# Cutting planes for integer programming based on lattice-free sets

Ricardo Fukasawa

Department of Combinatorics & Optimization
University of Waterloo

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Mixed Integer Programming (MIP):

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \in \mathbb{Z}^p \times \mathbb{R}^{n-p} \end{array}$$

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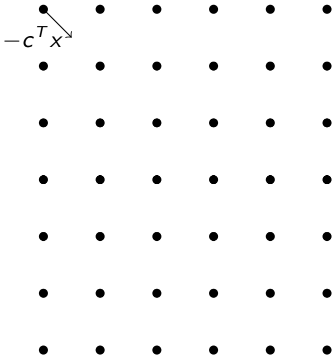

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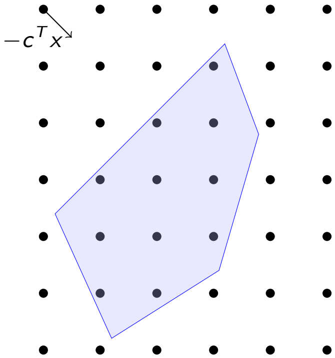

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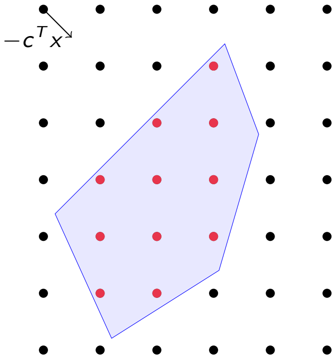

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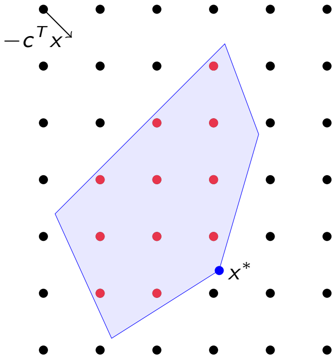

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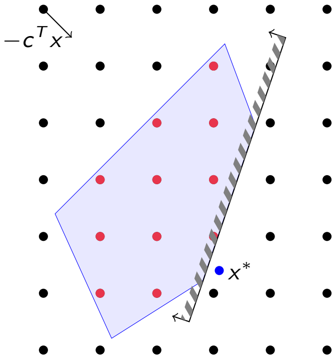

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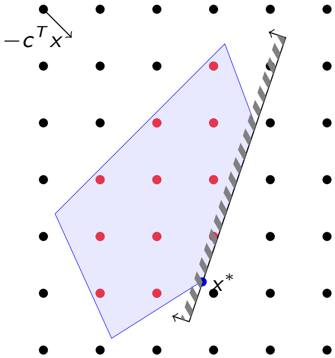

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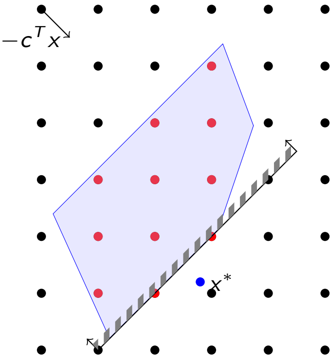

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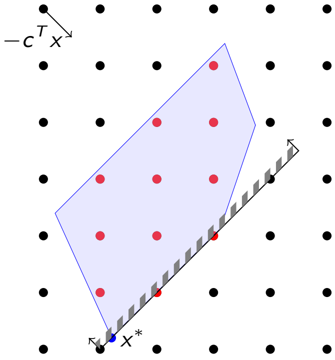

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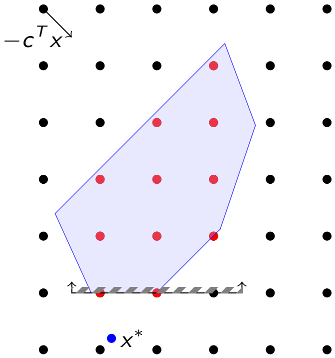

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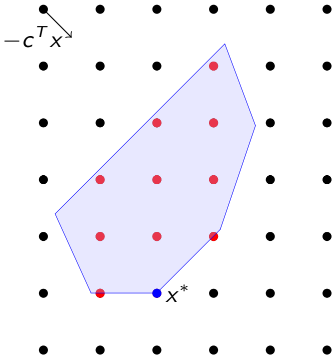

