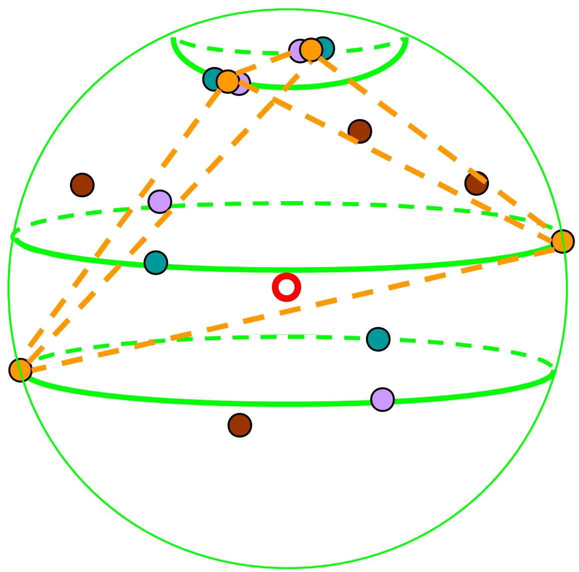
Colourful linear programming



Antoine Deza (McMaster)

based on joint work with

Frédéric Meunier (Paris Est)

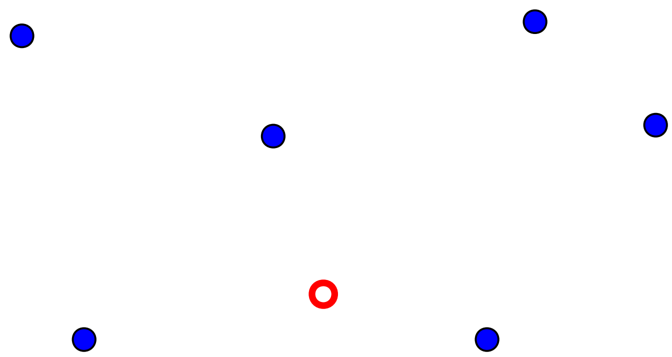
Pauline Sarrabezolles (Paris Est)

Tamon Stephen (Simon Fraser)

Tamás Terlaky (Lehigh)

Feng Xie (Microsoft)

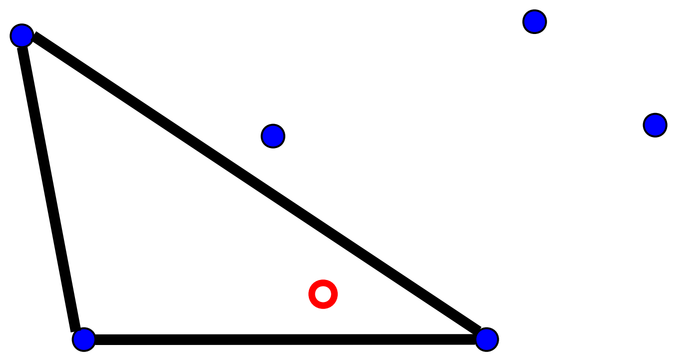
Carathéodory Theorem



Given a set S of n points in dimension d, then there exists an open simplex generated by points in S containing p

S, p *general position*

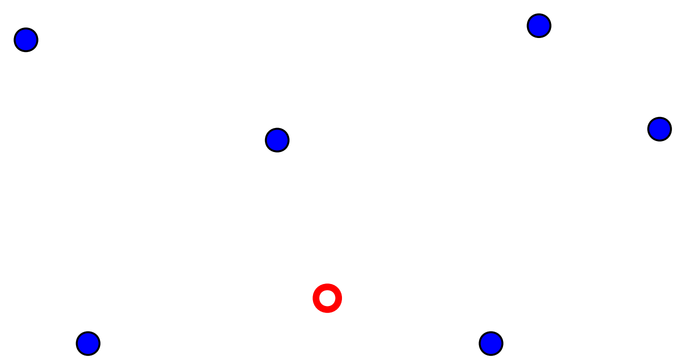
Carathéodory Theorem



Given a set S of n points in dimension d, then there exists an open simplex generated by points in S containing p

S, p general position

Simplicial Depth

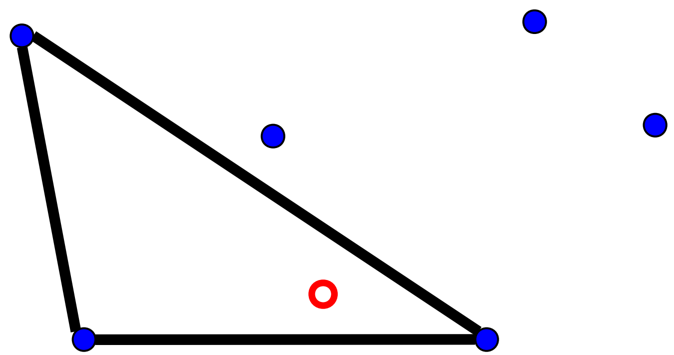


Given a set *S* of *n* points in dimension *d*, the *simplicial depth* of *p* is the number of open simplices generated by points in *S* containing *p* [Liu 1990]

S, *p* *general position*

Simplicial Depth

depth$_S$(*p*)=1

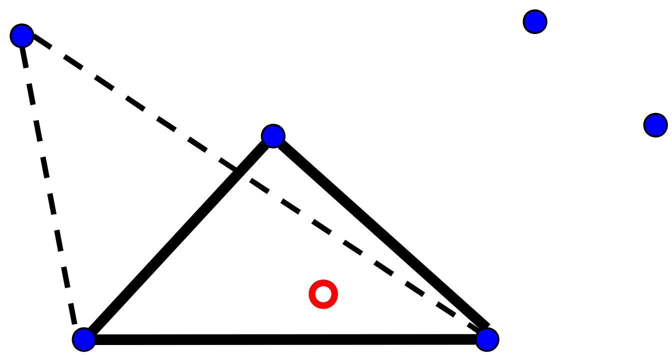


Given a set *S* of *n* points in dimension *d*, the *simplicial depth* of *p* is the number of open simplices generated by points in *S* containing *p* [Liu 1990]

S, *p general position*

Simplicial Depth

$depth_S(p)=2$

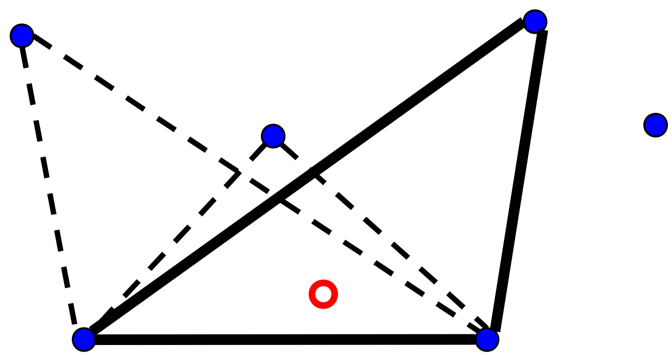


Given a set S of n points in dimension d, the *simplicial depth* of p is the number of open simplices generated by points in S containing p [Liu 1990]

S, p *general position*

Simplicial Depth

$depth_S(p)=3$

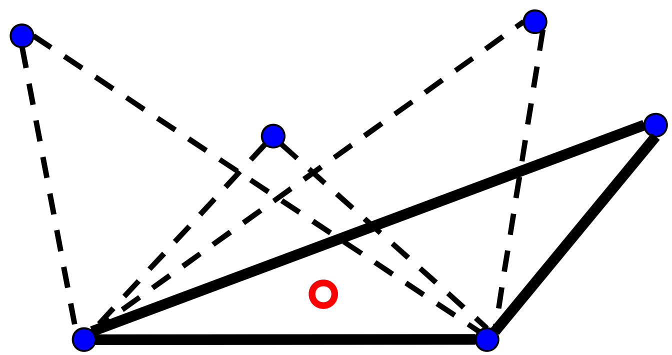


Given a set S of n points in dimension d, the *simplicial depth* of p is the number of open simplices generated by points in S containing p [Liu 1990]

S, p general position

Simplicial Depth

$depth_S(p) = 4$



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S, p *general position*

Simplicial Depth

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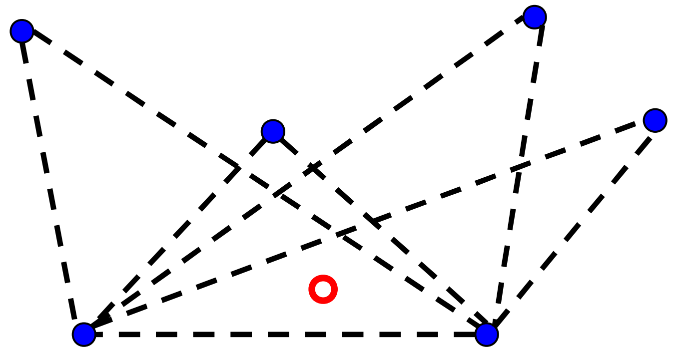
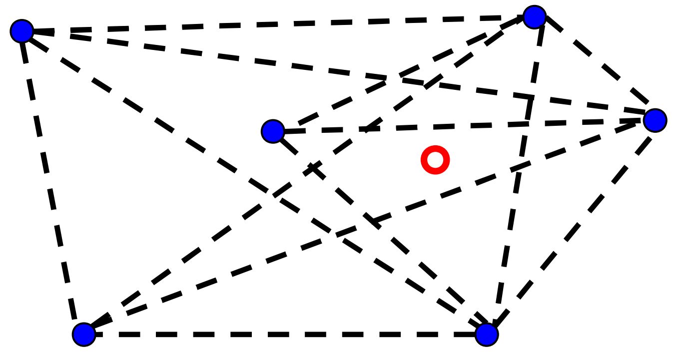


Given a set S of n points in dimension d, the *simplicial depth* of p is the number of open simplices generated by points in S containing p [Liu 1990]

S, p general position

Simplicial Depth

depth$_S$(p)=9



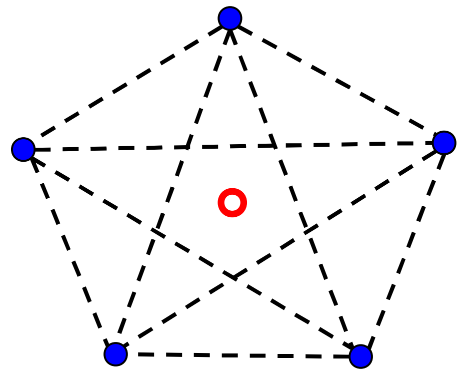
Given a set S of n points in dimension d, the *simplicial depth* of p is the number of open simplices generated by points in S containing p [Liu 1990]

S, p *general position*

Deepest Point in Dimension 2

Deepest point bounds in dimension 2 [Kárteszi 1955], [Boros, Füredi 1984], [Bukh, Matoušek, Nivasch 2010]

$$\frac{n^3}{27} + O(n^2) \leq \max_p depth_S(p) \leq \frac{n^3}{24} + O(n^2)$$



$depth_S(p)=5$

S, p general position

Deepest Point in Dimension d

Deepest point bounds in dimension *d* [Bárány 1982]

$$\frac{1}{(d+1)^{d+1}}\binom{n}{d+1} + O(n^d) \leq \max_p depth_S(p) \leq \frac{1}{2^d(d+1)!} n^{d+1} + O(n^d)$$

S, *p* *general position*

Deepest Point in Dimension d

Deepest point bounds in dimension d [Bárány 1982]

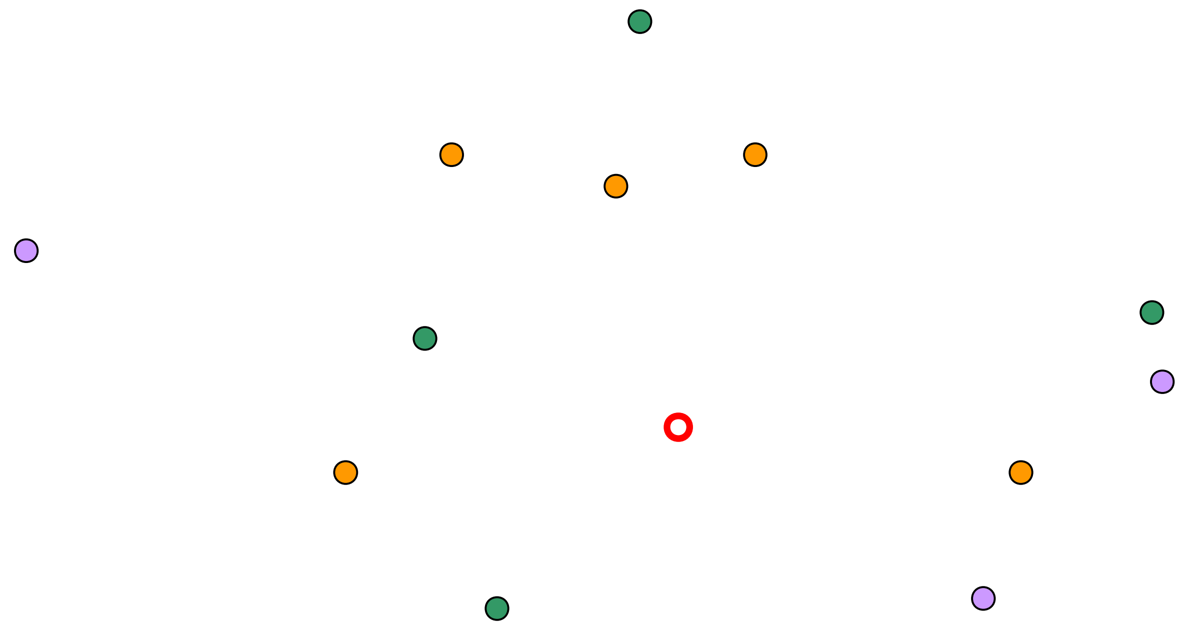
$$\frac{1}{(d+1)^{d+1}}\binom{n}{d+1} + O(n^d) \leq \max_{p} depth_S(p) \leq \frac{1}{2^d (d+1)!} n^{d+1} + O(n^d)$$

