$, *then the greatest* m*th successive* $\mathbf{C}$*-inradius of the pieces* $\mathbf{V}_i \cap \mathbf{K}$, $i = 1, 2, \ldots, n$ *is at least* $r_{\mathbf{C}}(\mathbf{K}, mn)$.

Theorem 7.3.1 *Let the ball* $\mathbf{B}$ *of the real Hilbert space* $\mathbb{H}$ *be covered by the convex bodies* $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n$ *in* $\mathbb{H}$. *Then*

$$\sum_{i=1}^{n} \mathrm{r}(\mathbf{C}_i \cap \mathbf{B}) \geq \mathrm{r}(\mathbf{B}).$$



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42. A. Bezdek, On a generalization of Tarski's plank problem, *Discrete Comput. Geom.* **38** (2007), 189–200.

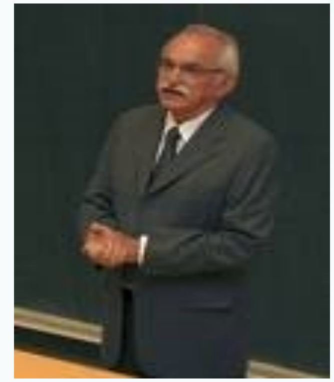
Problem 7.3.2 *Let the ball* $\mathbf{B}$ *be covered by the convex bodies* $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n$ *in an arbitrary Banach space. Prove or disprove that*

$$\sum_{i=1}^{n} \mathrm{r}(\mathbf{C}_i \cap \mathbf{B}) \geq \mathrm{r}(\mathbf{B}).$$

Theorem 7.3.3 *If the spherically convex bodies* $\mathbf{K}_1, \ldots, \mathbf{K}_n$ *cover the spherical ball* $\mathbf{B}$ *of radius* $r(\mathbf{B}) \geq \frac{\pi}{2}$ *in* $\mathbb{S}^d, d \geq 2$, *then*

$$\sum_{i=1}^{n} r(\mathbf{K}_i) \geq r(\mathbf{B}).$$

For $r(\mathbf{B}) = \frac{\pi}{2}$ *the stronger inequality* $\sum_{i=1}^{n} r(\mathbf{K}_i \cap \mathbf{B}) \geq r(\mathbf{B})$ *holds. Moreover, equality for* $r(\mathbf{B}) = \pi$ *or* $r(\mathbf{B}) = \frac{\pi}{2}$ *holds if and only if* $\mathbf{K}_1, \ldots, \mathbf{K}_n$ *are lunes with common ridge which have pairwise no common interior points.*



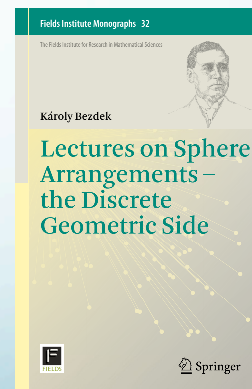
Rolf Schneider

Theorem 7.3.3 is a consequence of the following result proved by Schneider and the author in [83]. Recall that $\mathrm{Svol}_d(\ldots)$ denotes the spherical Lebesgue measure on $\mathbb{S}^d$, and recall that $(d+1)\omega_{d+1} = \mathrm{Svol}_d(\mathbb{S}^d)$.

Theorem 7.3.4 *If* $\mathbf{K}$ *is a spherically convex body in* $\mathbb{S}^d, d \geq 2$, *then*

$$\mathrm{Svol}_d(\mathbf{K}) \leq \frac{(d+1)\omega_{d+1}}{\pi} r(\mathbf{K}).$$

Equality holds if and only if $\mathbf{K}$ *is a lune.*



83. K. Bezdek and R. Schneider, Covering large balls with convex sets in spherical space, *Beiträge Algebra Geom.* **51/1** (2010), 229–235.

Indeed, Theorem 7.3.4 implies Theorem 7.3.3 as follows. If $\mathbf{B} = \mathbb{S}^d$; that is, the spherically convex bodies $\mathbf{K}_1, \ldots, \mathbf{K}_n$ cover $\mathbb{S}^d$, then

$$(d+1)\omega_{d+1} \leq \sum_{i=1}^{n} \mathrm{Svol}_d(\mathbf{K}_i) \leq \frac{(d+1)\omega_{d+1}}{\pi} \sum_{i=1}^{n} r(\mathbf{K}_i),$$

and the stated inequality follows. In general, when $\mathbf{B}$ is different from $\mathbb{S}^d$, let $\mathbf{B}' \subset \mathbb{S}^d$ be the spherical ball of radius $\pi - r(\mathbf{B})$ centered at the point antipodal to the center of $\mathbf{B}$. As the spherically convex bodies $\mathbf{B}', \mathbf{K}_1, \ldots, \mathbf{K}_n$ cover $\mathbb{S}^d$, the inequality just proved shows that

$$\pi - r(\mathbf{B}) + \sum_{i=1}^{n} r(\mathbf{K}_i) \geq \pi,$$

and the stated inequality follows. If $r(\mathbf{B}) = \frac{\pi}{2}$, then $\mathbf{K}_1 \cap \mathbf{B}, \ldots, \mathbf{K}_n \cap \mathbf{B}$ are spherically convex bodies and as $\mathbf{B}', \mathbf{K}_1 \cap \mathbf{B}, \ldots, \mathbf{K}_n \cap \mathbf{B}$ cover $\mathbb{S}^d$, the stronger inequality follows. The assertion about the equality sign for the case when $r(\mathbf{B}) = \pi$ or $r(\mathbf{B}) = \frac{\pi}{2}$ follows easily.

We close this section with the following question that bridges Theorem 7.3.3 to Theorem 7.3.1:

Problem 7.3.5 *Let the spherically convex bodies* $\mathbf{K}_1, \ldots, \mathbf{K}_n$ *cover the spherical ball* $\mathbf{B}$ *of radius* $r(\mathbf{B}) < \frac{\pi}{2}$ *in* $\mathbb{S}^d, d \geq 2$. *Then prove or disprove that*

$$\sum_{i=1}^{n} r(\mathbf{K}_i) \geq r(\mathbf{B}).$$