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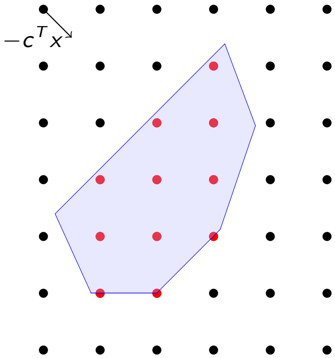
Valid inequalities/
Cutting planes/
Cuts



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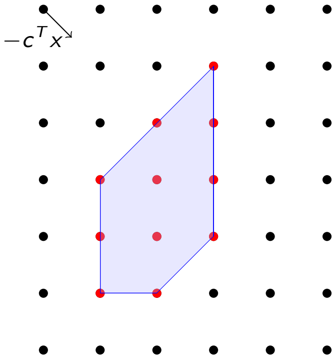


Want "strongest possible" valid inequalities (facet-defining): Get the convex hull of feasible solutions

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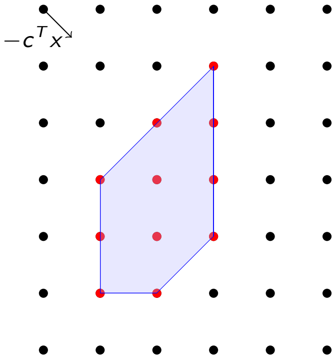



Want "strongest possible" valid inequalities (facet-defining): Get the convex hull of feasible solutions

Typically hard: Relax the problem and get facet-defining inequalities for the relaxation

# Most important cuts

Most important cutting planes used by commercial solvers:

- The Gomory mixed-integer cut (GMI).
- The Mixed Integer Rounding cut (MIR).
- Knapsack Cover and Lifted Knapsack Cover cuts.

Bixby et. al (1999), "Closing the GAP": three most important cuts

> Solution time increases by a factor of 2.52 without GMI cuts.
> Solution time increases by a factor of 1.83 without MIR cuts.
> Solution time increases by a factor of 1.4 without knapsack covers.

(geometric averages after comparing the relative performance of 9 different cutting planes on 106 problems with CPLEX 8.0)

## Multiple-row cutting planes

Assume that we have the optimal tableau of an LP relaxation of a MIP

$$\begin{array}{ll} \min & \bar{c}_N^T x_N \\ \text{s.t.} & x_B - \bar{A}_N x_N = \bar{b} \\ & x \geq 0 \\ & x \in \mathbb{Z}^p \times \mathbb{R}^{n-p} \end{array} \quad (1)$$

Now one can, in addition do the following relaxations:

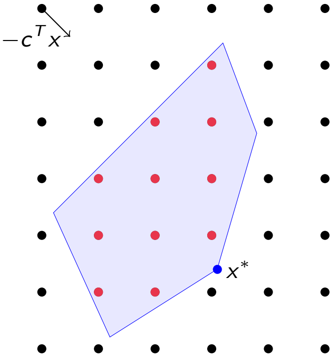
1. Pick a subset of rows associated with basic integer variables
2. Relax the nonnegativity of the basic variables

$$\begin{array}{ll} \min & \bar{c}_N^T x_N \\ \text{s.t.} & x_i - \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}_i, \forall i \in B' \subseteq B \\ & x_N \geq 0 \\ & x \in \mathbb{Z}^p \times \mathbb{R}^{n-p} \end{array} \quad (2)$$

(Gomory '69)

# Corner polyhedron

Intuitively, what we are doing is relaxing all constraints that are not tight at the current optimal LP solution.