- tight upper bound
- lower bound uses Colourful Carathéodory theorem

… breakthrough [Gromov 2010] & further improvements

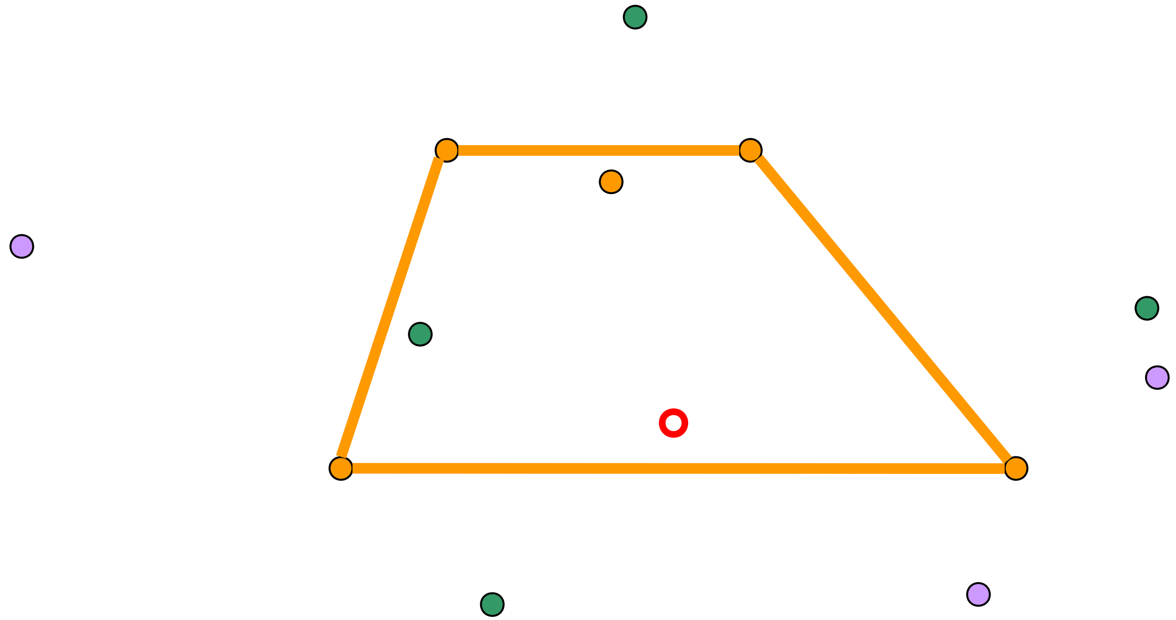
S, p general position

Colourful Carathéodory Theorem



Given colourful set $\boldsymbol{S} = S_1 \cup S2 \ldots \cup S_{d+1}$ in dimension d, and $p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$, there exists a colourful simplex containing p [Bárány 1982]

Colourful Carathéodory Theorem



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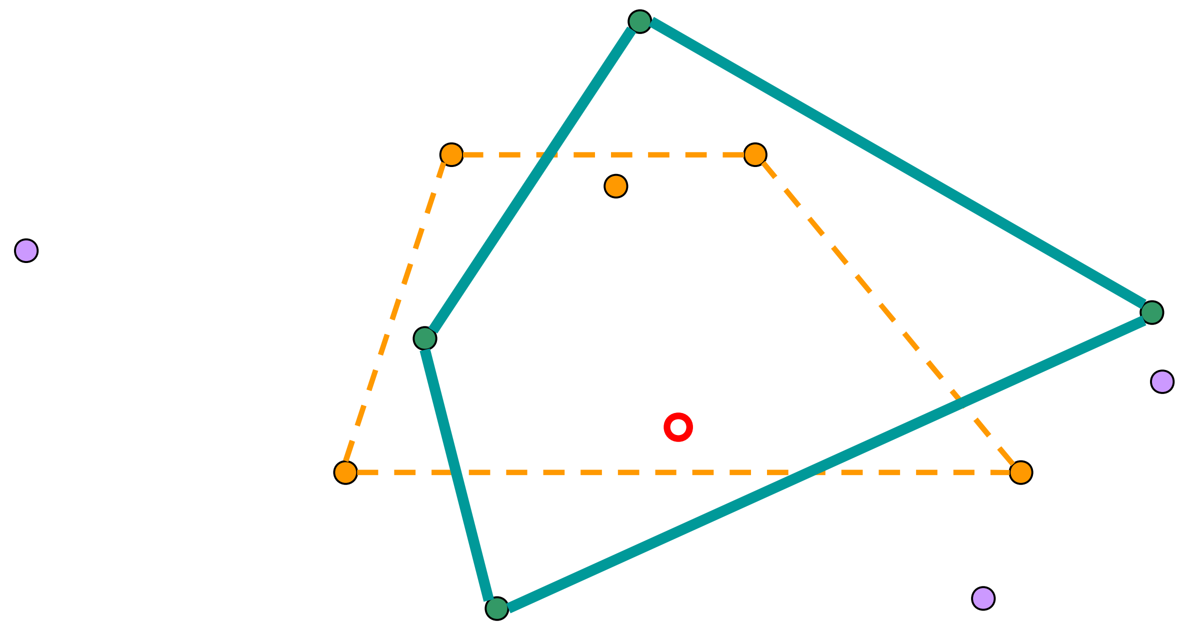

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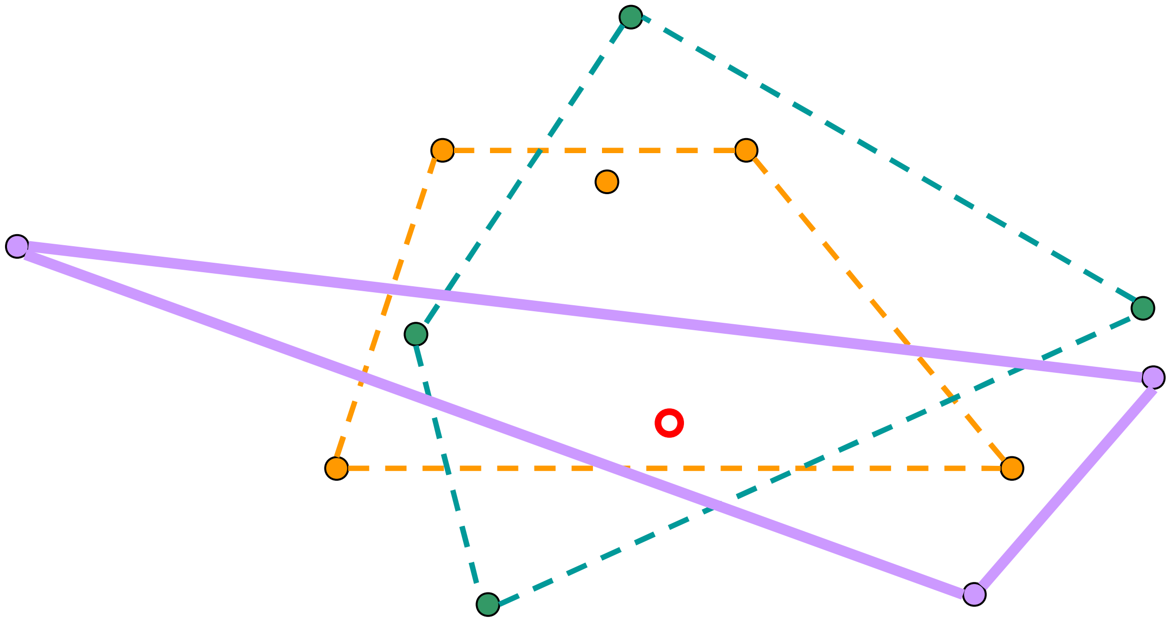

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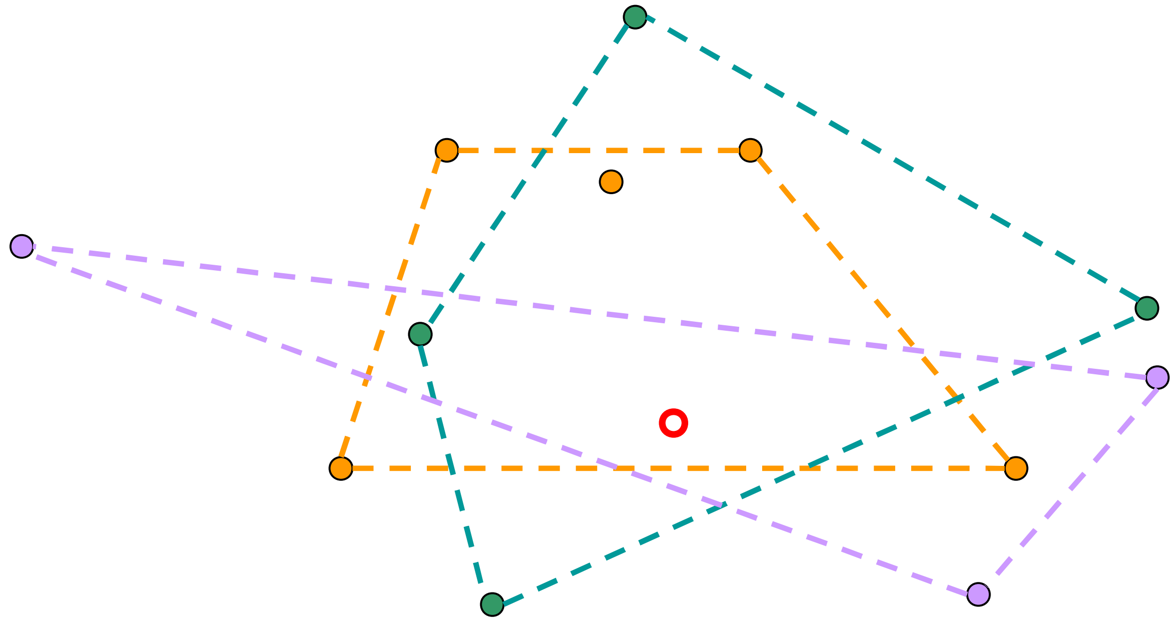

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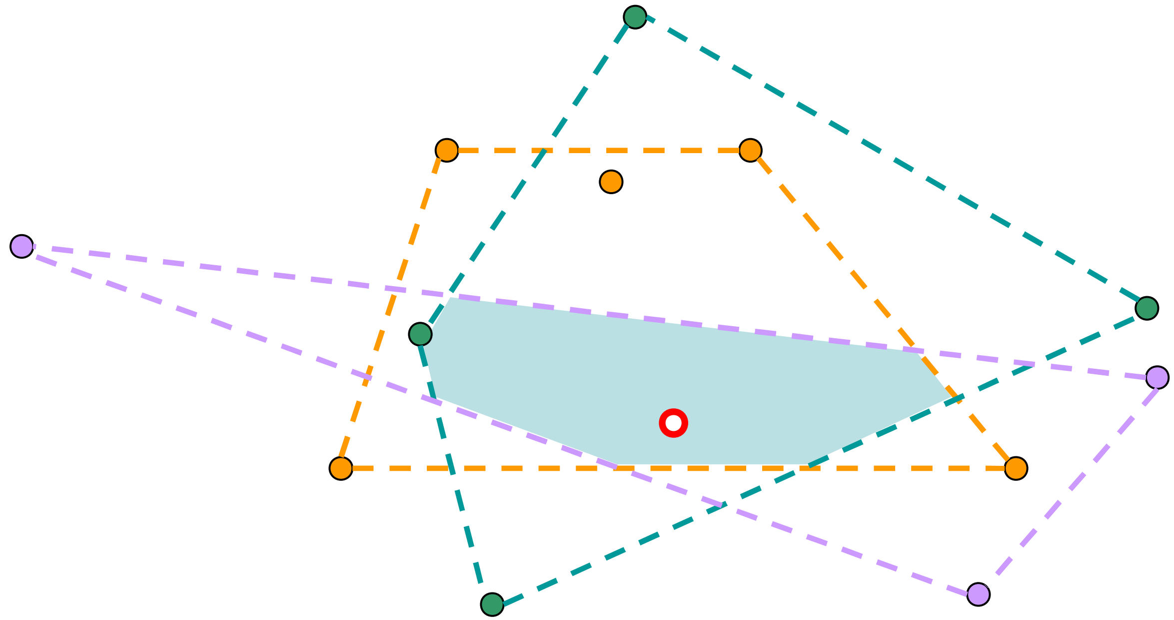

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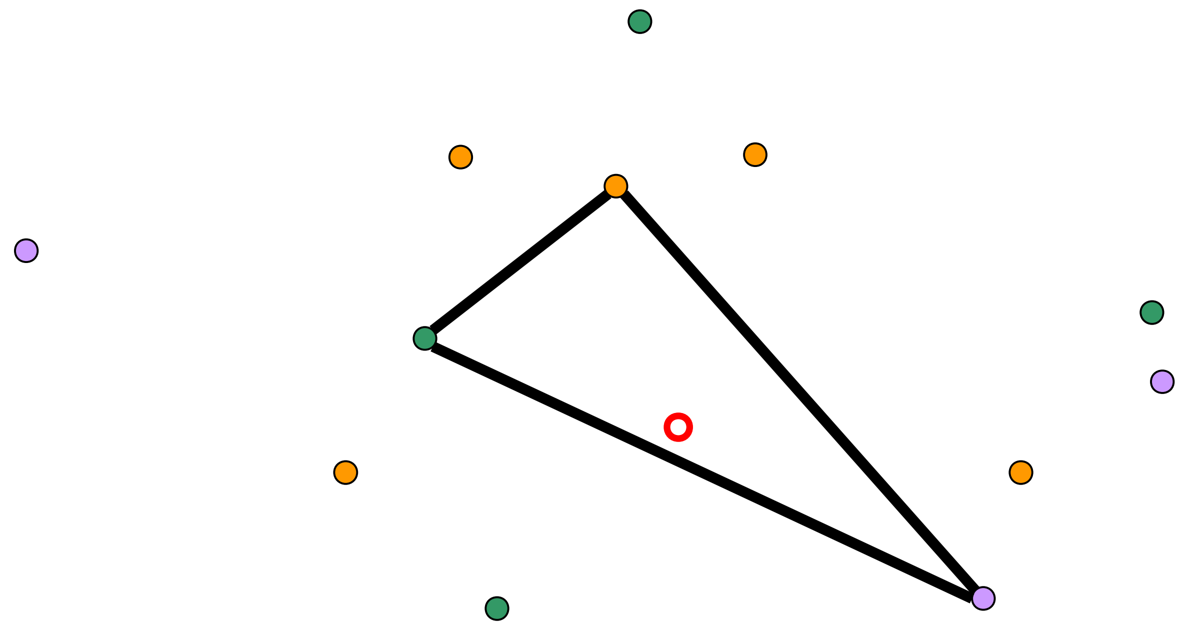
Colourful Simplicial Depth



