An update on polytopes with many symmetries

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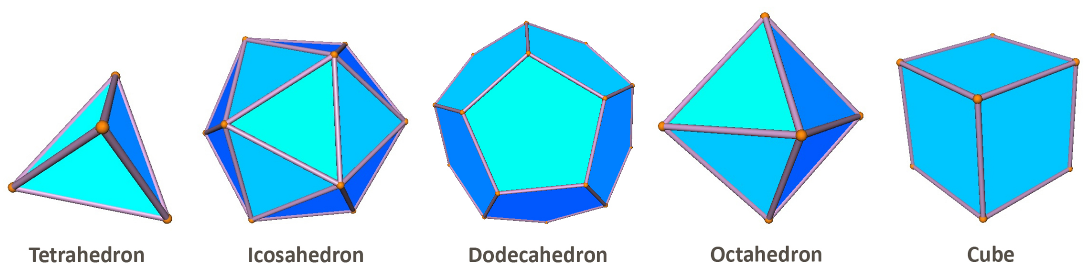
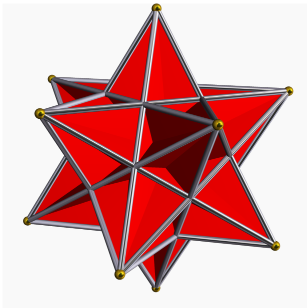
Polytopes

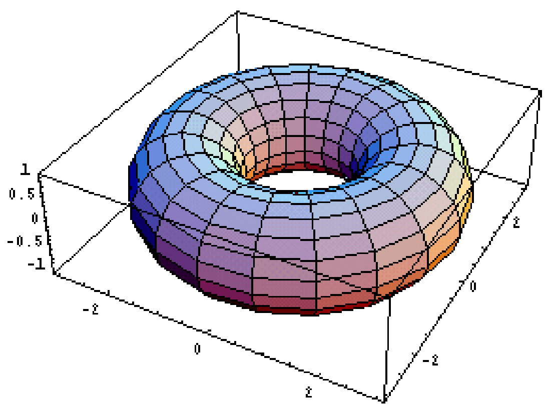
A polytope is a geometric structure with vertices, edges, and (usually) other elements of higher rank, and with some degree of uniformity and symmetry.

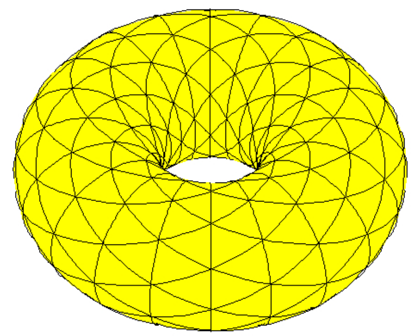
There are many different kinds of polytope, including both convex polytopes like the Platonic solids, and non-convex 'star' polytopes:

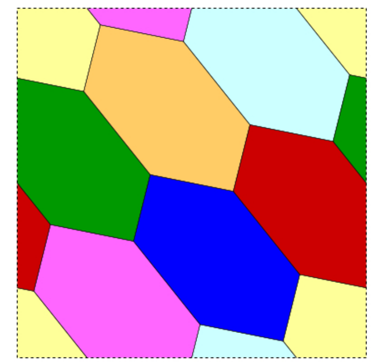




... as well as examples of rank 2, known as maps:



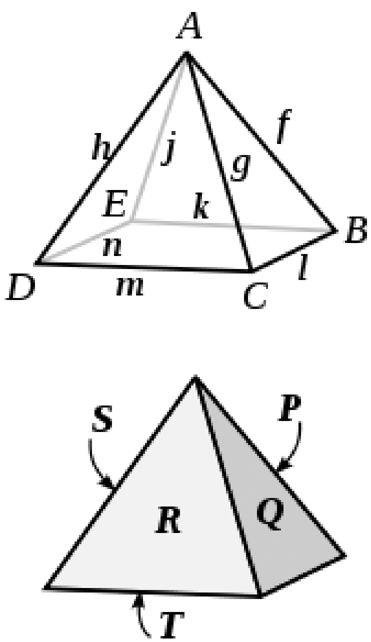


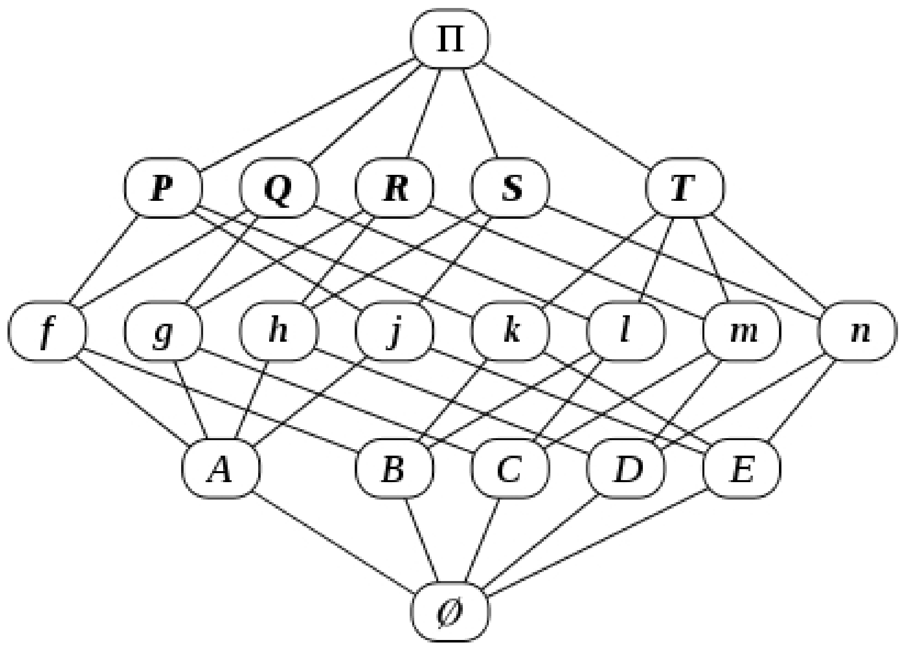


[Examples on every orientable surface of genus $g > 1$, and on non-orientable surfaces of genus p for infinitely many $p > 2$]

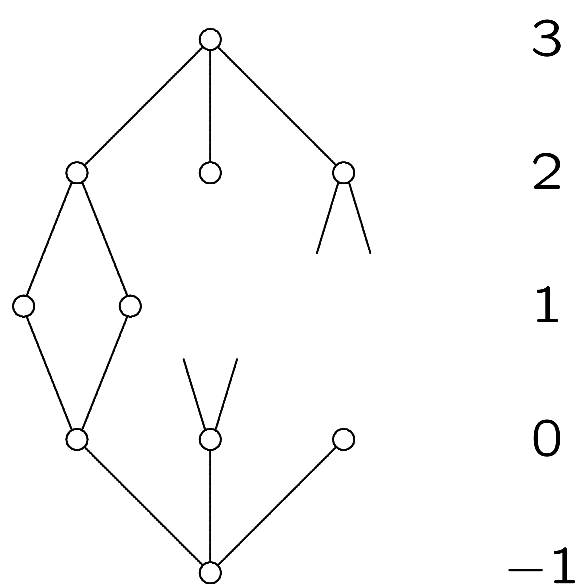
Abstract polytopes

An abstract polytopes is a generalised form of polytope, considered as a partially ordered set:





Definition



An abstract polytope of rank n is a partially ordered set $\mathcal{P}$ endowed with a strictly monotone rank function having range $\{-1, \ldots, n\}$. For $-1 \leq j \leq n$, elements of $\mathcal{P}$ of rank j are called the j-faces, and a typical j-face is denoted by F_j.

This poset must satisfy certain combinatorial conditions which generalise the properties of geometric polytopes.

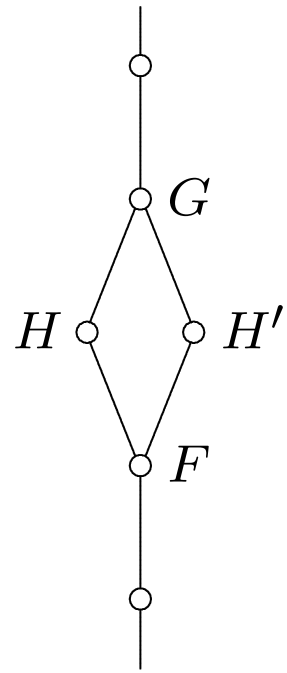
We require that $\mathcal{P}$ has a smallest (-1)-face F_{-1}, and a greatest n-face F_n, and that each maximal chain (or flag) of $\mathcal{P}$ has length $n+2$, e.g. $F_{-1} - F_0 - F_1 - F_2 - ... - F_{n-1} - F_n$.