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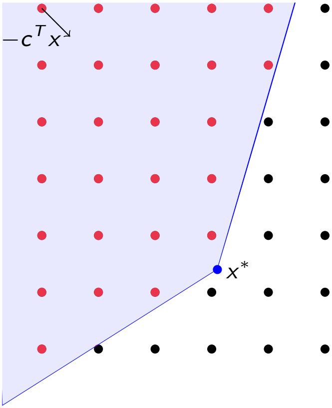

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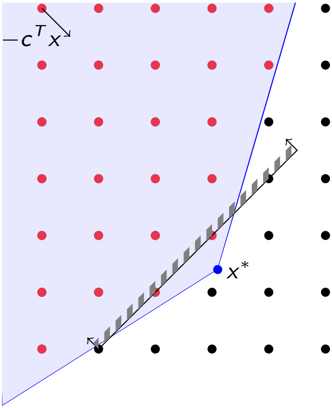

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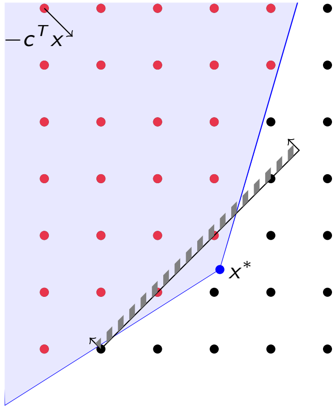


Still allows us to derive cutting planes for $x^*$, but much simpler to analyze.

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Now one can, in addition do the following relaxations:

1. Pick a subset of rows associated with basic integer variables
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3. Relax the integrality of the non-basic variables.
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This motivates the study of the following relaxation:

$$R_f^q(r^1, \ldots, r^k) = conv \left\{ (x, s) \in \mathbb{Z}^q \times \mathbb{R}_+^k : x = f + \sum_{j=1}^{k} r^j s_j \right\}$$

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Remark: If we have a basic feasible solution, we are at the point $(x,s)=(f,0)$.
If $f\in\mathbb{Z}^q$, then we are done, since we are at an integer feasible solution (and hence there is no cut to generate).
So we may assume $f\notin\mathbb{Z}^q$.

# Intersection Cut

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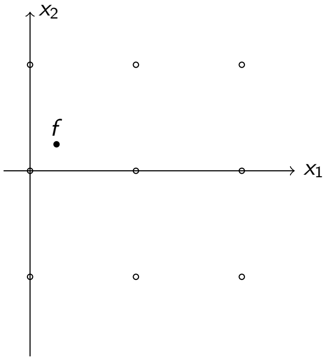


Figure: Picture of the $x$-space ($m = 2$)

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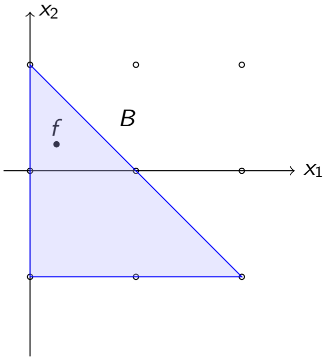


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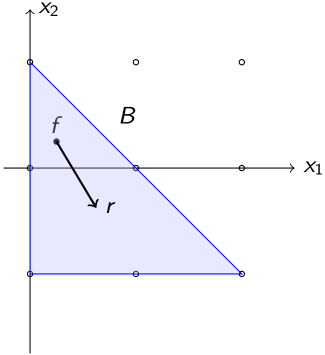


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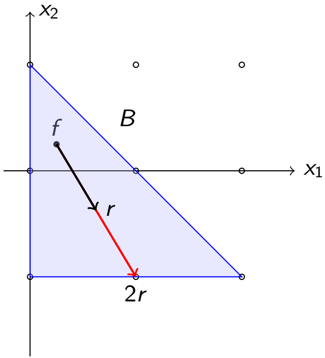


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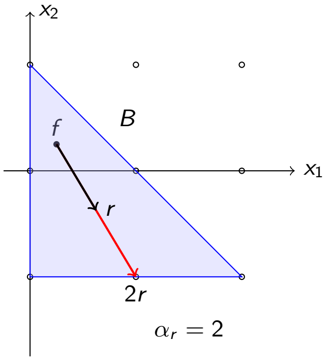


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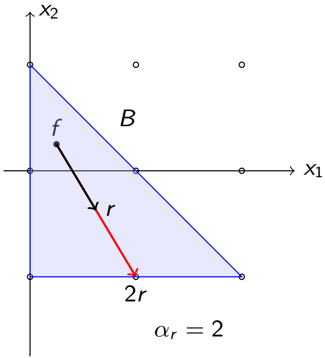