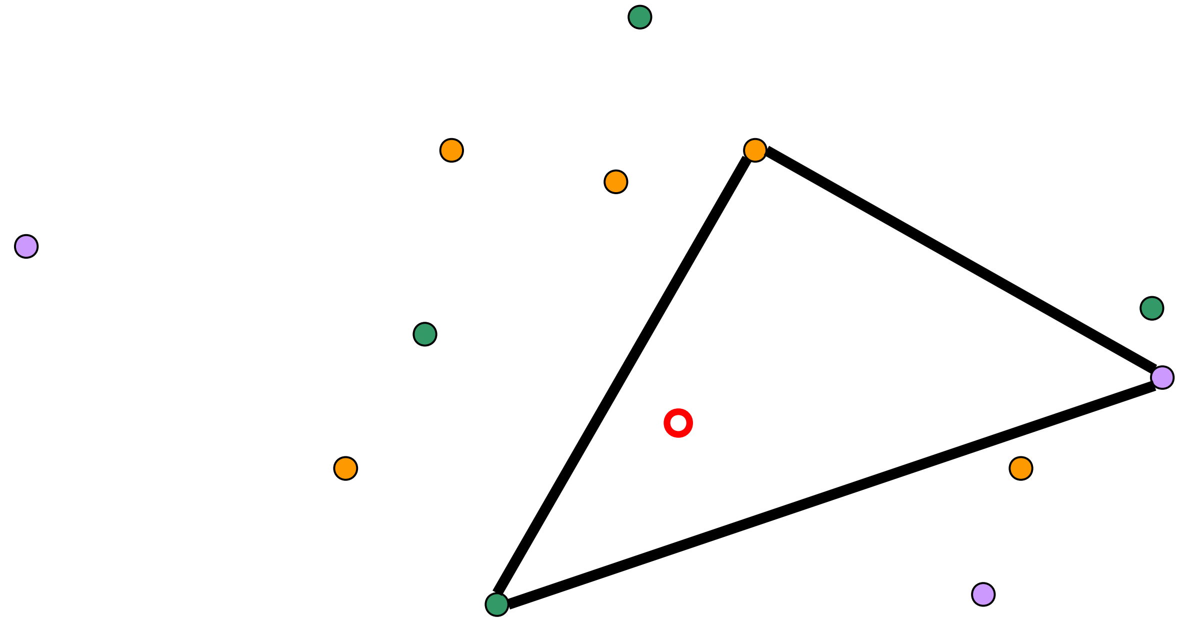
$depth_S(p)=1$

Given colourful set $\boldsymbol{S} = S_1 \cup S2 \ldots \cup S_{d+1}$ in dimension d, the *colourful simplicial depth* of p is the number of open colourful simplexes generated by points in $\boldsymbol{S}$ containing p

***S**, p general position*

$p \in \text{conv}(S_1) \cap \text{conv}(S2) \cap \ldots \cap \text{conv}(S_{d+1})$

Colourful Simplicial Depth



$depth_S(p)=2$

Given colourful set $\boldsymbol{S} = S_1 \cup S2\ldots \cup S_{d+1}$ in dimension d, the *colourful simplicial depth* of p is the number of open colourful simplexes generated by points in $\boldsymbol{S}$ containing p

$\boldsymbol{S}$, p *general position*

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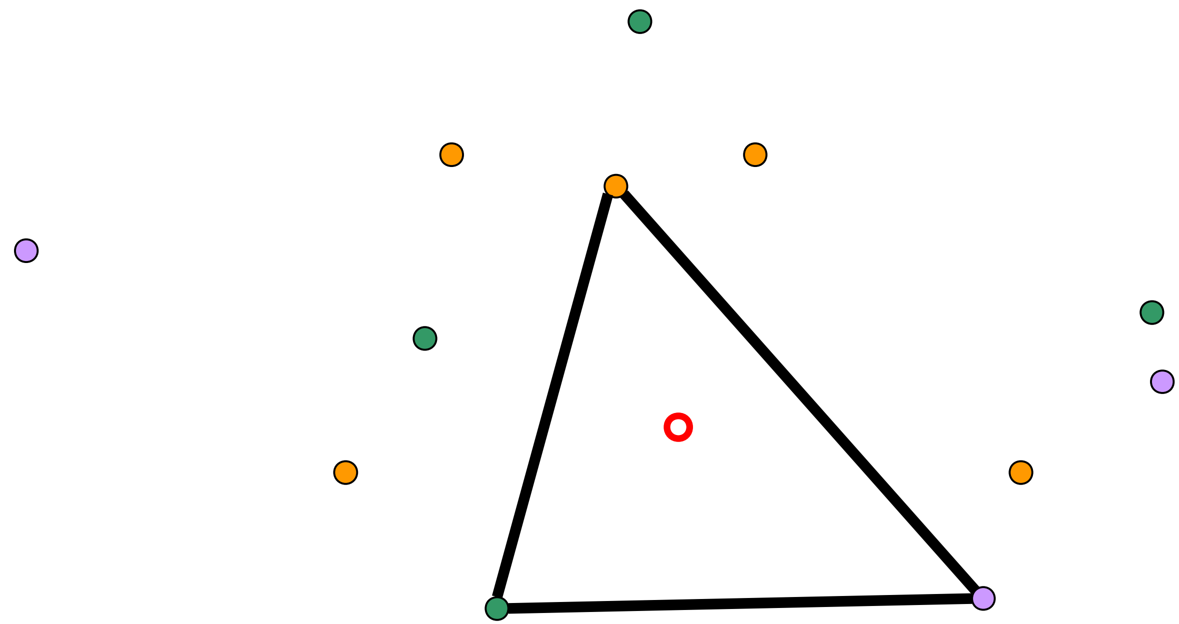
Colourful Simplicial Depth



$depth_S(p) = 3$

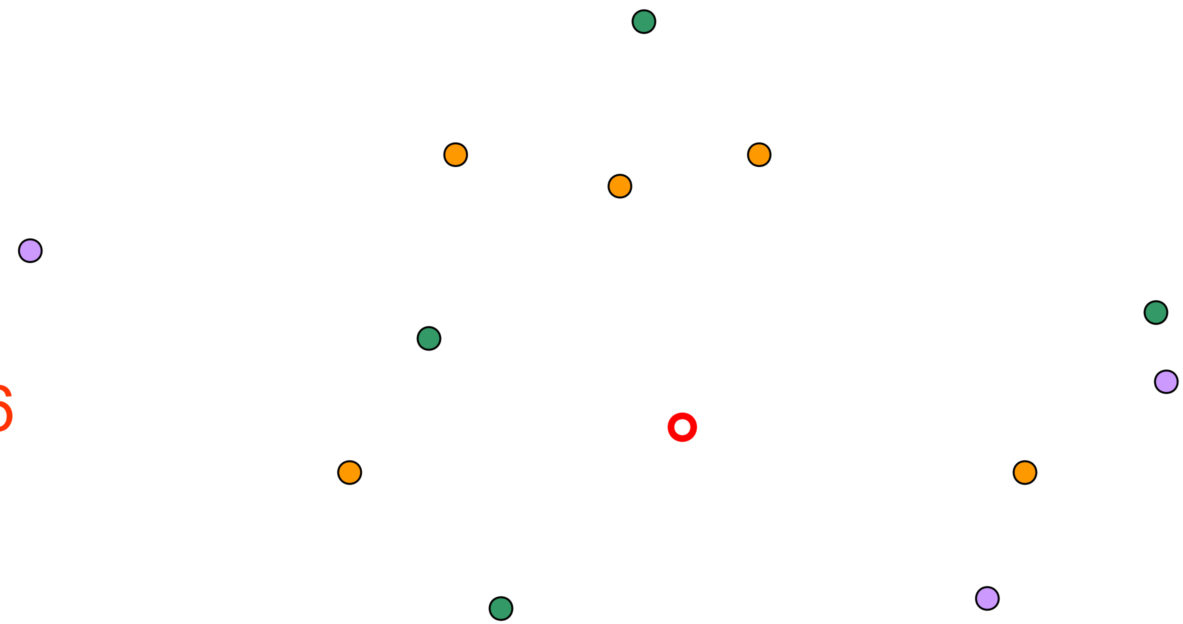
Given colourful set $\boldsymbol{S} = S_1 \cup S2 \ldots \cup S_{d+1}$ in dimension d, the *colourful simplicial depth* of p is the number of open colourful simplexes generated by points in $\boldsymbol{S}$ containing p

S, *p general position*

$p \in$ conv(S_1) $\cap$ conv($S2$) $\cap \ldots \cap$ conv(S_{d+1})

Colourful Simplicial Depth

$depth_S(p) = 16$



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Deepest Point in Dimension d

Deepest point bounds in dimension d [Bárány 1982]

$$\frac{\mu(d)}{(d+1)^{d+1}}\binom{n}{d+1} + O(n^d) \leq \max_{p} depth_S(p) \leq \frac{1}{2^d(d+1)!}n^{d+1} + O(n^d)$$

with $\mu(d) = \min_{S,p} depth_S(p)$

[Bárány 1982]: $\mu(d) \geq 1$

… breakthrough [Gromov 2010] & further improvements

S, p general position

Deepest Point in Dimension d

$$\max_{p} depth_S(p) \geq c_d \binom{n}{d+1}$$

[Bárány 1982] $\quad c_d \geq \frac{d+1}{(d+1)^{(d+1)}}$

[Wagner 2003] $\quad c_d \geq \frac{d^2+1}{(d+1)^{(d+1)}}$

[Gromov 2010] $\quad c_d \geq \frac{2d}{(d+1)!(d+1)}$

simpler proofs: [Karazev 2012], [Matoušek, Wagner 2012]

d=3: [Král', Mach, Sereni 2012]

S, *p* *general position*

Colourful Research Directions

- Generalize the sufficient condition of Bárány for the existence of a colourful simplex
- Improve lower bound for $\mu(d) = \min_{S,p} depth_S(p)$
- Computational approaches for $\mu(d)$ for small *d*.
- Obtain an efficient algorithm to find a colourful simplex : *Colourful Linear Programming Feasibility* problem

$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

S, *p* *general position* and $|S_1|,|S2|,\ldots,|S_{d+1}| \geq d+1$

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S, *p general position* and $|S_1|,|S2|,\ldots,|S_{d+1}| \geq d+1$

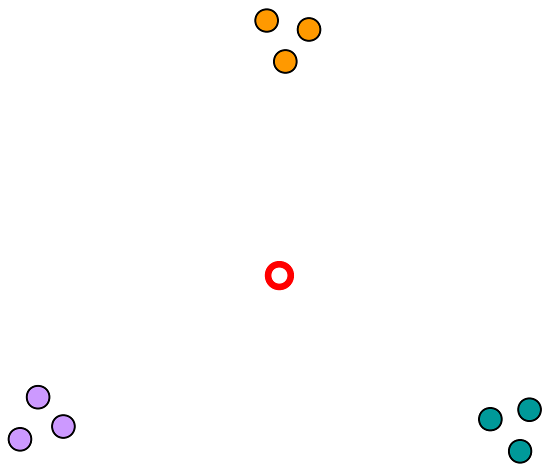
Colourful Carathéodory Theorems

[Bárány 1982] Given colourful set $\boldsymbol{S} = S_1 \cup S2 \ldots \cup S_{d+1}$ in dimension d, and $p \in \text{conv}(S_1) \cap \text{conv}(S2) \cap \ldots \cap \text{conv}(S_{d+1})$, then there exists a colourful simplex containing p

[Holmsen, Pach, Tverberg 2008] and [Arocha, Bárány, Bracho, Fabila, Montejano 2009] Given colourful set $\boldsymbol{S} = S_1 \cup S2 \ldots \cup S_{d+1}$ in dimension d, and $p \in \text{conv}(S_i \cup Sj)$ for $1 \leq i < j \leq d + 1$, then there exists a colourful simplex containing p

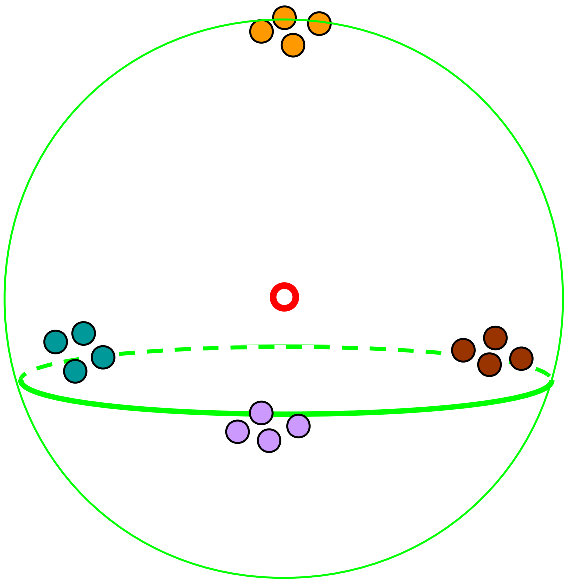
[Meunier, D. 2013] Given colourful set $\boldsymbol{S} = S_1 \cup S2 \ldots \cup S_{d+1}$ in dimension d, if for $1 \leq i < j \leq d + 1$ there exists $k \neq i$, $k \neq j$, such that for all $x_k \in S_k$ the ray $[x_k\, p)$ intersects $\text{conv}(S_i \cup Sj)$ in a point distinct from x_k, then there exists a colourful simplex containing p

Colourful Carathéodory Theorems



[Meunier, D. 2013] Given colourful set $\boldsymbol{S} = S_1 \cup S2 \ldots \cup S_{d+1}$ in dimension d, if for $1 \leq i < j \leq d+1$ there exists $k \neq i$, $k \neq j$, such that for all $x_k \in S_k$ the ray $[x_k\, p)$ intersects conv($S_i \cup S$j) in a point distinct from x_k, then there exists a colourful simplex containing p

