A complete certificate universal rigidity

Bob Connelly and Shlomo Gortler

Cornell University and Harvard University

connelly@math.cornell.edu and sjg@cs.harvard.edu

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Global Rigidity

Given a bar framework (G, p) in E^d, how do you tell when the bar (distance) constraints determine the configuration up to rigid congruences? When this happens (G, p) is called *globally rigid in* E^d.

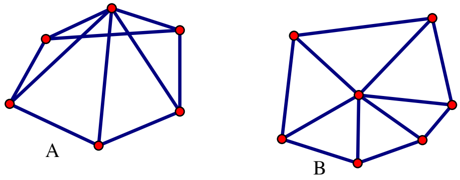


Figure: Framework A is globally rigid in E^2, as is Framework B, but Framework B has to solve a subset-sum problem for the angles at the central vertex to certify it. Framework A is globally rigid in all Euclidean spaces $E^D \supset E^2$ and can be certified by checking the positive definite property and rank of a symmetric $n \times n$ matrix.

Generic Global Rigidity

Theorem (Connelly (2005) and Gortler-Healy-Thurston (2010))

A bar framework (G, p) in E^d, for $n \geq d$, is globally rigid at a generic configuration $p = (p_1, \ldots, p_n)$ if and only if there is some configuration q, where the rigidity matrix $R(q)$ has maximal rank $nd - d(d+1)/2$ and a stress matrix Ω of maximal rank $n - (d+1)$.

Irony: (G, q) may not be globally rigid. Although "almost all" configurations are generic, it seems to be computationally infeasible to be able to detect the appropriately non-generic configurations.

But what about a non-generic configuration? This could be just about any configuration, since you don't know what generic means.

Universal Rigidity

There is a class of frameworks where the certificate for global rigidity is quite feasible.

Definition

- A framework (G, p) in E^d is *universally rigid* if it is globally rigid (or equivalently locally rigid) in any $E^D \supset E^d$.
- A *stress* $\omega = (\dots, \omega_{ij}, \dots)$ for the graph G is an assignment of a scalar $\omega_{ij} = \omega_{ji}$ for each pair of vertices $\{i, j\}$ in G, such that $\omega_{ij} = 0$, when there is no edge (member) between vertex i and vertex j.
- The *stress-energy* E_ω is a quadratic form associated to any stress ω defined on the space of all configurations p by

$$E_\omega(p) = \sum_{i<j} \omega_{ij}(p_i - p_j)^2.$$

Rigidity Modulo Affine Motions

The stress-energy has a matrix representation such that $E_\omega = \Omega \otimes I^D$, where Ω is an $n \times n$ symmetric matrix, where the $\{i, j\}$ coordinate is $-\omega_{ij}$, for $i \neq j$, and the row and column sums are 0.

Theorem (Connelly (1980))

If the framework (G, p) in E^d has a stress ω such that Ω is positive semi-definite (PSD) of rank $n - d - 1$, while p is one of the minimum (critical equilibrium) configurations for E_ω, then any other framework (G, q) with the same corresponding member lengths is such that q is an affine image of p.

Conic at infinity

Definition

For a framework (G, p) in E^d the vectors $\{p_i - p_j\}$, $\{i, j\}$ members in G, determine points in the projective space $\mathbb{RP}^{d-1}$ of lines through the origin in E^d. If those projective points lie on a conic, we say the *member directions for* (G, p) *lie on a conic at infinity*.

For example, in the plane a conic at infinity is just two points/directions.

Theorem

A framework (G, p) in E^d has a flex (continuous motion preserving the bar lengths) that consists of affine motions if and only if its member directions lie on a conic at infinity.

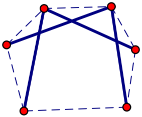
Super Stability

Theorem (Connelly (1980))

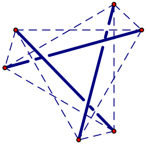
If the framework (G, p) in E^d has a stress ω such that Ω is positive semi-definite (PSD) of rank $n - d - 1$, while p is one of the minimum (critical equilibrium) configurations for E_ω, and the member directions do NOT lie on a conic at infinity, then it is universally rigid.