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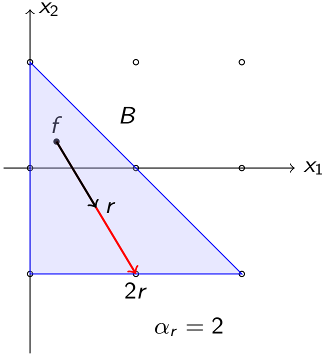


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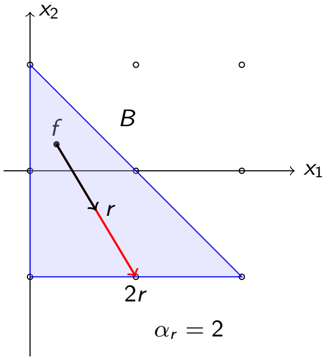


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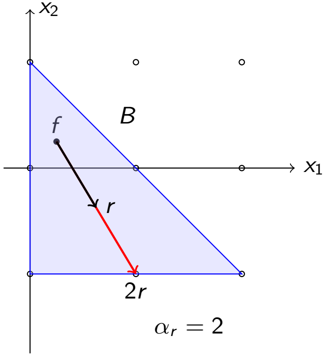
- All nontrivial facet-defining inequalities for $R_f(r^1, \ldots, r^k)$ can be obtained in this way

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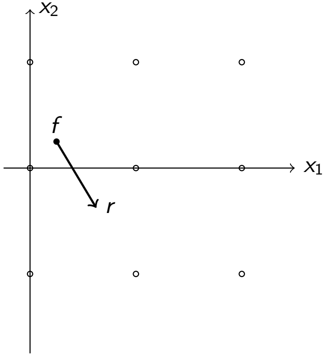


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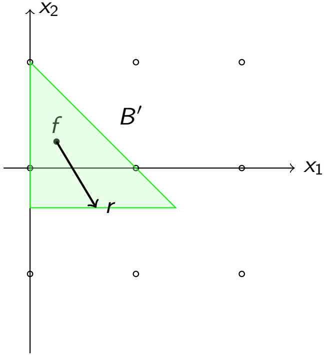


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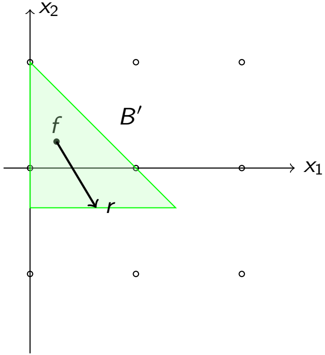


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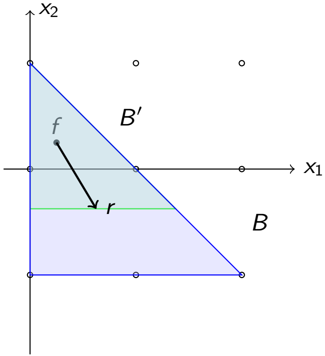


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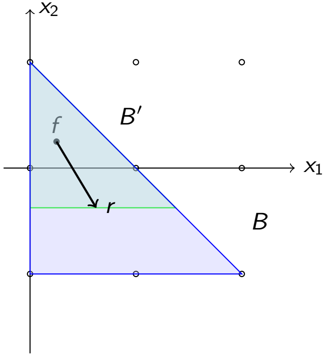


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## Intersection Cut

- $\psi_{B'}(r) = 1$
- $\psi_B(r) = \frac{1}{2}$
- So $B \supseteq B'$ implies $\psi_B(r) \leq \psi_{B'}(r)$: Larger set gives better coefficients

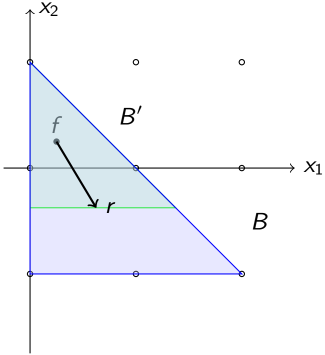


Figure: Picture of the $x$-space ($m = 2$)