The faces of rank $0, 1$ and $n-1$ are called the vertices, edges and facets (or co-vertices) of the polytope, respectively.

Two flags are called adjacent if they differ by just one face.

We require that $\mathcal{P}$ is strongly flag-connected, that is, any two flags Φ and Ψ of $\mathcal{P}$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$ such that each two successive faces Φ_{i-1} and Φ_i are adjacent, and $\Phi \cap \Psi \subseteq \Phi_i$ for all i.

Finally, we require the following homogeneity property, which is often called the diamond condition:



Whenever $F \leq G$, with rank$(F) = j-1$ and rank$(G) = j+1$, there are exactly two faces H of rank j such that $F \leq H \leq G$.

Symmetries of abstract polytopes

An automorphism of an abstract polytope $\mathcal{P}$ is an order-preserving bijection $\mathcal{P} \to \mathcal{P}$.

Just as for maps (on surfaces), every automorphism is uniquely determined by its effect on any given flag.

Regular polytopes

The number of automorphisms of an abstract polytope $\mathcal{P}$ is bounded above by the number of flags of $\mathcal{P}$.

When the upper bound is attained, we say that $\mathcal{P}$ is regular:

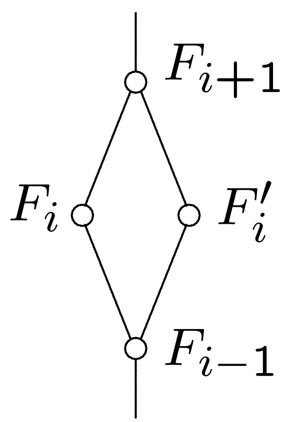
An abstract polytope $\mathcal{P}$ is regular if its automorphism group $\mathrm{Aut}\,\mathcal{P}$ is transitive (and hence regular) on the flags of $\mathcal{P}$.

Involutory 'swap' automorphisms

Let $\mathcal{P}$ be a regular abstract polytope, and let Φ be any flag $F_{-1} - F_0 - F_1 - F_2 - ... - F_{n-1} - F_n$. Call this the base flag.

For $0 \leq i \leq n-1$, there is an automorphism ρ_i that maps Φ to the adjacent flag Φ^i (differing from Φ only in its i-face).

Then also ρ_i also takes Φ^i to Φ (by the diamond condition), so ρ_i swaps Φ with Φ^i, hence ${\rho_i}^2$ fixes Φ, so ρ_i has order 2:



... ρ_i swaps F_i with F_i'
and fixes every other F_j

Connection with Coxeter groups

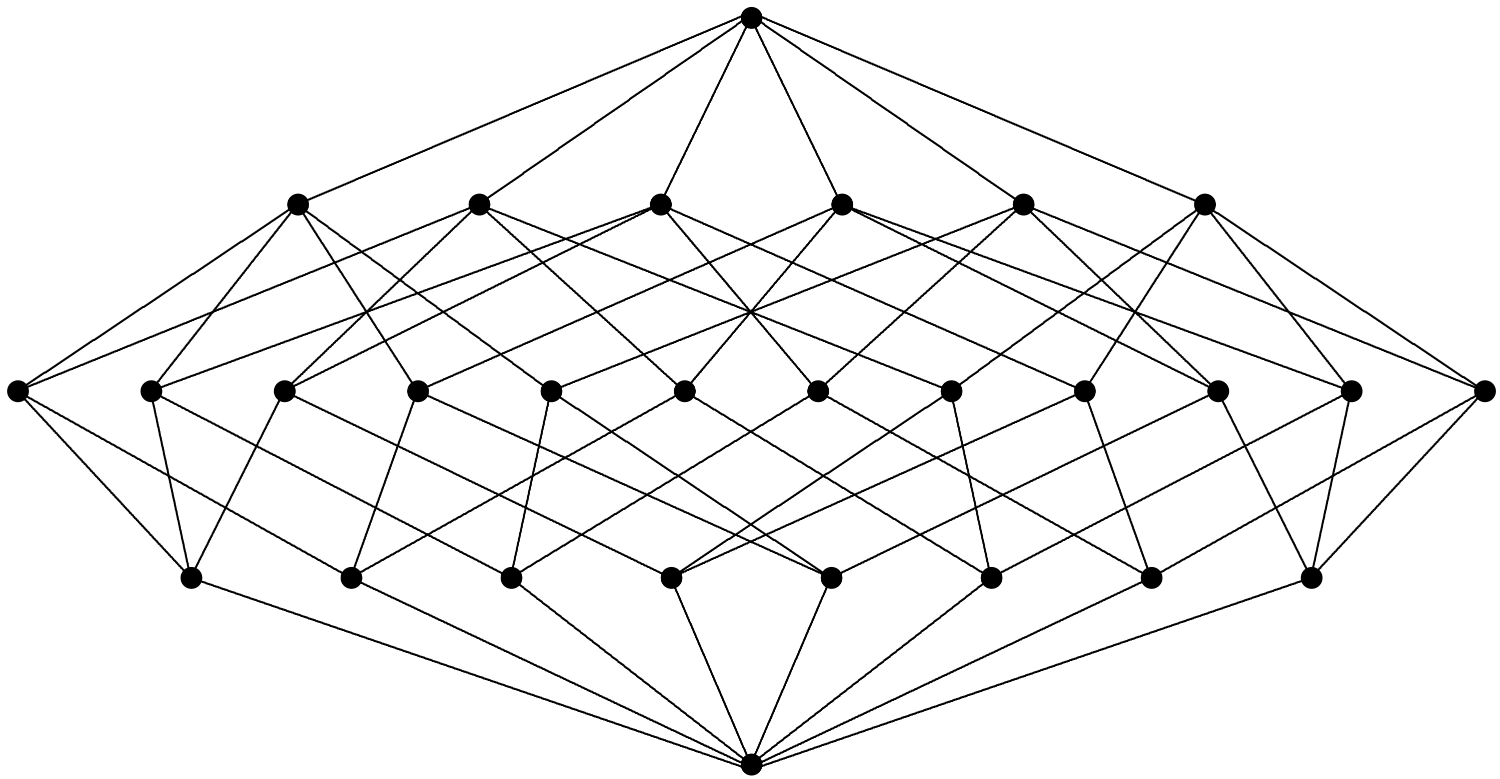
The automorphism group of any regular polytope $\mathcal{P}$ of rank n is generated by the 'swap' automorphisms $\rho_0, \rho_1, \ldots, \rho_{n-1}$, which satisfy the following relations

- $\rho_i^2 = 1$ for $0 \leq i \leq n-1$,
- $(\rho_{i-1}\rho_i)^{k_i} = 1$ for $1 \leq i \leq n-1$,
- $(\rho_i\rho_j)^2 = 1$ for $0 \leq i < i+1 < j \leq n-1$.

These are precisely the defining relations for the Coxeter group $[\,k_1, k_2, .., k_{n-1}]$ (with Schläfli symbol $\{\,k_1, k_2, .., k_{n-1}\}$). In particular, $\mathrm{Aut}\,\mathcal{P}$ is a quotient of this Coxeter group.

We usually call $\{\,k_1, k_2, ..., k_{n-1}\,\}$ the type of the polytope.

Example: the cube (a 3-polytope of type $\{3, 4\}$)



$\mathrm{Aut}(Q_3) \cong S_4 \times C_2$ is a quotient of the $[3, 4]$ Coxeter group

Stabilizers and cosets

$$\begin{aligned}
\mathrm{Stab}_{\mathrm{Aut}\,\mathcal{P}}(F_0) &= \langle \rho_1, \rho_2, \rho_3, \ldots, \rho_{n-2}, \rho_{n-1} \rangle \\
\mathrm{Stab}_{\mathrm{Aut}\,\mathcal{P}}(F_1) &= \langle \rho_0, \rho_2, \rho_3, \ldots, \rho_{n-2}, \rho_{n-1} \rangle \\
\mathrm{Stab}_{\mathrm{Aut}\,\mathcal{P}}(F_2) &= \langle \rho_0, \rho_1, \rho_3, \ldots, \rho_{n-2}, \rho_{n-1} \rangle \\
&\vdots \\
\mathrm{Stab}_{\mathrm{Aut}\,\mathcal{P}}(F_{n-2}) &= \langle \rho_0, \rho_1, \rho_2, \ldots, \rho_{n-3}, \rho_{n-1} \rangle \\
\mathrm{Stab}_{\mathrm{Aut}\,\mathcal{P}}(F_{n-1}) &= \langle \rho_0, \rho_1, \rho_2, \ldots, \rho_{n-3}, \rho_{n-2} \rangle
\end{aligned}$$