Definition

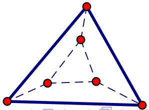
A framework that satisfies the conditions of the Theorem above is called *super stable*.



Cauchy Polygon



Snelson



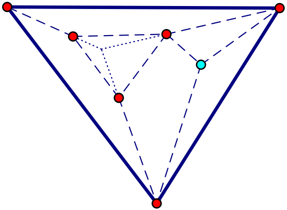
Desargues

More Universally Rigid Frameworks

Are all universally rigid frameworks super stable?

More Universally Rigid Frameworks

Are all universally rigid frameworks super stable? **No!**

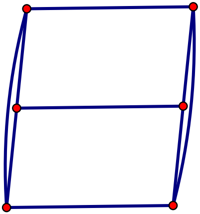


The members adjacent to the blue vertex all have zero stress, and the critical stress matrix has rank $7 - 3 - 1 = 3$, one less than needed for superstability. Nevertheless, after removing that vertex, the rest of the framework is a Desargues' configuration and is super stable by itself. Then attaching that extra vertex preserves universal rigidity. This is an example of a spider web, and all rigid spider webs are universally rigid by this method.

Definition (Alfakih (2007))

A framework (G,p) is called *dimensionally rigid* if the dimension of affine span of the vertices of $p = (p_1, \ldots, p_n)$ is maximal among all configurations of (G,q) with corresponding bar lengths the same as (G,p).

Note that a dimensionally rigid framework may not even be rigid.



Universal Rigidity Revisited

Corollary

If the framework (G, p) in E^d has a stress ω such that Ω is positive semi-definite (PSD) of rank $n - d - 1$, while p is one of the minimum (critical equilibrium) configurations for E_ω, then any other framework (G, q) with the same corresponding member lengths is such that q is an affine image of p and thus is dimensionally rigid.

Theorem (Alfakih (2007))

If the framework (G, p) in E^d is dimensionally rigid in E^d, and (G, q) has corresponding member lengths the same, then q is an affine image of p. So if, additionally, the member directions of (G, p) do not lie on a conic at infinity, then (G, p) is universally rigid.

Affine sets

The space of all configurations in E^D, $\mathcal{C}$, is naturally identified with the $(\mathbb{R}^D)^n = \mathbb{R}^{Dn}$.

Definition

A subset $\mathcal{A} \subset \mathcal{C}$ is called an *affine set*, if it is the finite intersection

$$\{p \in \mathcal{C} \mid \sum_{ij} \lambda_{ij}(p_i - p_j) = 0\},$$

for some set $\{\ldots, \lambda_{ij} = \lambda_{ji}, \ldots\}$.

For example, any set of three collinear points p_1, p_2, p_3, where p_2 is the midpoint of p_1 and p_3, is an affine set. Or a configuration of four points of a parallelogram, possibly degenerate, is another example. In general, an affine set is a subset of the configuration space E^D that is determined by linear constraints on configuration vectors such that it is closed under arbitrary affine transformations.

Universal Configurations for Affine Sets

Definition

A configuration p is *universal for an affine set* $\mathcal{A}$ if its affine span is of maximal dimension among all configurations q in $\mathcal{A}$.

Lemma (Universality Property)

If the configuration p is universal for the affine set $\mathcal{A}$, and q is another configuration in $\mathcal{A}$, then q is an affine image of p.

Proof.

Define $\tilde{p}$ to be another configuration where $\tilde{p}_i = (p_i, q_i)$ in $\mathbb{R}^D \times \mathbb{R}^D$ for $i = 1, \ldots, n$. The configuration $\tilde{p}$ is also in $\mathcal{A}$ since all its coordinates satisfy the same equations. Since projection is an affine linear map and the affine span of p is maximal, the dimension of the affine span of $\tilde{p}$ must also be maximal, and the projection between their spans must be an isomorphism. So the map $p \to \tilde{p} \to q$ provides the required affine map. $\square$

The Rigidity Map and the Measurement Cone

Definition

The *rigidity map* $f : \mathcal{C} \to R^m$ is the function defined by

$$f(p) = (\dots, (p_i - p_j)^2, \dots),$$

where $\{i, j\}$ is the edge in G corresponding to the coordinate of $\mathcal{M}$.