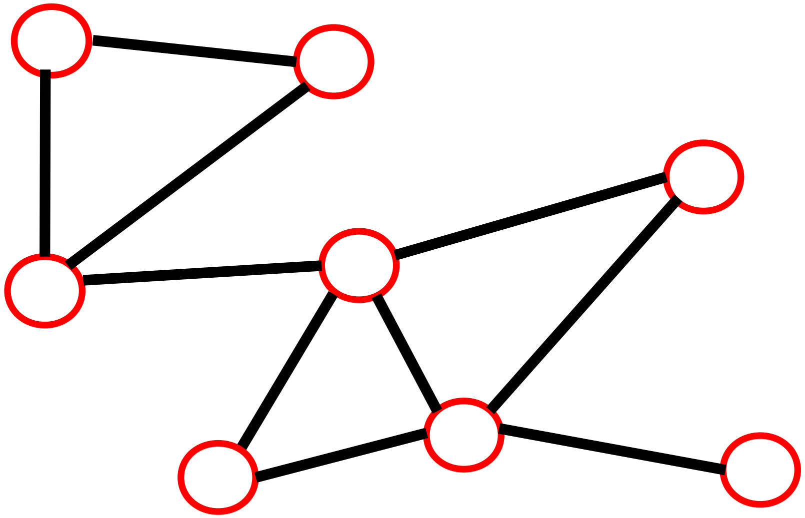
Colourful Carathéodory Theorems



[Meunier, D. 2013] Given colourful set $\boldsymbol{S} = S_1 \cup S2 \ldots \cup S_{d+1}$ in dimension d, if for $i \neq j$ the open half-space containing p and defined by an i-facet of a colourful simplex intersects $S_i \cup S$j, then there exists a colourful simplex containing p

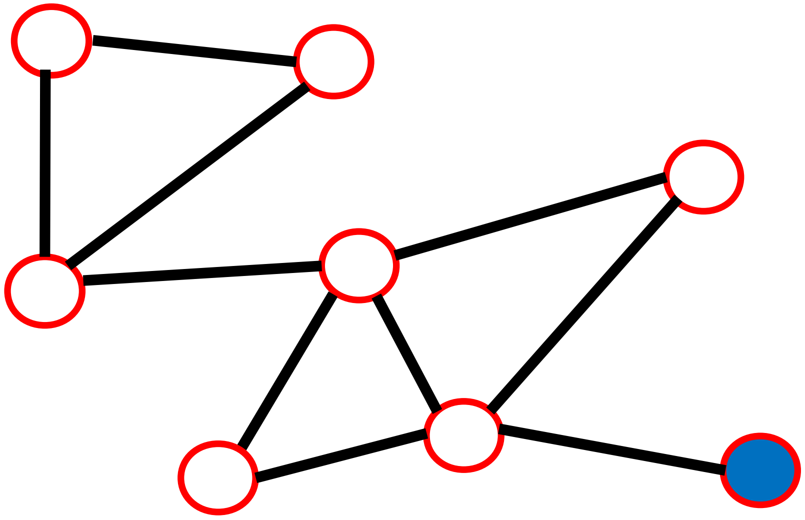
- *further generalization in dimension 2*

Given One, Get Another One



In a graph, if there is a vertex with an odd degree…

Given One, Get Another One



In a graph, if there is a vertex with an ***odd degree***…

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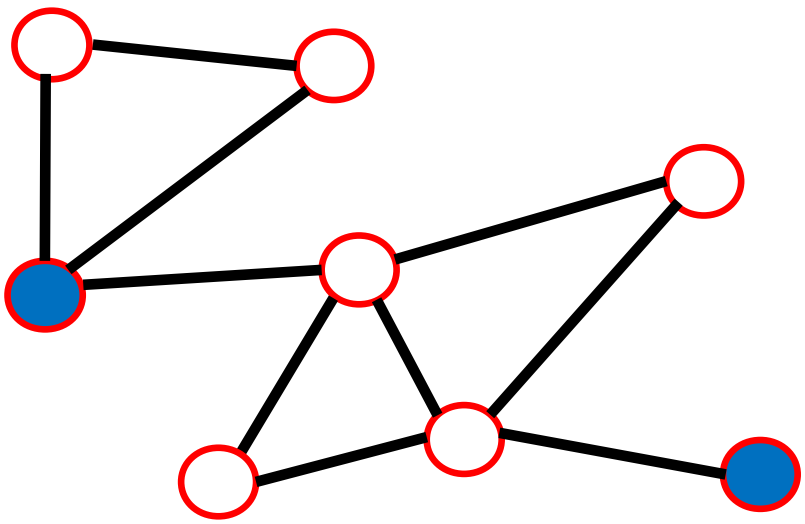

In a graph, if there is a vertex with an ***odd degree***… then there is ***another one***

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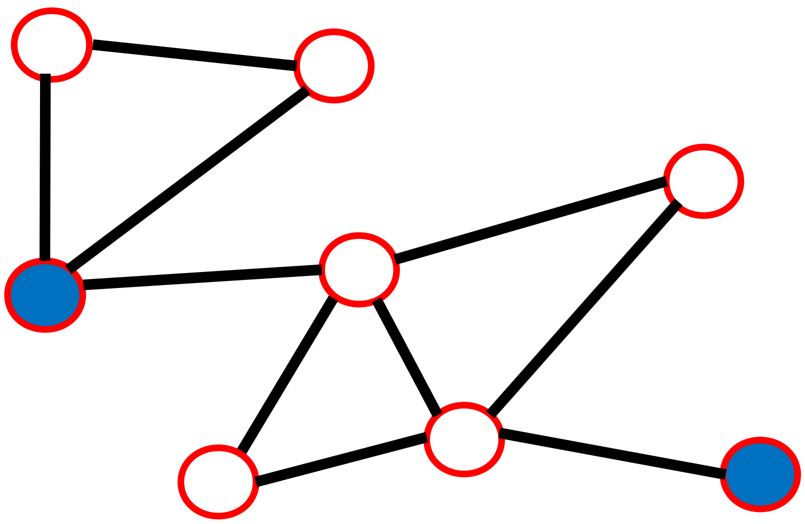


In a graph, if there is a vertex with an ***odd degree***… then there is ***another one***

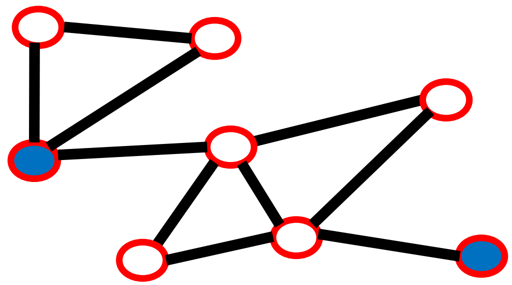
- *Duoid / oik* room partitioning Todd 1974, Edmonds 2009]

(*Exchange algorithm*: generalization of Lemke-Howson for finding a Nash equilibrium for a 2 players game)

- *Polynomial Parity Argument* PPA(D) [Papadimitriou 1994]

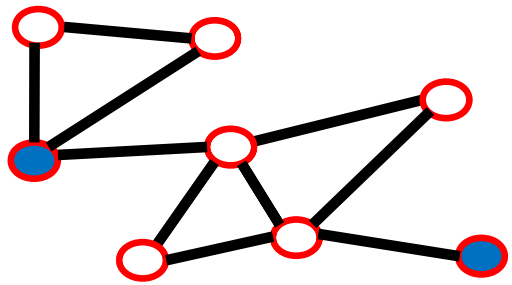
(Hamiltonian circuit in a cubic graph, Borsuk-Ulam, …)

Given One, Get Another One



[Meunier, D. 2013] Given colourful set $\boldsymbol{S} = S_1 \cup S_2 \ldots \cup S_{d+1}$ in dimension d with $|S_i|=2$, if there is a colourful simplex containing p then there is another one

Given One, Get Another One



[Meunier, D. 2013] Any condition implying the existence of a colourful simplex containing p actually implies that the number of such simplices is at least d+1

$\boldsymbol{S}$, p *general position* and $|S_1|,|S2|,\ldots,|S_{d+1}| \geq d+1$

Colourful Research Directions

- Generalize the sufficient condition of Bárány for the existence of a colourful simplex
- Improve lower bound for $\mu(d) = \min_{S,p} depth_S(p)$
- Computational approaches for $\mu(d)$ for small d.
- Obtain an efficient algorithm to find a colourful simplex : *Colourful Linear Programming Feasibility* problem

$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}}\binom{n}{d+1} + O(n^d)$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

S, *p general position* and $|S_1|, |S2|, \ldots, |S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

[Bárány 1982] $$1 \leq \mu(d)$$

$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \text{conv}(S_1) \cap \text{conv}(S2) \cap \ldots \cap \text{conv}(S_{d+1})$

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Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

[Bárány 1982]

$$d+1 \leq \mu(d)$$

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Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

[Bárány 1982]

$$d + 1 \leq \mu(d)$$

[D.,Huang,Stephen,Terlaky 2006]

$$2d \leq \mu(d) \leq d^2 + 1$$

$$\mu(d) \text{ even for odd } d$$

$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

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[D.,Huang,Stephen,Terlaky 2006] $2d \leq \mu(d) \leq d^2+1$

$\mu(d)$ even for odd d

[Bárány, Matoušek 2007] $\max(3d, \frac{d^2+d)}{5}) \leq \mu(d)$ for $d \geq 3$

$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \text{conv}(S_1) \cap \text{conv}(S2) \cap \ldots \cap \text{conv}(S_{d+1})$

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$\mu(d)$ even for odd d

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$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

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S, *p general position* and $|S_1|, |S2|, \ldots, |S_{d+1}| \geq d+1$

Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

[Bárány 1982] $d+1 \le \mu(d)$

[D.,Huang,Stephen,Terlaky 2006] $2d \le \mu(d) \le d^2+1$

$\mu(d)$ even for odd d

[Bárány, Matoušek 2007] $3d \le \mu(d)$ for $d \ge 3$

[Stephen,Thomas 2008] $\left\lfloor \frac{(d+2)^2}{4} \right\rfloor \le \mu(d)$ for $d \ge 8$

$$\max_p depth_S(p) \ge \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

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[Bárány, Matoušek 2007] $\max(3d, \frac{d^2+d}{5}) \le \mu(d)$ for $d \ge 3$

[Stephen,Thomas 2008] $\left\lfloor \frac{(d+2)^2}{4} \right\rfloor (d+2)2/4 \le \mu(d)$ for $d \ge 8$

[D.,Stephen, Xie 2011] $\left\lceil \frac{(d+1)^2}{2} \right\rceil \le \mu(d)$ for $d \ge 4$

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$$\mu(d) = \min_{S,p} depth_S(p)$$

$$\mu(1) = 2 \qquad \mu(2) = 5 \qquad \mu(3) = 10$$

$$\left\lceil \frac{(d+1)^2}{2} \right\rceil \le \mu(d) \le d^2 + 1 \quad \text{for } d \ge 4$$

$\mu(d)$ even for odd d

conjecture: $\mu(d) = d^2 + 1$

$$\max_p depth_S(p) \ge \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \mathrm{conv}(S_1) \cap \mathrm{conv}(S2) \cap \ldots \cap \mathrm{conv}(S_{d+1})$

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Colourful Simplicial Depth Bounds

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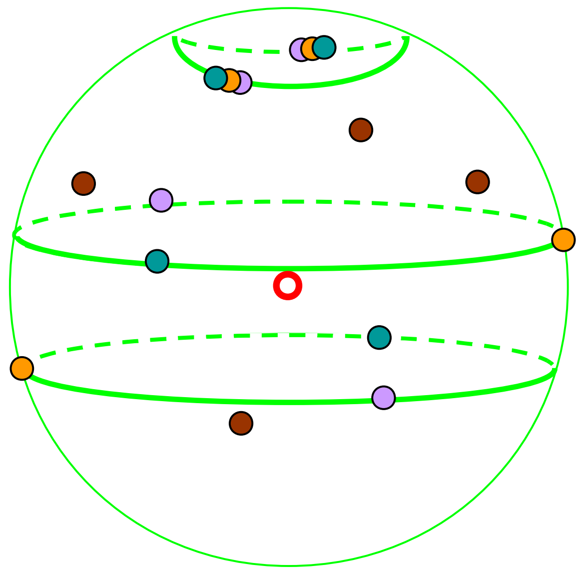

$$3d \le \mu(d) \le d^2 + 1$$

$p \in \text{conv}(S_1) \cap \text{conv}(S2) \cap \ldots \cap \text{conv}(S_{d+1})$

S, p *general position* and $|S_1|,|S2|,\ldots,|S_{d+1}| \ge d+1$

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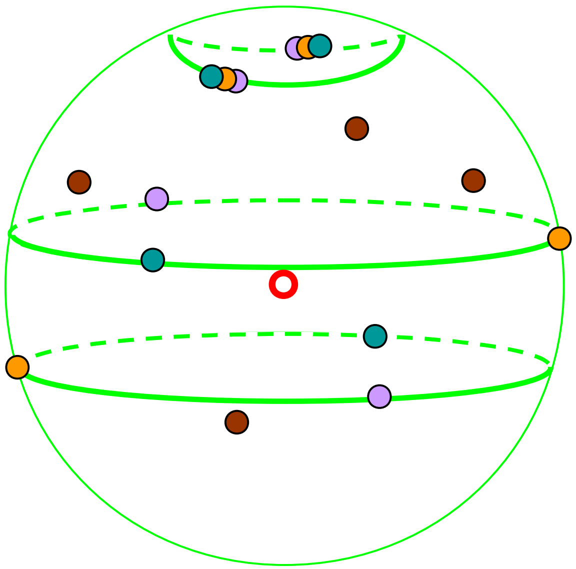

$$9 \le \mu(3) \le 10$$

$p \in \text{conv}(S_1) \cap \text{conv}(S2) \cap \ldots \cap \text{conv}(S_{d+1})$

S, *p general position* and $|S_1|,|S2|,\ldots,|S_{d+1}| \ge d+1$

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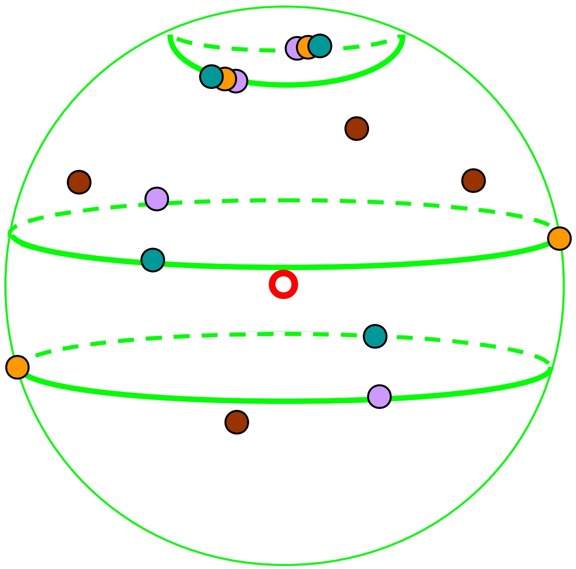