# Intersection Cut

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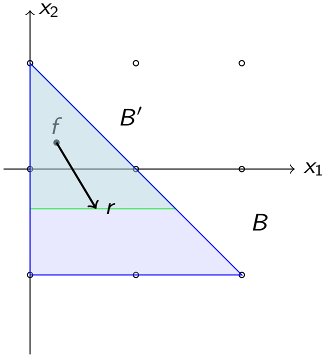


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- Borozan and Cornuéjols '09, Basu, Conforti, Cornuéjols, Zambelli '10: Minimal inequalities for a semi-infinite relaxation come from maximal lattice-free convex sets.

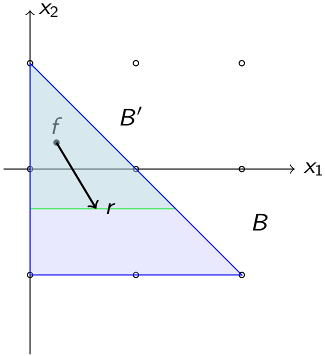



Figure: Picture of the $x$-space ($m = 2$)

# Maximal lattice-free convex sets

## Theorem (Lovasz '89)

*A set $K \subseteq \mathbb{R}^q$ is a maximal $\mathbb{Z}^q$-free convex set if and only if*

- *Either $K$ is a polyhedron of the form $K = P + L$, where $P$ is a polytope, $L$ is a rational linear space, $dim(P) + dim(L) = q$, $K$ does not contain any point of $\mathbb{Z}^q$ in its interior and there is a point of $\mathbb{Z}^q$ in the relative interior of each facet of $K$*
- *or $K$ is an irrational hyperplane*

## Corollary

*Maximal lattice-free convex sets in $\mathbb{R}^q$ are polyhedra with at most $2^q$ facets. (Also follows from Doignon '73, Bell '77, Scarf '77)*

## Proof.

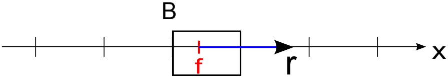
Suppose that there are more than $2^q$ facets, then there are two facets with points $x^1$, $x^2$ in their respective relative interior such that they have the same parity. But then, $(x^1 + x^2)/2$ is a lattice point in the interior. □

## Why is $R_f^q(r^1, \ldots, r^k)$ a good relaxation?

Consider the case where $q = 1$:

$$R_f^1(r^1, \ldots, r^k) = conv\{(x,s) \in \mathbb{Z} \times \mathbb{R}_+^k : x = f + \sum_{j=1}^{k} r^j s_j\}$$

In this case, lattice free convex sets are simply intervals. Consider then the lattice free interval $B = [\lfloor f \rfloor, \lceil f \rceil]$.




$$\psi_B(r^j) = \begin{cases} \frac{r^j}{1-\hat{f}} & , \text{ if } r^j > 0 \\ \frac{-r^j}{\hat{f}} & , \text{ if } r^j \leq 0 \end{cases}$$

In terms of the original constraint:

$$x + \sum_{j=1}^{k} a^j s_j = f$$

$$\psi_B(a^j) = \begin{cases} \frac{-a^j}{1-\hat{f}} & , \text{ if } a^j < 0 \\ \frac{a^j}{\hat{f}} & , \text{ if } a^j \geq 0 \end{cases}$$

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If our original relaxation was:

$$\{(x, s) \in \mathbb{Z} \times \mathbb{Z}_+^p \times \mathbb{R}_+^{k-p} : x + \sum_{j=1}^{k} a^j s_j = f\}$$

We can "lift" the nonbasic integer variables and get the following inequality:

$$\sum_{j=1,\ldots,p:\hat{a^j} \leq \hat{f}} \frac{\hat{a^j}}{\hat{f}} + \sum_{j=1,\ldots,p:\hat{a^j} > \hat{f}} \frac{1 - \hat{a^j}}{1 - \hat{f}} + \sum_{j=p+1,\ldots,k:a^j \geq 0} \frac{a^j}{\hat{f}} - \sum_{j=p+1,\ldots,k:a^j < 0} \frac{a^j}{1 - \hat{f}} \geq 1$$

we get exactly the Gomory Mixed-Integer (GMI) cut.