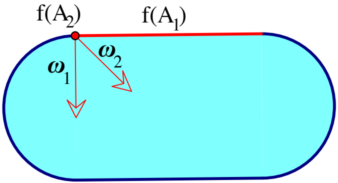
For a graph G, the *measurement cone* $\mathcal{M} = f(\mathcal{C}) \subset R^m$. So the configurations p and q have the same member lengths for G if and only if $f(p) = f(q)$.

Theorem

For $D \geq n$, and any affine set $\mathcal{A}$, the image $f(\mathcal{A}) \subset \mathcal{M}$ is convex.

The Convexity Argument

For an affine set $\mathcal{A}$, a supporting hyperplane $\mathcal{H}$ for $f(\mathcal{A})$ corresponds to the zero-set/kernel of an appropriate PSD stress-energy form E_ω, since $E_\omega(p) = \omega f(p)$, where ω is a stress for (G, p).



So we can find a flag of affine sets $\mathcal{C} = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \ldots \mathcal{A}_k$ in configuration space that corresponds to a flag of faces $\mathcal{M} = f(\mathcal{A}_0) \supset f(\mathcal{A}_1) \supset f(\mathcal{A}_2) \supset \ldots f(\mathcal{A}_k)$ in the measurement cone that converges to the face that contains $f(p)$. This is called *facial reduction* in Borwein-Wolkowicz.

The Main Theorem

Definition

We say that a sequence of affine sets $\mathcal{C} = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \dots \mathcal{A}_k$ are *stress supported* if it has a corresponding sequence of stress energy functions, for the graph G, $E_1, \dots, E_k$, such that each E_i is restricted to $\mathcal{A}_{i-1}$, is PSD and $\mathcal{A}_i = E_i^{-1}(0)$. We call this sequence, an *iterated affine sequence*.

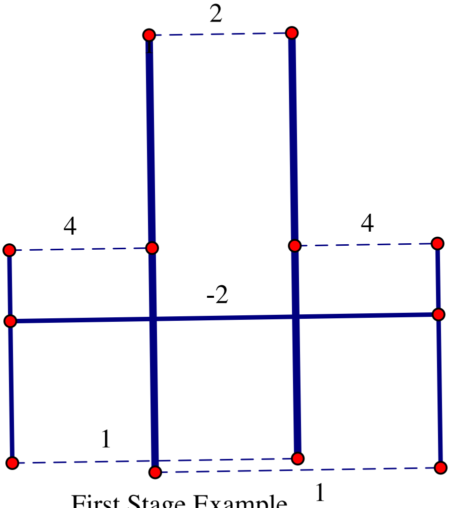
Theorem

A framework (G, p) is dimensionally rigid if and only if there is a stress supported iterated affine sequence $\mathcal{C} = \mathcal{A}_0 \supset \mathcal{A}_1 \supset \mathcal{A}_2 \cdots \supset \mathcal{A}_k$, where the p is a universal configuration for $\mathcal{A}_k$.

Corollary

A framework (G, p) is universally rigid if and only if, in addition, the member directions do not lie on a conic at infinity.