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$$\mu(d) \text{ even for odd } d$$

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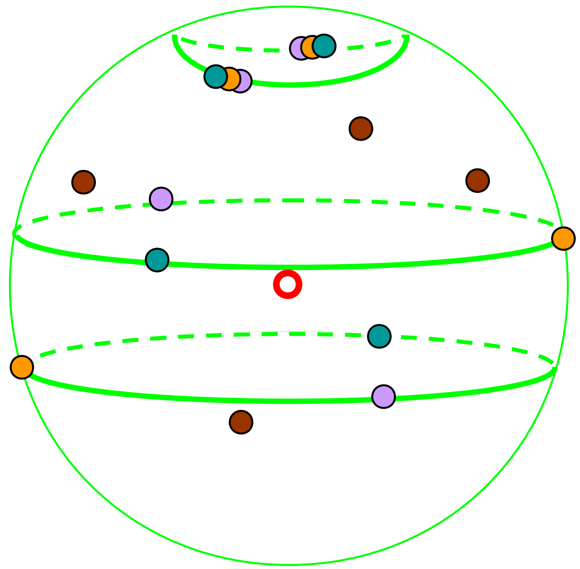

$$\mu(3) = 10$$

$p \in \text{conv}(S_1) \cap \text{conv}(S2) \cap \ldots \cap \text{conv}(S_{d+1})$

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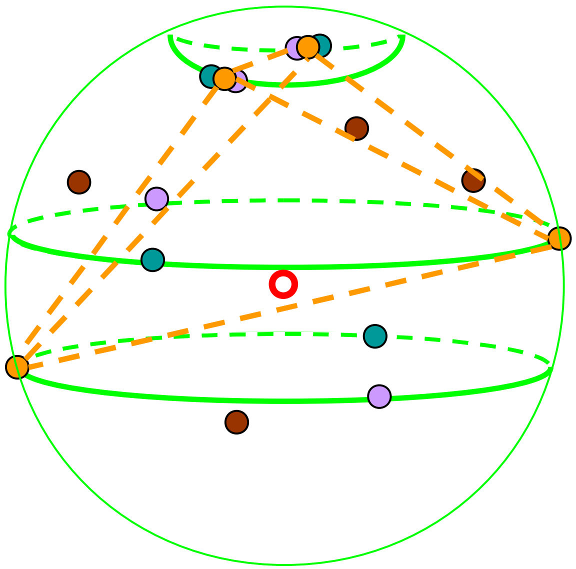

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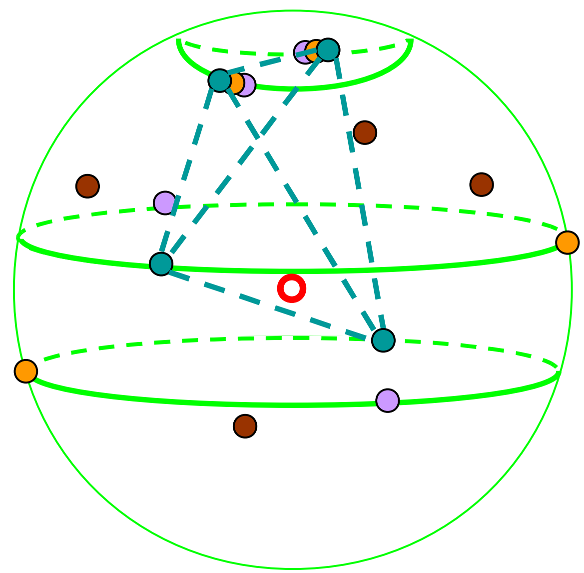

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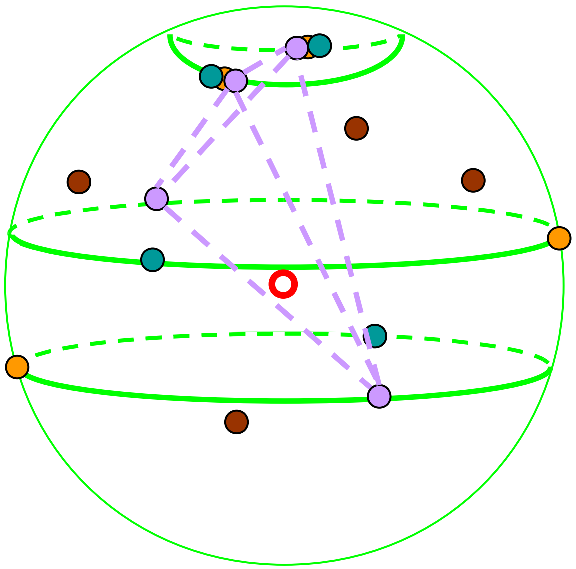

$$\mu(3) = 10$$

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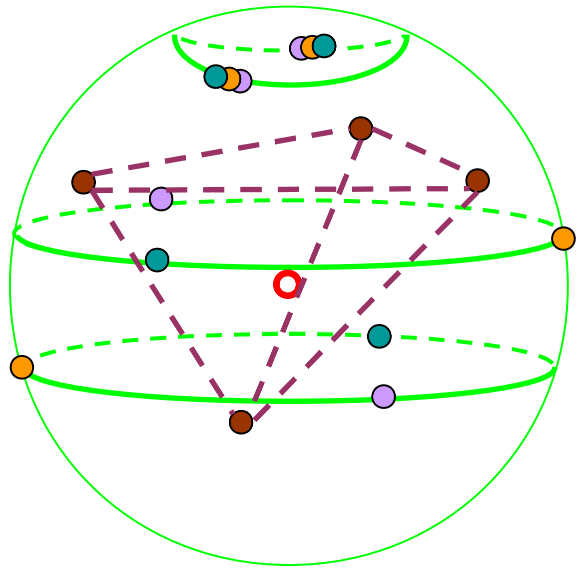

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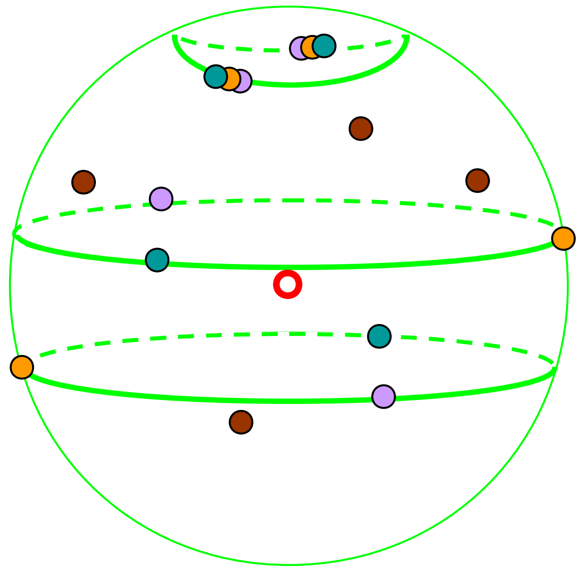
Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

$depth_S(p)=10$



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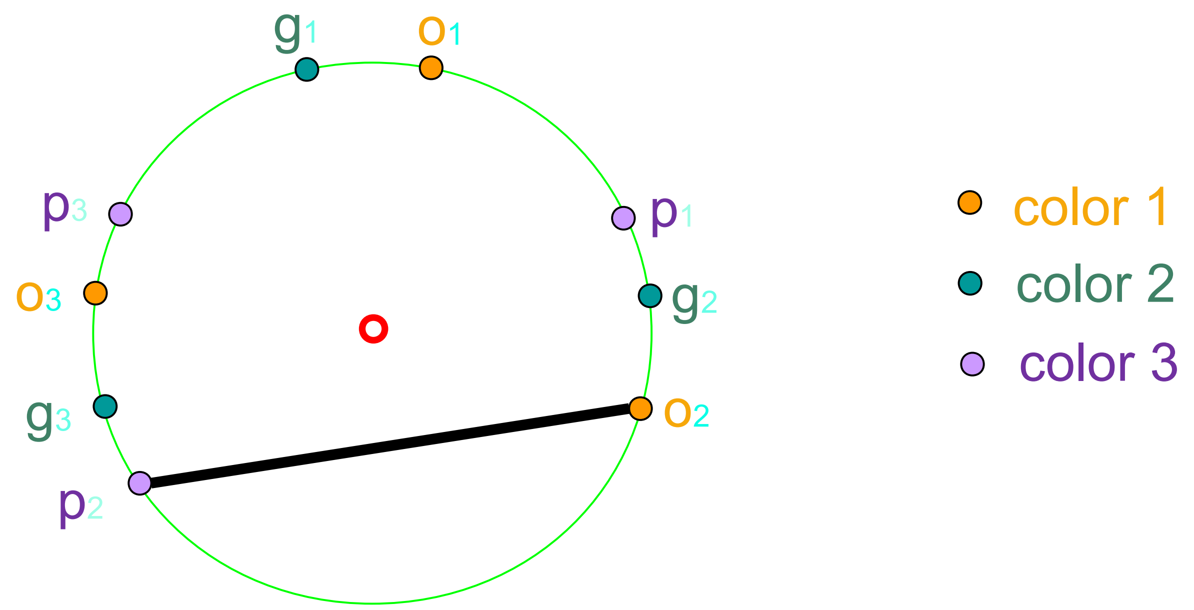
Transversal

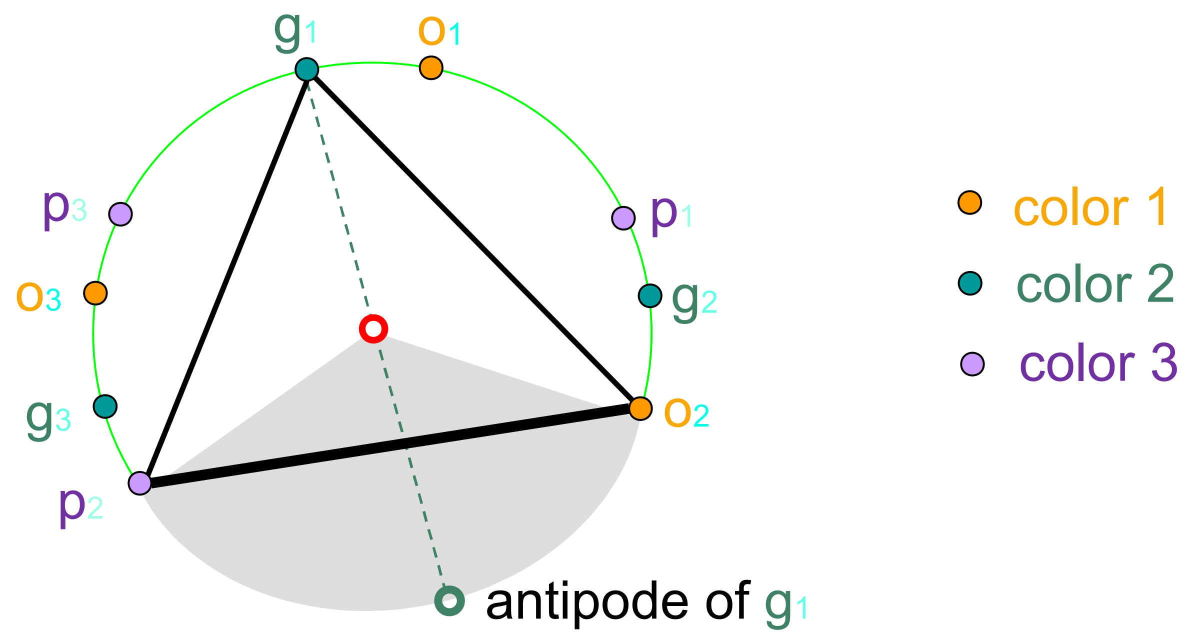
colourful set of d points (one colour missing)



$\hat{2}$-transversal (o_2,p_2)

Transversal

colourful set of d points (one colour missing)