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So this is a way to generalize GMI cuts to multiple rows.

## Generating stronger cuts

Assume that we have the optimal tableau of an LP relaxation of a MIP

$$\begin{array}{ll} \min & \bar{c}_N^T x_N \\ \text{s.t.} & x_B - \bar{A}_N x_N = \bar{b} \\ & x \geq 0 \\ & x \in \mathbb{Z}^p \times \mathbb{R}^{n-p} \end{array} \tag{5}$$

Now one can, in addition do the following relaxations:

1. Pick a subset of rows associated with basic integer variables
2. Relax the integrality of the non-basic variables
3. Relax the nonnegativity of the basic variables

$$\begin{array}{ll} \min & \bar{c}_N^T x_N \\ \text{s.t.} & x_i - \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}_i, \forall i \in B' \subseteq B \\ & \\ & x_N \geq 0 \\ & x_i \in \mathbb{Z}, \forall i \in B' \subseteq B \end{array} \tag{6}$$

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# A motivating example

## Example

Let $r_1, r_1, r_2, r_3, r_4, r_5, f \in \mathbb{R}^2$ be as in the picture to the right, and consider the following set,

$$X = \left\{(x, s) \in \mathbb{Z}^2 \times \mathbb{R}^5_+ \ : \ x = f + \sum_{j=1}^{5} r_j s_j\right\}$$

Cornuejóls and Margot (2009) and Andersen et al. (2007):

$$s_1 + s_2 + s_3 + s_4 + s_5 \geq 1$$

is valid and facet-defining for $X$.
However, using the non-negativity of the $x$ variables in $X_+ = X \cap \mathbb{R}^+_7$, it is possible to show that the following stronger inequality:

$$s_1 + s_2 + s_3 - s_5 \geq 1$$

is valid (and facet defining) for $X_+$.

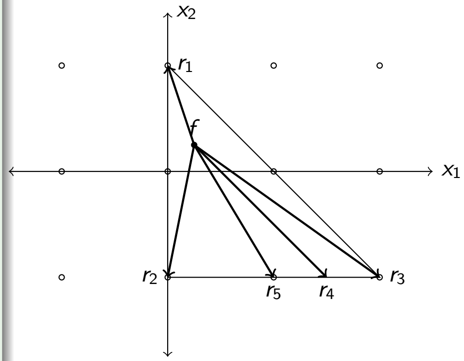


Figure:

# Cuts based on $S$-free sets

In general, we are interested now on:

$$conv \left\{ (x, s) \in S \times \mathbb{R}^k_+ : x = f + \sum_{j=1}^{k} r^j s_j \right\}$$

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This last paper in particular generalizes Lovász' results and the Borozan and Cornuéjols theorem.

# Maximal lattice-free convex sets in $\mathbb{R}^2$

Lovasz (89): Maximal lattice-free convex sets in $\mathbb{R}^2$ are either irrational lines or

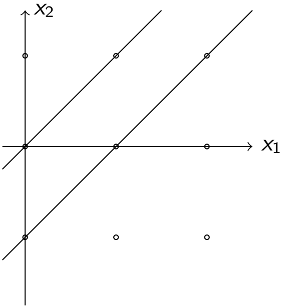


Figure: Split

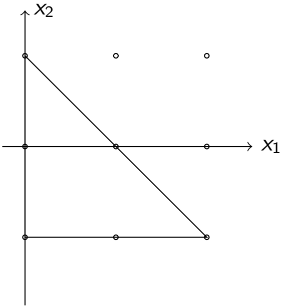


Figure: Triangle

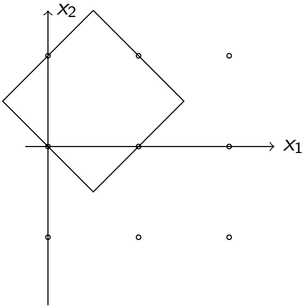


Figure: Quadrilateral

All with at least one integer point in the relative interior of each edge.