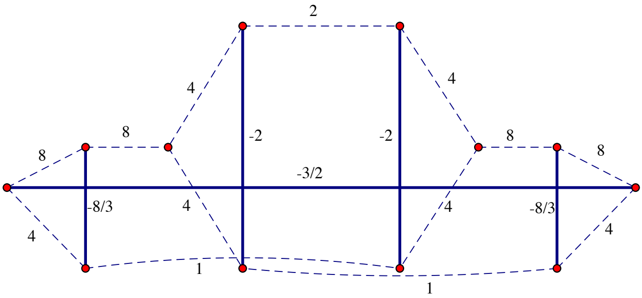
Examples



First Stage Example

Another Example

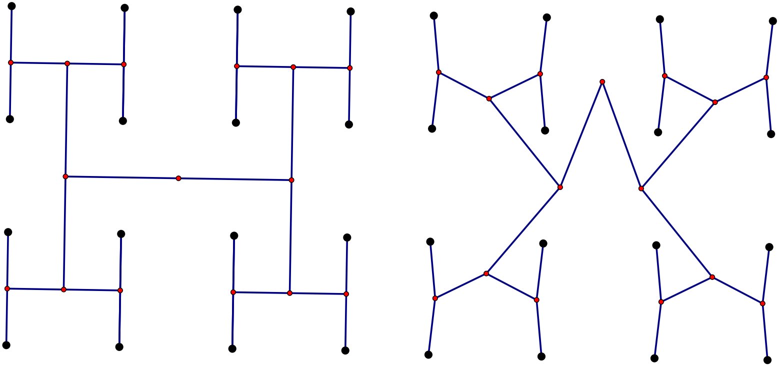
This is a uniformly rigid two-step example with no subgraph that is super stable.



Semi-Definite Programming

Another approach to determining universal rigidity is to take the given member lengths and apply an algorithm that uses semi-definite programming (SDP) to find a configuration with a maximal dimensional affine span for the given edge lengths, starting with (G, p). If it returns the configuration p again, you can conclude that (G, p) is universally rigid. The problem is that this process only converges to a dimensionally rigid example, and the measure of success is how close the calculated lengths are to the given lengths, which can be problematic as the following example shows. The question of whether there is an "algorithm" to "compute" universal rigidity is partly tied up with the question of how the configuration itself is defined. Is the problem itself well-defined?

A Disturbing Example



The black vertices are pinned, while the members on the right have been increased by less than 0.5%.

SDP Applied to Stresses

However, one can use SDP to find PSD stresses for a given configuration. If the space of critical PSD stresses for a given framework is open in the space of all stresses, then a random search for such a stress will be successful with positive probability, at least. The 4-pole examples above have only a one-dimensional space of equilibrium stresses, which assures success.

Another approach is to create configurations with PSD stresses. This method, part of the general process known as "form finding", can be used to find super stable highly symmetric configurations for tensegrities.

Tensegrities

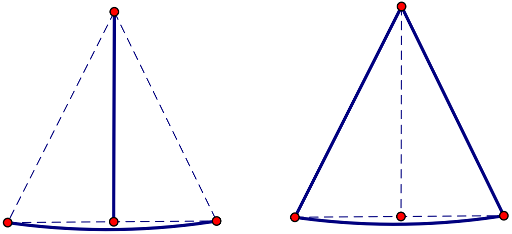
Definition

A *tensegrity* is a framework (G, p), where each edge (member) of G is defined to be a strut, cable, or bar. Struts are not allowed to decrease in length, cables are not allowed to increase in length, and bars are not allowed to change in length.

In the figures here, cables are indicated by dashed line segments, while struts and bars are indicated by solid line segments. If the stress energy forms have a stress where $\omega_{ij} > 0$, then that member can be assumed to be a cable, and when $\omega_{ij} < 0$, it can be assumed to be a strut. The main result can be extended to the case of tensegrities.

A Tensegrity Example

The following is an example of a tensegrity where some of the cables an struts designations can be reversed, while in each case the tensegrity is universally rigid in two steps. The degenerate triangles are indicated such that the strut is bent slightly.



The second step energy form is PSD even though it is identically zero and the stress for one example is the negative of the other. But there are enough directions to apply the last step to eliminate the affine motions.

A Projective Transformations

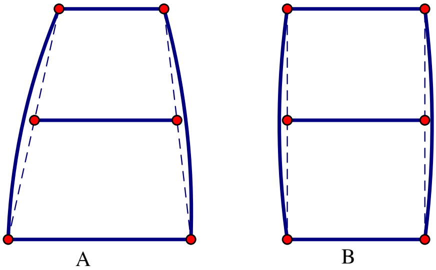
From the proof of the main theorem, it is easy to see that a projective transformation preserves the definiteness of the energy forms. This gives the following.

Theorem

If a framework (G, p) dimensionally rigid and the configuration q is a non-singular projective image of p, then (G, q) is dimensionally rigid as well.

The following example shows that this is not true for universal rigidity.

The Orchard Ladder



Framework/tensegrity A is dimensionally rigid in the plane, and framework/tensegrity B is a non-singular projective image which is not dimensionally rigid, since it has only two member directions, and thus has an affine flex in the plane.

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