As $\mathcal{P}$ is flag-transitive, Aut $\mathcal{P}$ acts transitively on i-faces for all i, so i-faces can be labelled with cosets of $\mathrm{Stab}_{\mathrm{Aut}\,\mathcal{P}}(F_i)$, for all i, and incidence is given by non-empty intersection.

Also this can be reversed, giving a construction for regular polytopes from smooth quotients of (string) Coxeter groups, as for regular maps, but under certain extra assumptions ...

The Intersection Condition

When $\mathcal{P}$ is regular, the generators ρ_i for $\mathrm{Aut}\,\mathcal{P}$ satisfy an extra condition known as the intersection condition, namely

$$\langle \rho_i : i \in I \rangle \ \cap \ \langle \rho_i : i \in J \rangle \ = \ \langle \rho_i : i \in I \cap J \rangle$$

for every two subsets I and J of the index set $\{0, 1, \ldots, n-1\}$.

Conversely, this condition on generators $\rho_0, \rho_1, \ldots, \rho_{n-1}$ of a quotient of a Coxeter group $[\,k_1, k_2, .., k_{n-1}]$ ensures the diamond condition and strong flag connectedness. Hence:

If G is a finite group generated by n elements $\rho_0, \rho_1, \ldots, \rho_{n-1}$ which satisfy the defining relations for a string Coxeter group of rank n, with orders of the ρ_i and products $\rho_i \rho_j$ preserved, and these generators ρ_i satisfy the intersection condition, then there exists a regular polytope $\mathcal{P}$ with $\mathrm{Aut}\,\mathcal{P} \cong G$.

Chiral polytopes

If the automorphism group Aut $\mathcal{P}$ has two orbits on flags, such that adjacent flags always lie in different orbits, then the polytope $\mathcal{P}$ is said to be chiral. (This is like a chiral map: if f and f' are the two faces containing a given arc (v, e), then the flags (v, e, f) and (v, e, f') lie in different orbits.)

The automorphism group of every chiral n-polytope $\mathcal{P}$ is a smooth quotient of the orientation-preserving subgroup $[k_1, k_2, .., k_{n-1}]^+ = \langle \rho_0\rho_1, \rho_1\rho_2, ..., \rho_{n-2}\rho_{n-1} \rangle$ of the relevant Coxeter group $[k_1, k_2, .., k_{n-1}]$, and $\{k_1, k_2, .., k_{n-1}\}$ is its type. There is also an analogue of the intersection condition, which can be used to construct examples.

Construction of small regular polytopes

All ‘small’ regular polytopes can be found/constructed via their automorphism groups, which are (smooth) quotients of string Coxeter groups $[\,k_1, k_2, .., k_{n-1}]$.

Michael Hartley used the database of small finite groups to find all regular polytopes with N flags, where $1 \le N \le 2000$ but $N \neq 1024, 1536$. See `www.abstract-polytopes.com/atlas`.

This approach, however, is limited by the database, and the very large numbers of 2-groups and $\{2,3\}$-groups of small order. A much more effective approach is to use ‘low index subgroup’ methods (applied to the Coxeter groups).

Atlas of small chiral and regular polytopes

[Joint work with Dimitri Leemans (2012/13)]

We now have complete lists of all regular polytopes with up to 4000 flags and all chiral polytopes with up to 4000 flags.

These come from an initial computation for rank 3, and then increasing ranks, using the following consequence of the intersection condition:

if $\mathcal{P}$ is a regular/chiral polytope of type $\{k_1, k_2, ..., k_{n-1}\}$, and its facets have m flags, then $\mathcal{P}$ has at least mk_{n-1} flags.