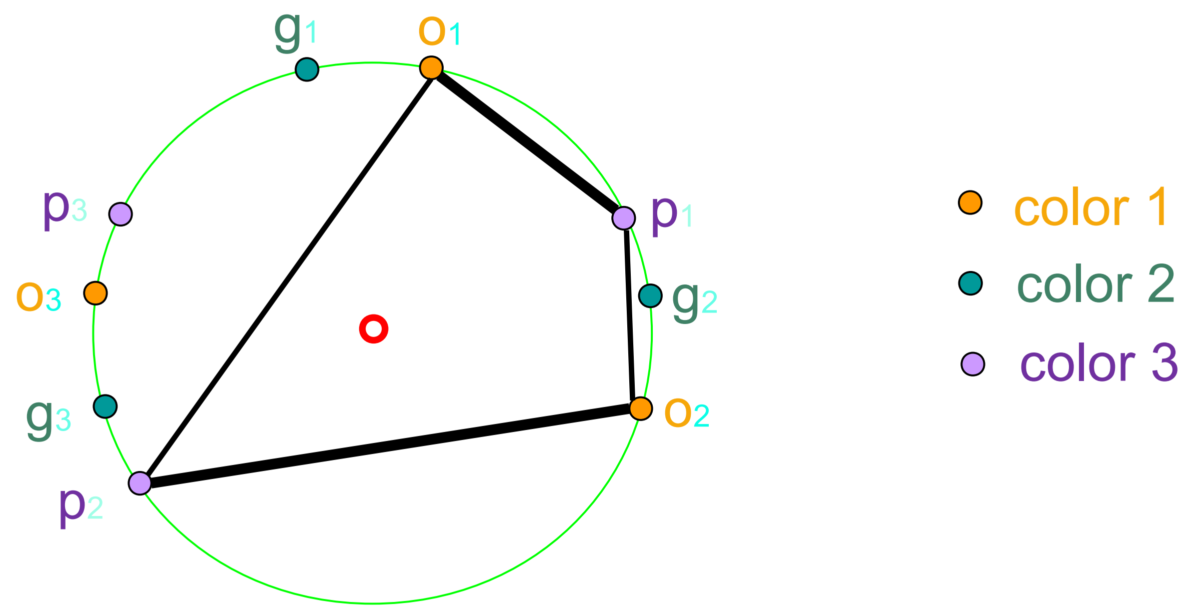


$\hat{2}$-transversal (o_2,p_2) *spans* the antipode of g_1

iff (o_2,p_2,g_1) is a colourful simplex

Combinatorial (topological) Octahedra

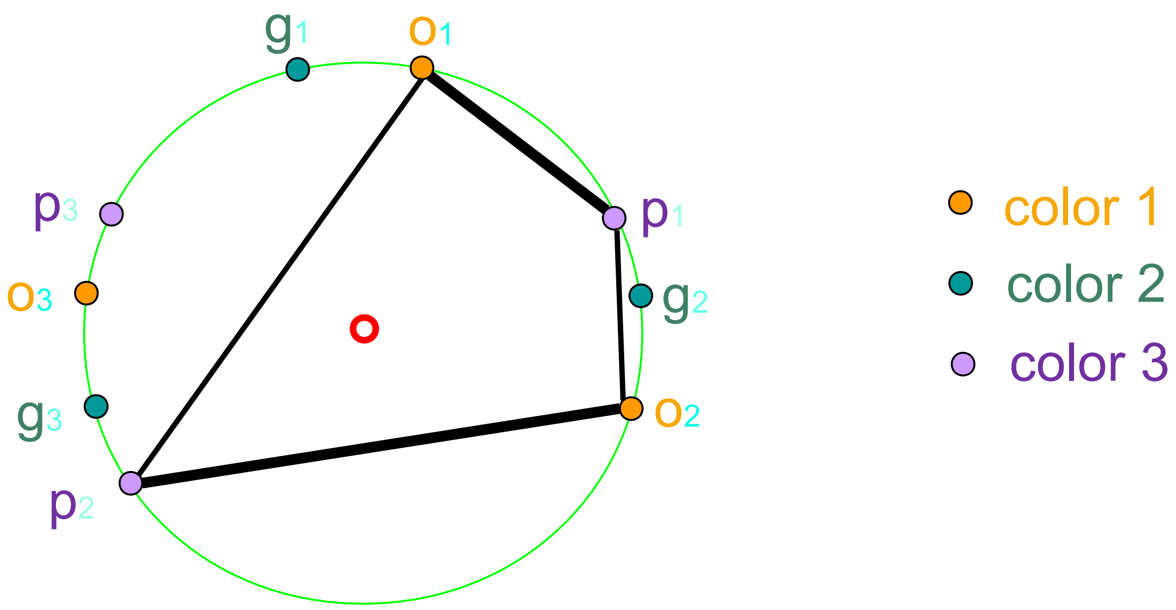
pair of disjoint $\hat{i}$-transversals



octahedron $[(o_1,p_1),(o_2,p_2)]$

Octahedron Lemma

origin-*containing* octahedra

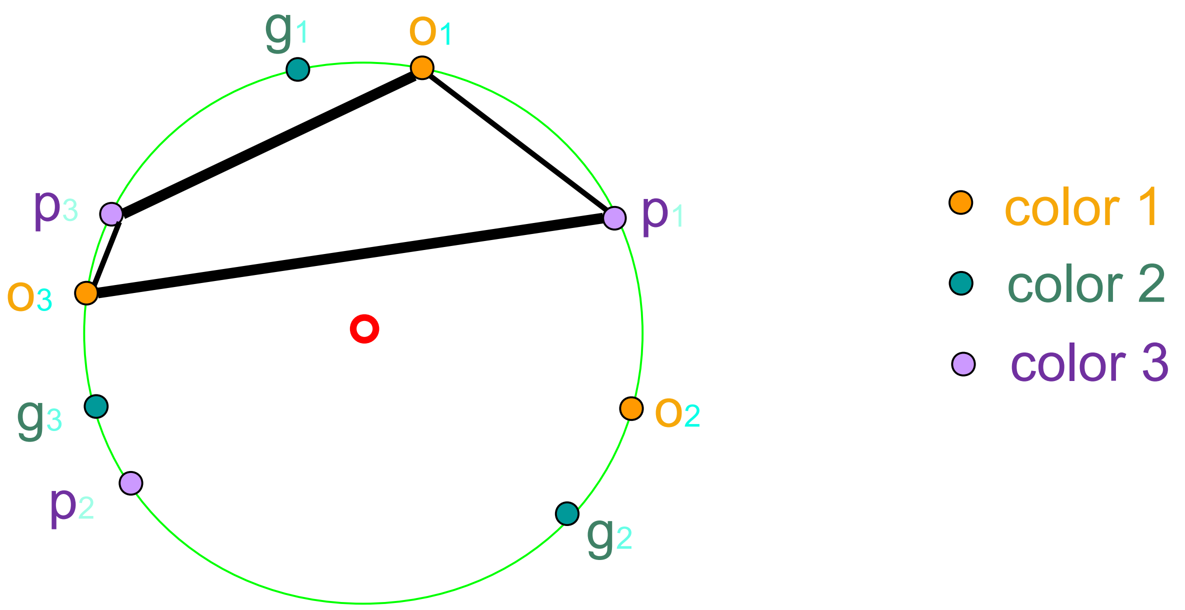


octahedron [(o_1,p_1),(o_2,p_2)]

2^d colourful faces span the whole sphere if it contains the origin (creating d+1 colourful simplexes)

Octahedron Lemma

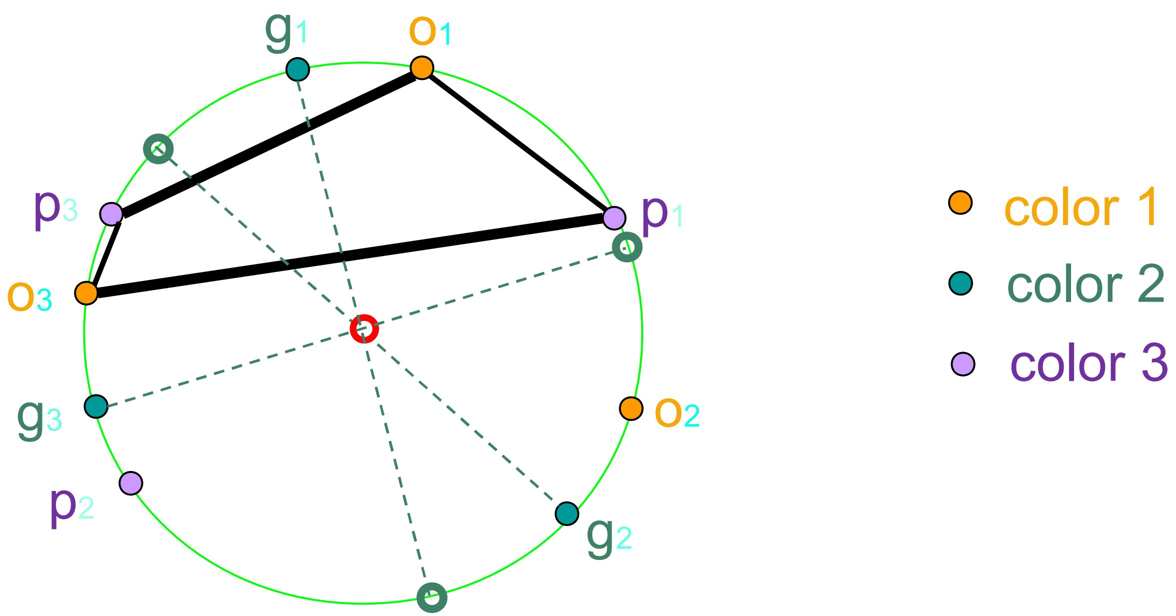
octahedron *not containing* the origin



octahedron [(o_1,p_3),(o_3,p_1)] does not contain *p*

Octahedron Lemma

octahedron *not containing* the origin

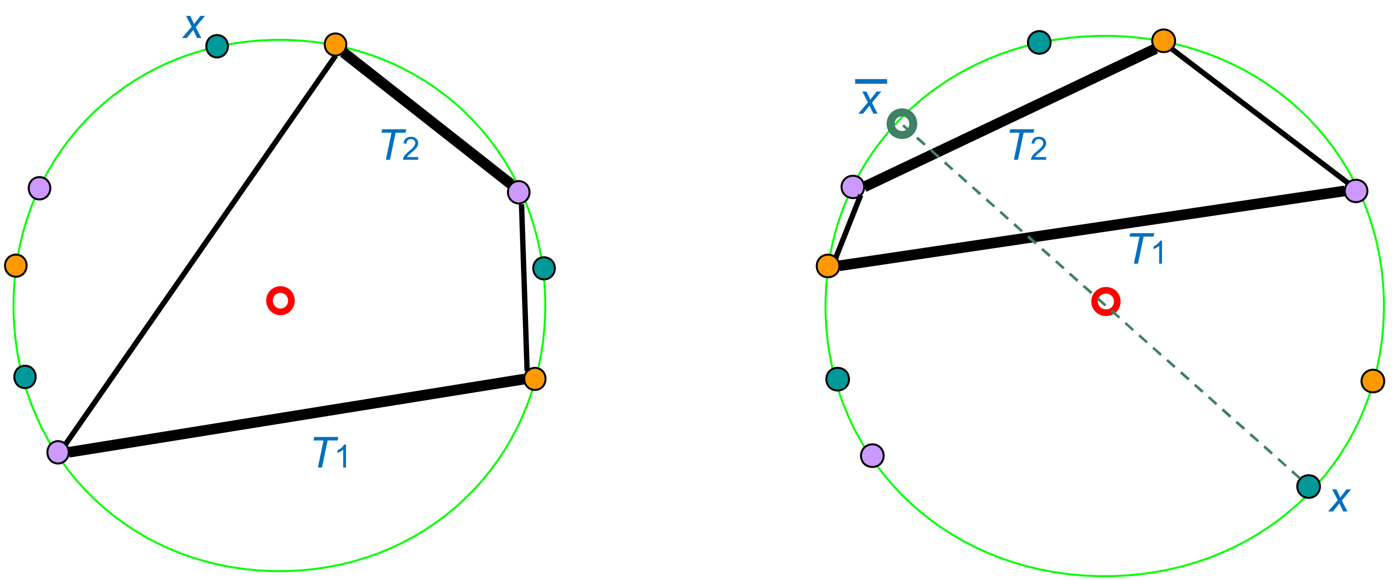


octahedron [(o_1,p_3),(o_3,p_1)] spans any antipode an *even* number of times

Octahedron Lemma

Given 2 disjoint transversals T_1 and T_2, and T_1 spans $\bar{x}$ *(antipode of x)*,

- either octahedron (T_1,T_2) contains p,
- or there exists a transversal $T \neq T_1$ consisting of points from T_1 and T_2 that spans $\bar{x}$.



Colourful Research Directions

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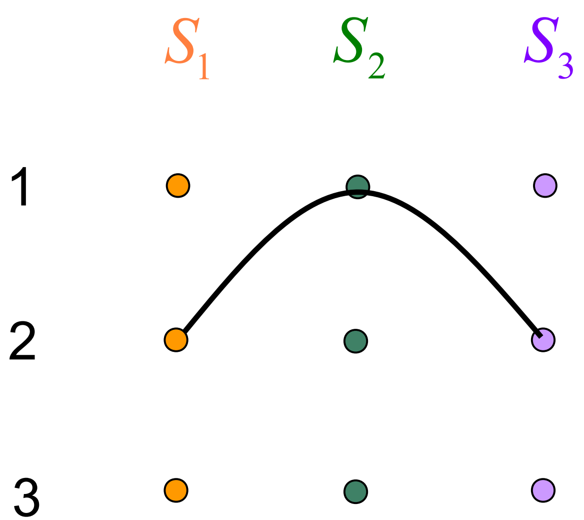
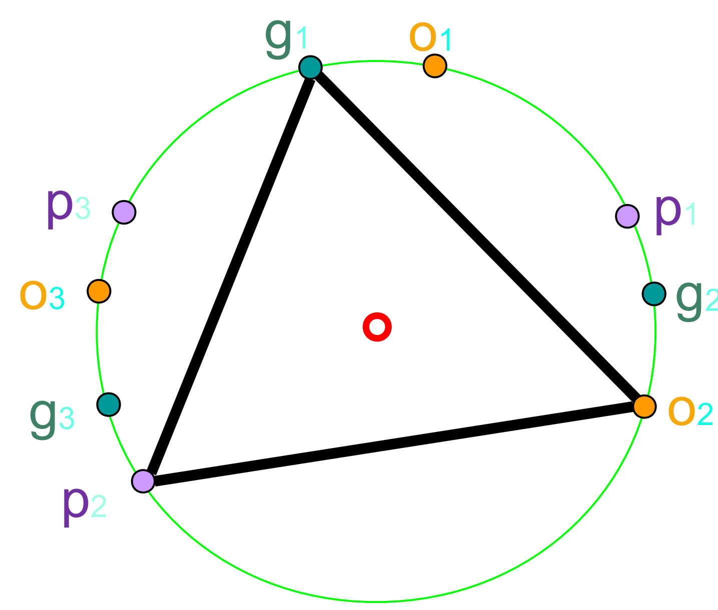
Computational Approach

(d+1)-uniform (d+1)-partite *hypergraph* representation of *colourful point configurations*