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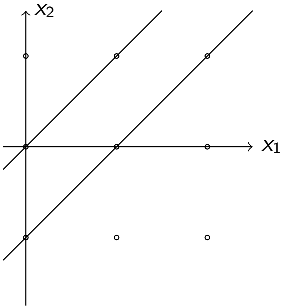


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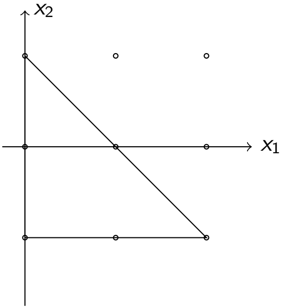


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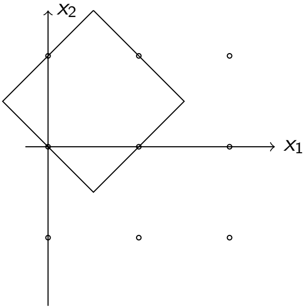
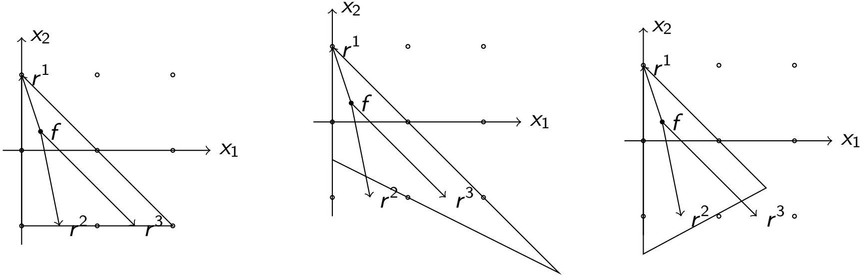



Figure: Quadrilateral

All with at least one integer point in the relative interior of each edge.
Problem: There is an infinite number of them. How do we generate all possible inequalities?

# Facet-defining inequalities

Maximal lattice-free convex sets give rise to minimal inequalities. However, may not be facet-defining.



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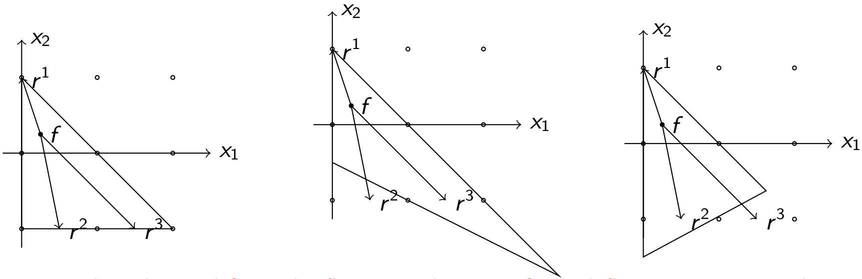


The inequality obtained from the first triangle is not facet-defining, since it can be obtained as a convex combination of the other two inequalities.

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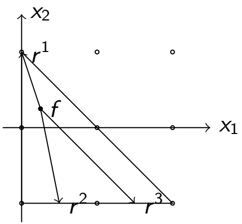
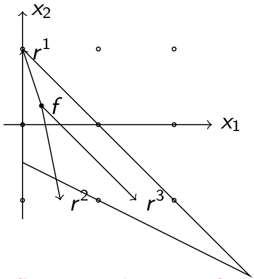
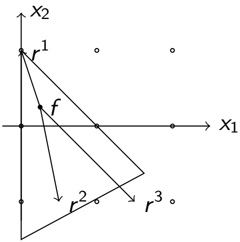


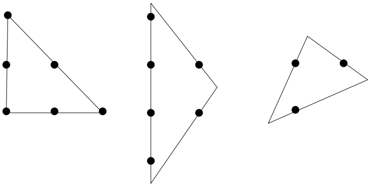




The inequality obtained from the first triangle is not facet-defining, since it can be obtained as a convex combination of the other two inequalities.

We are interested only in the maximal lattice-free sets that generate facet-defining inequalities.

# Triangles

Dey and Wolsey (08), three types of triangles (up to unimodular transformation):



Type 1 Type 2 Type 3

Each type of lattice-free set (split, triangle, quadrilateral) gives rise to a class of inequalities.