Up to 4000 flags, the largest rank for regular is 6, and the largest rank for chiral is 5. [Website yet to be created.]

The smallest regular polytopes

For each $n \geq 3$, what are the regular n-polytopes with the smallest numbers of flags? [Daniel Pellicer (Oaxaca, 2010)]

Answer [MC (*Adv. Math.* (2013), described at Fields (2011)]

For every $n \geq 9$, the smallest regular n-polytope is a unique polytope of type $\{4, \overset{n-1}{\ldots}, 4\}$, with $2 \cdot 4^{n-1}$ flags. The smallest ones are known (exactly) also for $n \leq 8$.

Lemma: If $\mathcal{P}$ is a regular n-polytope, of type $\{k_1, \ldots, k_{n-1}\}$, then # of flags of P $= |\mathrm{Aut}(\mathcal{P})| \geq 2k_1k_2 \ldots k_{n-1}$.

If this lower bound is attained, we say that $\mathcal{P}$ is tight.

Tight regular polytopes

If $\mathcal{P}$ is a tight regular polytope of type $\{k_1, \ldots, k_{n-1}\}$, then every regular sub-polytope of $\mathcal{P}$ is tight.

Also if $n \geq 3$ then $2k_1k_2 \ldots k_{n-1} = |\mathrm{Aut}(\mathcal{P})|$ is even (since two of the generators of $\mathrm{Aut}(\mathcal{P})$ are commuting involutions), so at least one k_i is even, indeed no two consecutive k_i can be odd. [Observations made by Gabe Cunningham]

Theorem [GC (2013)] If m and k are not both odd, then there exists a tight regular 3-polytope of type $\{m, k\}$.

Gabe also conjectured that if k is odd and $m > 2k$ (or vice versa), there is no tight regular 3-polytope of type $\{m, k\}$.

Tight regular polytopes (cont.)

Theorem [MC & GC (2013)] There exists a tight regular 3-polytope of type $\{m, k\}$ if and only if

- m and k are both even, or
- m is odd and k divides $2m$, or k is odd and m divides $2k$.

The proof relies on a connection with some work by MC and Tom Tucker on regular Cayley maps for cyclic groups, or more generally, on groups having a factorisation $G = AB$ where the subgroups A and B are cyclic and $A \cap B = \{1\}$.

Corollary [MC & GC (2013)] There exists a tight orientable regular polytope of type $\{k_1, \ldots, k_{n-1}\}$ if and only if k_j is an even divisor of $2k_i$ whenever k_i is odd and $j = i \pm 1$.

More on the intersection condition

$$\langle\, \rho_i : i \in I \,\rangle \cap \langle\, \rho_i : i \in J \,\rangle = \langle\, \rho_i : i \in I \cap J \,\rangle$$

The intersection condition involves up to $\binom{2^n}{2}$ pairs (I, J).

Question: How many of these need to be checked?

For some of them, the IC is always satisfied (e.g. if $I \subseteq J$).

For rank 2, we need only check the pair $(I, J) = (\{1\}, \{2\})$, and for that, the IC never fails when the quotient is smooth.

For rank $n > 2$, there are inductive processes for determining a minimal set of pairs that need to be checked (see P. McMullen and E. Schulte, *Abstract Regular Polytopes*, Cambridge (2002)).

The rank 3 case

Rank 3 polytopes are simply **non-degenerate maps** (where non-degeneracy follows the diamond condition).

Theorem: Let G be any finite group generated by three involutions a, b, c such that ab, bc, ac have orders k, m, 2, where $k \geq 3$ and $m \geq 3$. Then either G is the automorphism group of a regular 3-polytope, or G has non-trivial cyclic normal subgroup N (contained in $\langle ab \rangle$ or $\langle bc \rangle$).

Sketch proof. By smoothness, there is really only one pair (I, J) to check, namely $(\{0, 1\}, \{1, 2\})$. If the IC fails for that pair, then some non-trivial element of $\langle ab \rangle$ or $\langle bc \rangle$ generates a normal subgroup of G. □

Corollary 1: If G is a finite simple group, or more generally, has no non-trivial cyclic normal subgroups (e.g. A_n or S_n for some n), then G is the automorphism group of a regular 3-polytope of type $\{m, k\}$ whenever G is a smooth quotient of the $[m, k]$ Coxeter group.