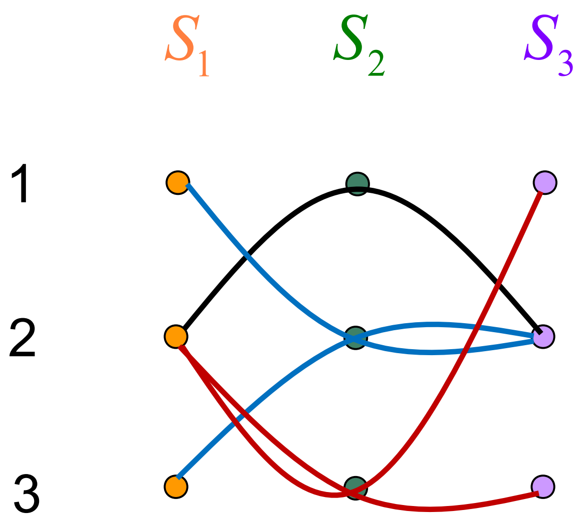


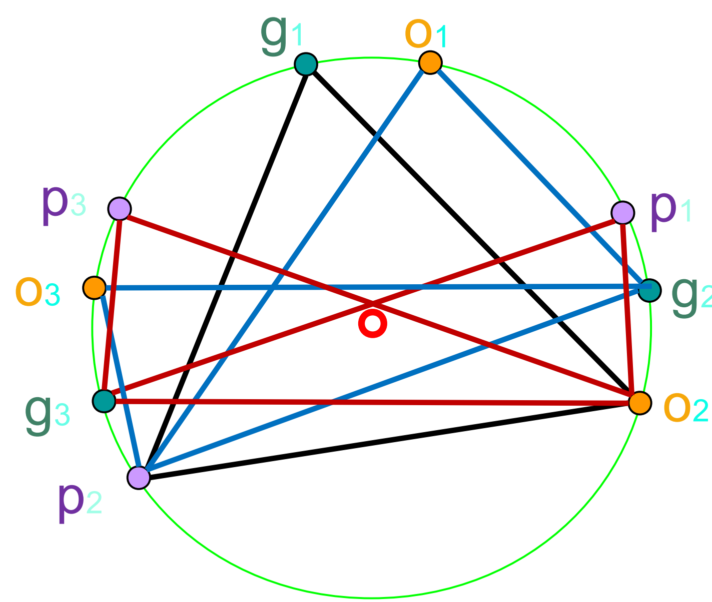
edge: colourful simplex containing *p*

❖ *combinatorial setting suggested by Imre Bárány*

Computational Approach

(*d*+1)-uniform (*d*+1)-partite *hypergraph* representation of *colourful point configurations*





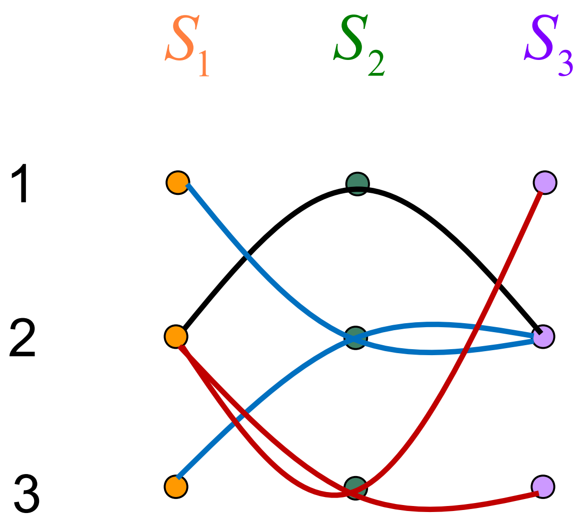
necessary conditions:

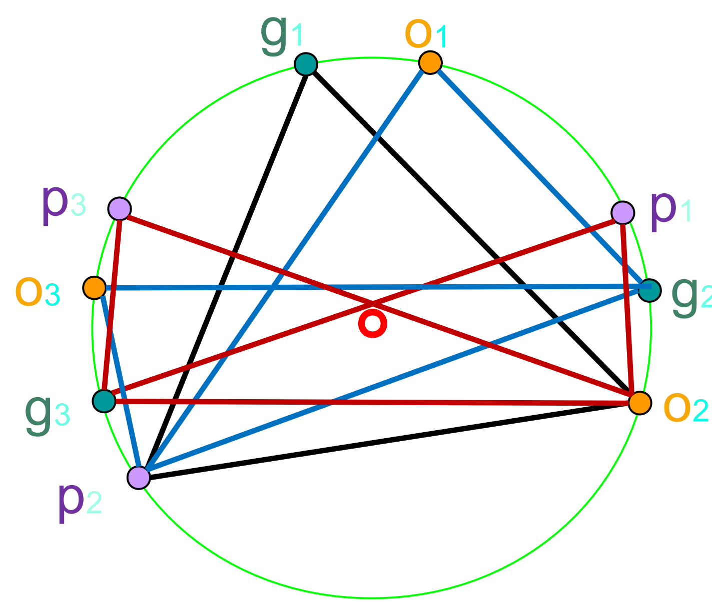
- every vertex belongs to at least 1 edge.
- even number of edges induced by subsets X_i of S_i of size 2

❖ *reformulation of the Octahedron Lemma*

Computational Approach

(d+1)-uniform (d+1)-partite *hypergraph* representation of *colourful point configurations*





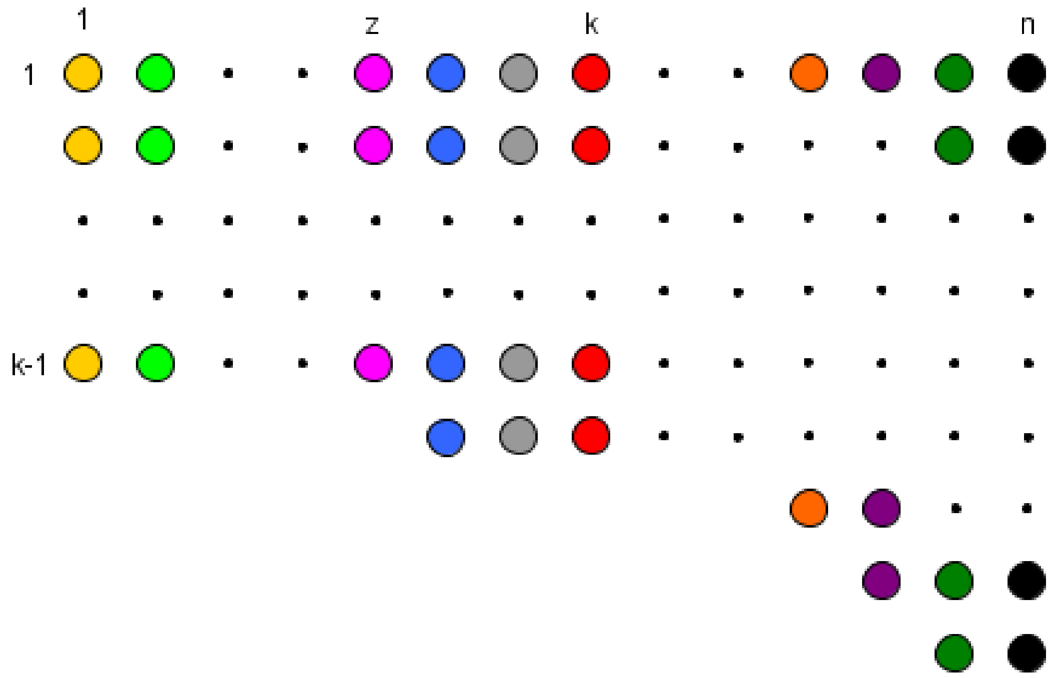
if no hypergraph with t or less hyper-edges satisfies the 2 necessary conditions, then $\mu(d) > t$

$\Rightarrow$ computational proof that $\mu(4) \geq 14$ [D., Stephen, Xie 2013]

❖ *isolated edge argument needed*

Octahedral Systems

n-uniform n-partite hypergraph $(S_1,\ldots,S_n,E)$ with $|S_i| \geq 2$ such that the number of edges induced by subsets X_i of S_i of size 2 for i=1,...n is even



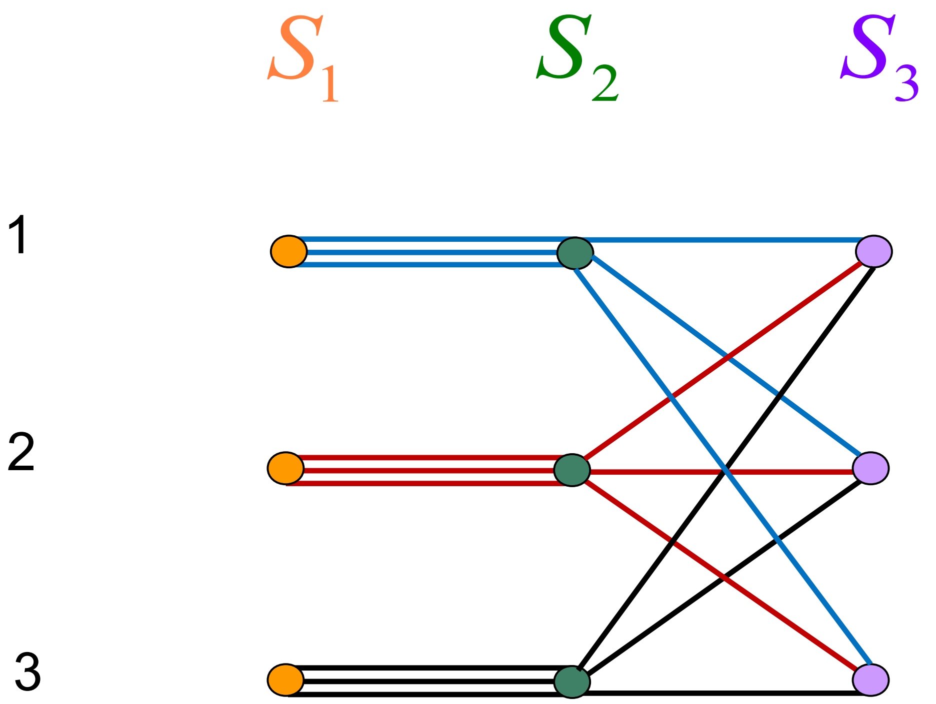
Octahedral Systems

- *even number of edges* if all $|S_i|$ are even for $i = 1, \ldots, n$
- *symmetric difference* of 2 octahedral systems is octahedral
- existence of *non-realizable* octahedral system

  without isolated vertex
- *number* of octahedral systems: $2^{\prod_{1}^{n} |S_i| - \prod_{1}^{n} (|S_i| - 1)}$

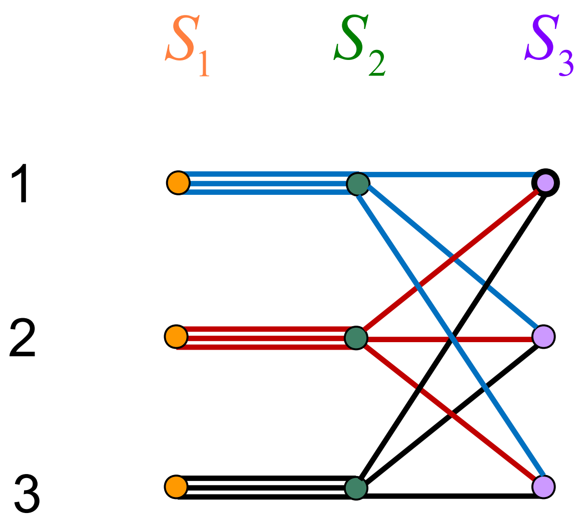
[D., Meunier, Sarrabezolles 2013]

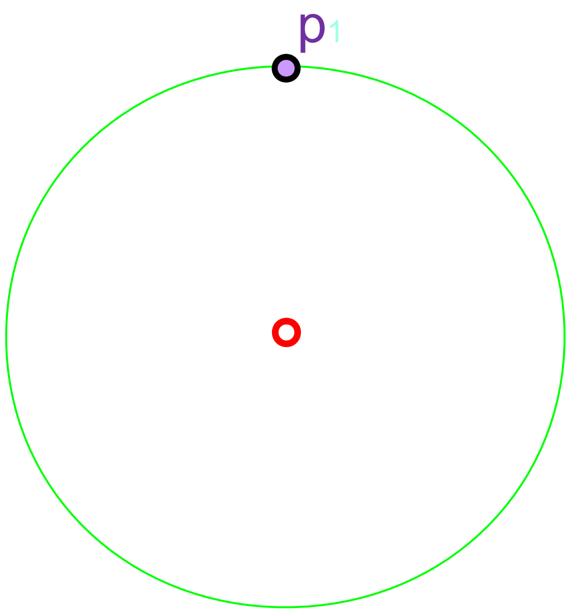
Octahedral Systems



a *non-realizable* octahedral system

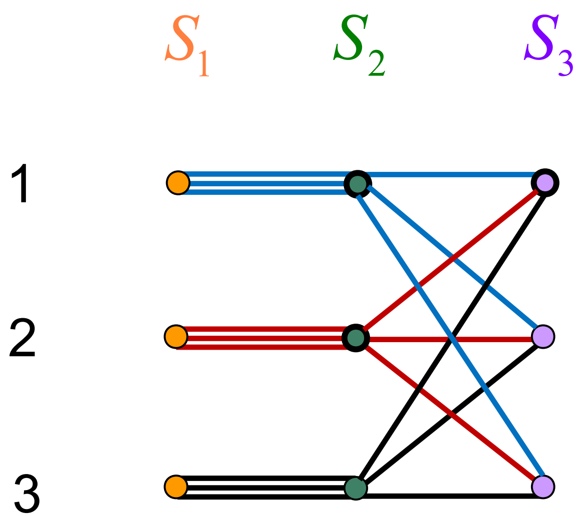
Octahedral Systems

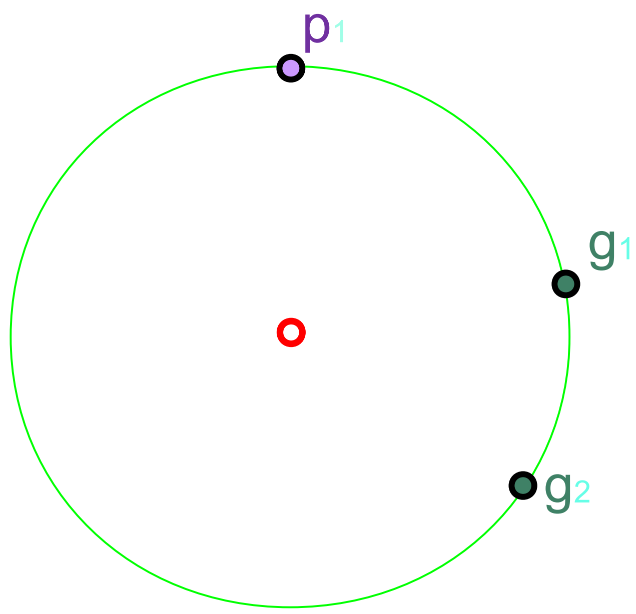




a *non-realizable* octahedral system

Octahedral Systems





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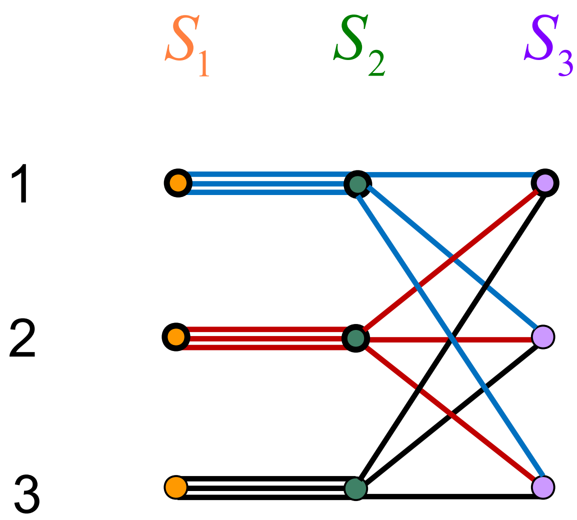
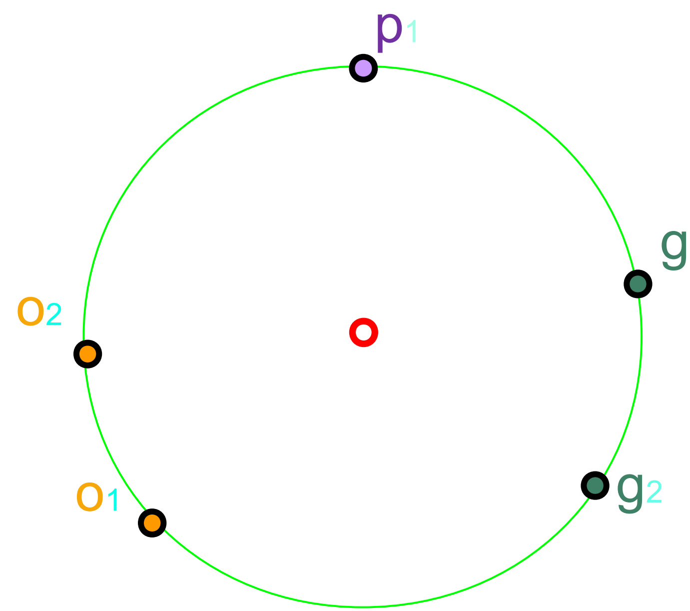
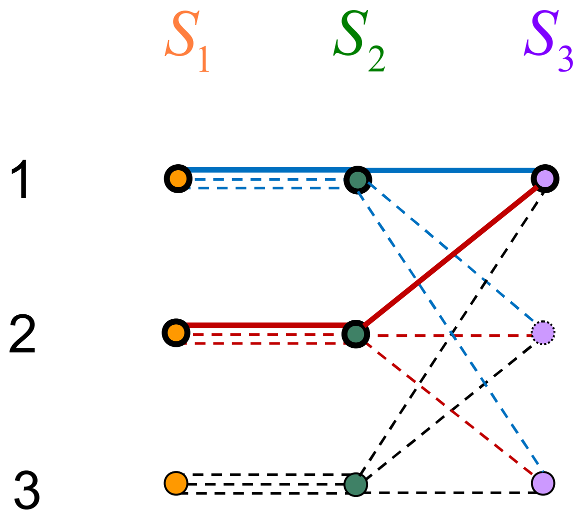
Octahedral Systems

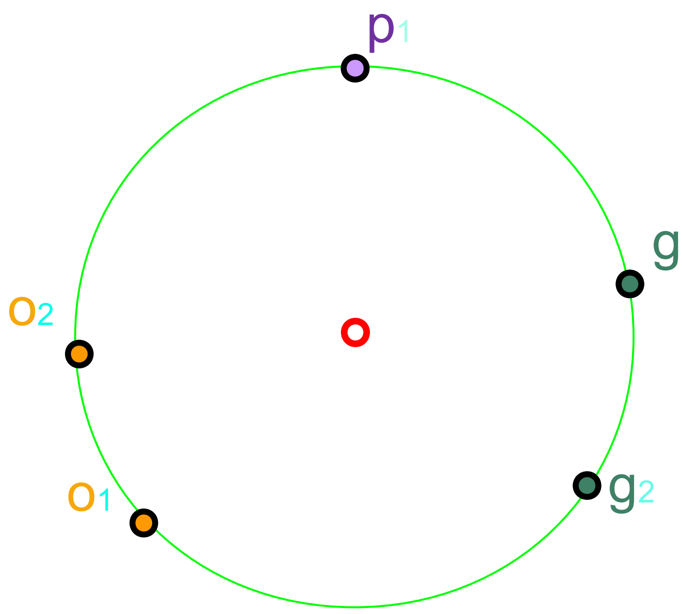




a *non-realizable* octahedral system

Octahedral Systems





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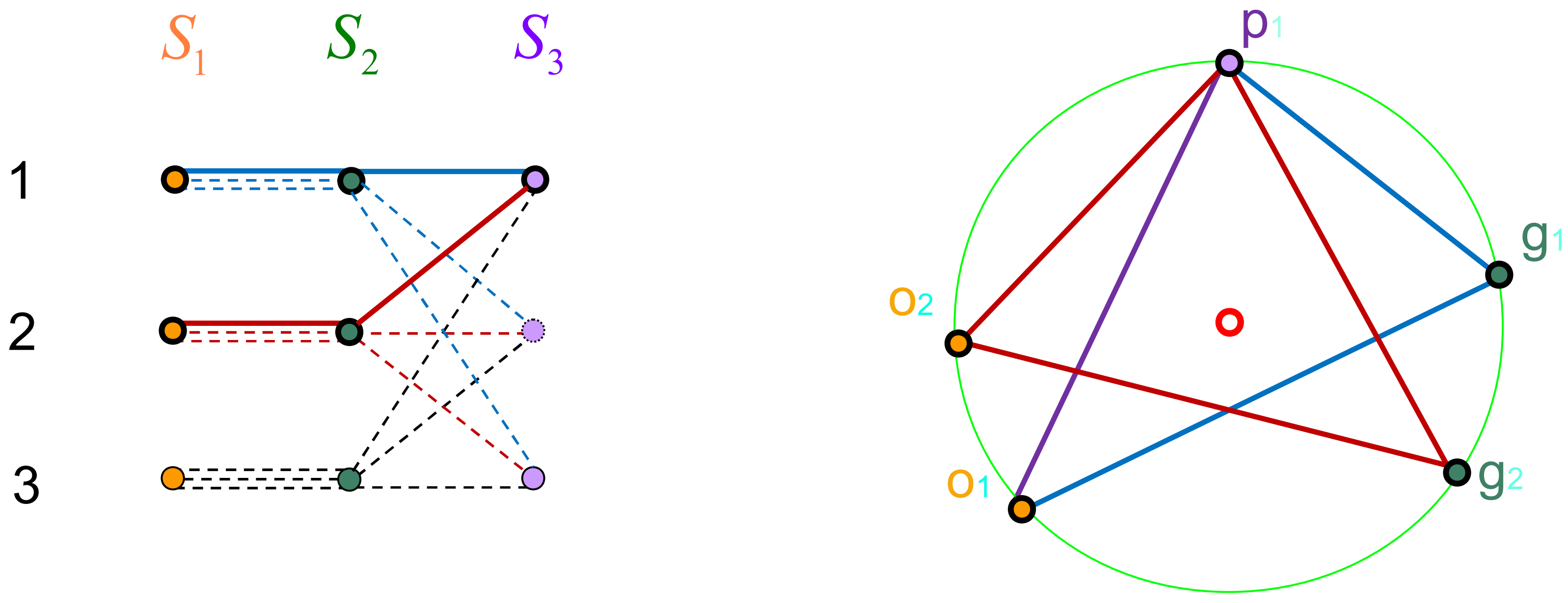
Octahedral Systems



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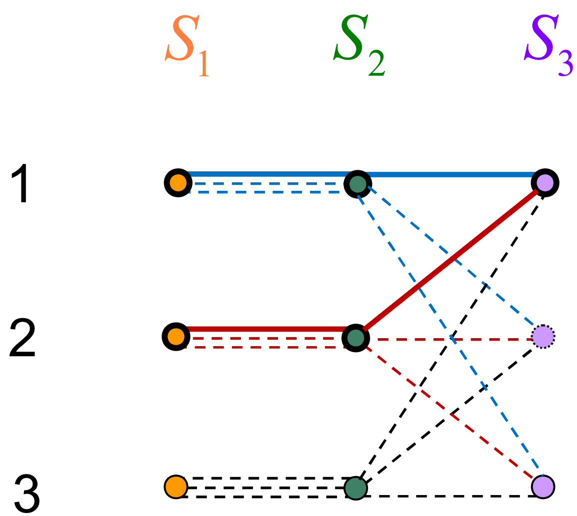
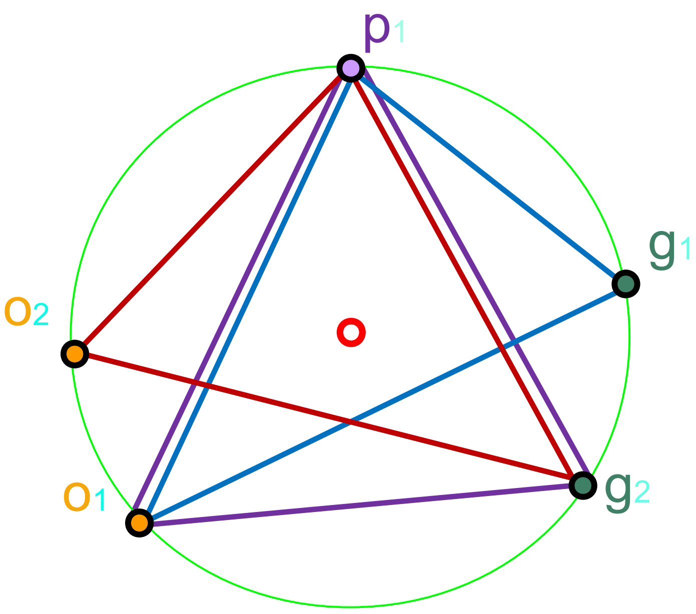
Octahedral Systems





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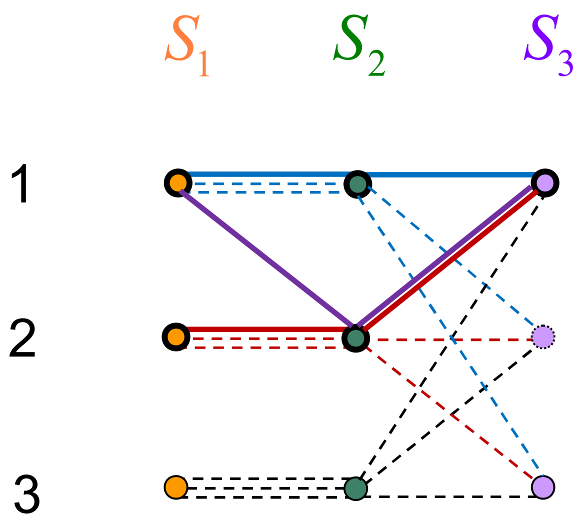
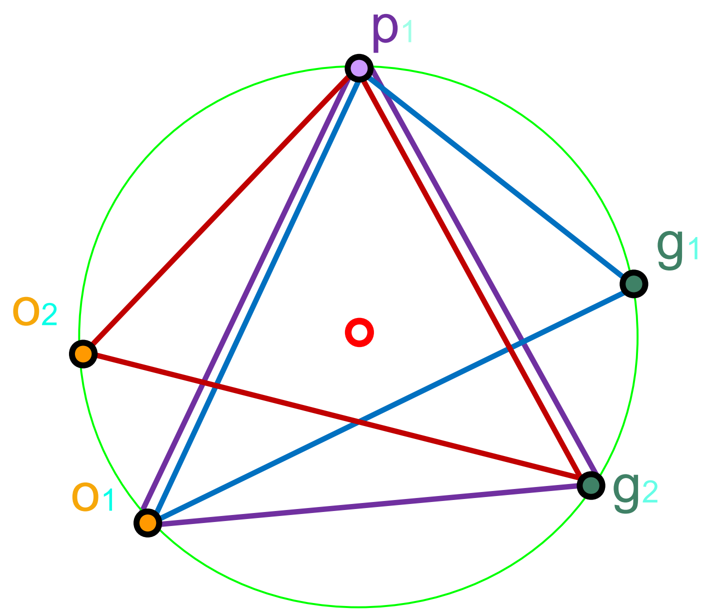
Octahedral Systems





a *non-realizable* octahedral system

Octahedral Systems

- octahedral system without isolated vertex, $|S_1| = \ldots = |S_n| = m$ has at least $m(m+5)/2 - 11$ edges, implying:

$$\mu(d) \geq (d+1)(d+6)/2 - 11$$

- further analysis: $\mu(4) = 17$

[D., Meunier, Sarrabezolles 2013]

Colourful Research Directions

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- Improve lower bound for $\mu(d) = \min_{S,p} depth_S(p)$
- Computational approaches for $\mu(d)$ for small *d*.
- Obtain an efficient algorithm to find a colourful simplex : *Colourful Linear Programming Feasibility* problem

[Bárány, Onn 1997] and [D., Huang, Stephen, Terlaky 2008]

$$\max_p depth_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

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Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} depth_S(p)$$

$\mu(1) = 2 \quad \mu(2) = 5 \quad \mu(3) = 10 \quad \mu(4) = 17$

$$(d+1)(d+6)/2 - 11 \le \mu(d) \le d^2 + 1 \quad \text{for } d \ge 5$$

$\mu(d)$ even for odd d

$$22 \le \mu(5) \le 26$$

Colourful Simplicial Depth Bounds

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$$22 \leq \mu(5) \leq 26$$

✓ ***thank you***

Tverberg Theorem

n points can be partitioned into $\left\lfloor \frac{n-1}{d+1} \right\rfloor + 1$ colours, with a point p in convex hull intersection. [Tverberg 1966]

$\binom{\left\lfloor \frac{n-1}{d+1} \right\rfloor + 1}{d+1}$ combinations to choose d+1 *colours*.

If each combination has at least μ colourful simplices. [Bárány 82]

$$\max_{p} depth_S(p) \geq \mu \binom{\left\lfloor \frac{n-1}{d+1} \right\rfloor + 1}{d+1} = \mu \binom{n}{d+1} + O(n^d)$$

S, p *general position*