Corollary 2: For every non-negative integer g, there exists a polytopal regular map on an orientable surface of genus g. (In other words, for every such g there exists a fully regular orientable map of genus g that is also a 3-polytope.)

Proof. There exists a family of groups G_n of order $16n$ (for $n \in \mathbb{Z}^+$), with each G_n being a smooth quotient of the $[4, 2n]$ Coxeter group, and they satisfy the IC since they have no cyclic normal subgroups of the kind given by the theorem.

These are 'Accola-Maclachlan' maps $\mathcal{AM}_n$ (of genus $n-1$).

The rank 4 case (Joint work with **Deborah Oliveros**)

For rank 4, easy observations show there are just four pairs (I, J) for which the intersection condition has to be checked:

$(I, J) = (\{0, 1\}, \{1, 2\})$, as in the rank 3 test;

$(I, J) = (\{0, 1, 2\}, \{3\})$;

$(I, J) = (\{0, 1, 2\}, \{2, 3\})$;

$(I, J) = (\{0, 1, 2\}, \{1, 2, 3\})$.

For some types, many (and sometimes all) of these cases can be easily eliminated. For example, if the type is $\{k_1, k_2, k_3\}$ and the IC fails for $(\{0, 1, 2\}, \{2, 3\})$, then $\langle \rho_0, \rho_1, \rho_2 \rangle \cap \langle \rho_2, \rho_3 \rangle$ is a subgroup of $\langle \rho_2, \rho_3 \rangle \cong D_{k_3}$ strictly containing $\langle \rho_2 \rangle \cong C_2$. So if k_3 is prime then this is $\langle \rho_2, \rho_3 \rangle$, and $\rho_3 \in \langle \rho_0, \rho_1, \rho_2 \rangle$.

Amazing Theorem 1 (for type {3,5,3})

Every smooth homomorphism ψ from the $[3, 5, 3]$ Coxeter group onto a finite group G gives a regular 4-polytope $\mathcal{P}$ of type $\{3, 5, 3\}$ with $\mathrm{Aut}(\mathcal{P}) \cong G$, except in precisely one case, where $G = \mathrm{PSL}(2, 11) \times C_2$ and the ψ-images $\rho_0, \rho_1, \rho_2, \rho_3$ in G of the standard Coxeter generators satisfy the relation $(\rho_0\rho_1\rho_2)^5 = (\rho_1\rho_2\rho_3)^5$.

The proof shows that the IC holds for all four of the critical pairs (I, J), except $(\{0, 1, 2\}, \{1, 2, 3\})$, for which a failure requires $(\rho_0\rho_1\rho_2)^5 = (\rho_1\rho_2\rho_3)^5$ to be a central involution.

Remarkably, adding that relation to the $[3, 5, 3]$ Coxeter group gives quotient $\mathrm{PSL}(2, 11) \times C_2$, and this is the only possible exception.

Amazing Theorem 2 (for type {5,3,5})

Every smooth homomorphism ψ from the $[5,3,5]$ Coxeter group onto a finite group G gives a regular 4-polytope $\mathcal{P}$ of type $\{5,3,5\}$ with $\mathrm{Aut}(\mathcal{P}) \cong G$, except in precisely one case, where $G = \mathrm{PSL}(2,19) \times C_2$ and the ψ-images $\rho_0, \rho_1, \rho_2, \rho_3$ in G of the standard Coxeter generators satisfy the relation $(\rho_0\rho_1\rho_2)^5 = (\rho_1\rho_2\rho_3)^5$.

Curiously, Dimitri Leemans and Egon Schulte showed in 2007 that the only regular polytopes of rank 4 or more with automorphism group $\mathrm{PSL}(2,q)$ for some q are Grünbaum's 11-cell of type $\{3,5,3\}$ for $\mathrm{PSL}(2,11)$, and Coxeter's 57-cell of type $\{5,3,5\}$ for $\mathrm{PSL}(2,19)$.

There's a **similar theorem for {4,3,5}**, with no exceptions.

Corollary: For all but finitely many positive integers n, the alternating group A_n and the symmetric group S_n are automorphism groups of at least one regular 4-polytope of each of the types $\{3, 5, 3\}$, $\{4, 3, 5\}$ and $\{5, 3, 5\}$.

Proof. It is known (MC, Martin & Torstensson (2006)) that all but finitely many A_n are smooth quotients of $[3, 5, 3]$. The same can be shown to hold also for S_n, and for the types $\{4, 3, 5\}$ and $\{5, 3, 5\}$ as well.

Note that this does not hold for the other locally spherical type $\{k_1, k_2, k_3\}$ for which the Coxeter group $[k_1, k_2, k_3]$ is infinite, namely $\{4, 3, 4\}$, because the $[4, 3, 4]$ Coxeter group is solvable, having a free abelian normal subgroup of index 48 with quotient $S_4 \times C_2$. But because of the latter fact, the regular 4-polytopes of type $\{4, 3, 4\}$ are all known.

Obvious question

Is it true that whenever the Coxeter group $[k_1, \ldots, k_d]$ is infinite and insoluble, all but finitely many alternating and symmetric groups are the automorphism group of at least one regular $(d+1)$-polytope of type $\{k_1, k_2, k_d\}$?

For rank 3, this is more-or-less known to be true (by work of Everitt (2000) on alternating quotients of Fuchsian groups).

What about higher ranks? [Open question]

Constructions for chiral polytopes

Until 8 years ago, the only known finite chiral polytopes had ranks 3 and 4. Then Isabel Hubard, Tomaž Pisanski & MC found some (small) examples of rank 5, and then later, Alice Devillers & MC constructed examples of ranks 6, 7 and 8.

This was all surpassed spectacularly by the following

Theorem [Daniel Pellicer (2010)]: For every $d \geq 3$, there exists a finite chiral polytope of rank d.

It is easy to prove that if $\mathcal{P}$ is a chiral polytope of rank d, then its sub-polytopes of rank $d-2$ are regular (not chiral), so a recursive construction is impossible. Daniel Pellicer's proof involved a construction for chiral polytopes with prescribed regular facets. But these can be very large!

Chiral polytopes of type $\{3,3,\ldots,3,k\}$

The regular d-simplex is a regular polytope of rank d and type $\{3, 3, ..., 3\}$, with automorphism group S_{d+1}.

Alice Devillers and MC constructed chiral polytopes of ranks 6, 7 and 8 with type $\{3, 3, ..., 3, k\}$ for various k, having facets isomorphic to the regular 5-, 6- and 7-simplex respectively.

These chiral polytopes can be analysed by expressing the automorphism group as a transitive permutation group, and then considering the (sub-)orbits of the stabiliser of a facet. Since each facet is a regular d-simplex, of type $\{3, 3, ..., 3\}$, its stabiliser (in the automorphism group) is the alternating group A_{d+1}. This gives a way to construct new examples — namely from permutation representations of A_{d+1}.

Theorem [Hubard, O'Reilly-Regueiro, Pellicer & MC]
For all but finitely many n, there exists a chiral 4-polytope $\mathcal{P}$ of type $\{3,3,k\}$ for some k, with Aut $\mathcal{P} \cong$ Alt(n) or Sym(n).

'Almost' theorem [same people as above]
For all $d \geq 4$, there are infinitely many n for which there exists a chiral d-polytope $\mathcal{P}$ of type $\{3,3,..,3,k\}$ for some k, with Aut $\mathcal{P} \cong$ Alt(n) or Sym(n).

Conjecture/challenge [MC]
For all $d \geq 4$, 'infinitely many' can be replaced by 'all but finitely many'.

Thank You

Title: An update on polytopes with many symmetries

Speaker: Marston Conder, University of Auckland, N.Z.

Abstract: In this talk I'll give a brief update on some recent discoveries about regular and chiral polytopes, including

- the smallest regular polytopes of given rank
- simplifications/applications of the intersection condition
- computer-assisted determination of all regular and chiral polytopes with up to 4000 flags
- conditions on the Schläfli type $\{p_1, p_2, ..., p_n\}$ for the existence of a tight regular n-polytope (with $2p_1p_2 \ldots p_n$ flags)
- chiral polytopes of type $\{3, 3, \ldots, 3, k\}$ (with some A_n or S_n as automorphism group).

Pieces of these involve joint work with Gabe Cunningham, Isabel Hubard, Dimitri Leemans, Deborah Oliveros, Eugenia O'Reilly Regueiro and Daniel Pellicer.