Want to be able to generate all facet-defining cuts in a given class of inequalities.

# Cornuéjols and Margot (09)

## Question

Which maximal lattice-free convex sets give rise to facet-defining inequalities for $R_f^2(r^1, \ldots, r^k)$?

## Definition

We say a maximal lattice-free convex set is <u>compatible</u> if its "corner" lies in the half line $f + \alpha r^i$ for some $i \in \{1, \ldots, k\}$, $\alpha \geq 0$.

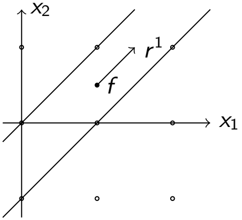



Figure: Compatible Split

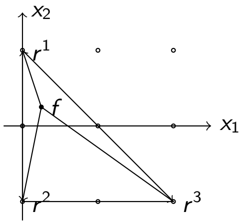



Figure: Compatible Triangle

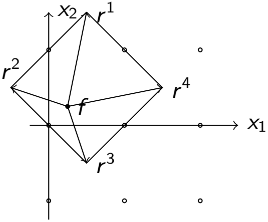



Figure: Compatible Quadrilateral

# Cornuéjols and Margot (09)

**Theorem (Cornuéjols and Margot (09))**

*All nontrivial facet-defining inequalities of $R_f^2(r^1, \ldots, r^k)$ are intersection cuts generated by:*

- *Compatible splits*
- *Compatible triangles*
- *Compatible quadrilaterals*
- *Splits that satisfy a certain Ray condition*
- *Triangles that satisfy a certain Ray condition*

# How do we get these cuts?

Cornuéjols and Margot give a way to identify, given a maximal lattice-free convex set $B$, if the associated intersection cut defines a facet of $R_f(r^1, \ldots, r^k)$.

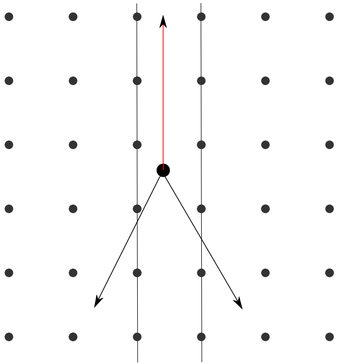
**Question**

Given $R_f(r^1, \ldots, r^k)$, how can we construct the associated maximal lattice-free convex sets that give facet-defining inequalities?

(Chen, Cook, F., Steffy)

## Compatible splits

For each $i = 1, \ldots, k$, check if the line $f + \alpha r^i$ contains an integer point. If not, we can generate a compatible split.



Let $ax_1 + bx_2 = c$ be the equation of the line $f + \alpha r^i$, with $a, b \in \mathbb{Z}$ relative prime.

$f + \alpha r^i$ contains an integer point if and only if $c \notin \mathbb{Z}$.

The associated compatible split is:

$$\lfloor c \rfloor \leq ax_1 + bx_2 \leq \lceil c \rceil$$

## Compatible triangles

For every triple $\{i, j, l\} \subseteq \{1, \ldots, k\}$, generate a compatible triangle. How?

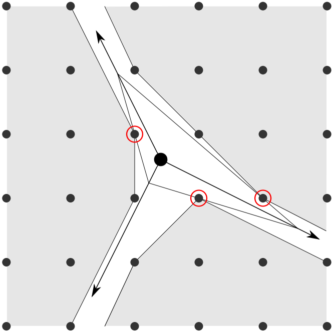
- Every edge of a maximal lattice-free set must contain an integer point in its relative interior.
- Consider an edge $e$ with corners in the half-lines generated by $r^1, r^2$.

For every triple of possible points in the integer hull, try to find a compatible triangle.

# Compatible triangles

For every triple $\{i, j, l\} \subseteq \{1, \ldots, k\}$, generate a compatible triangle. How?

- Every edge of a maximal lattice-free set must contain an integer point in its relative interior.
- Consider an edge $e$ with corners in the half-lines generated by $r^1, r^2$.
- Then $e$ must contain an extreme point of the convex hull of integer points in $f + cone(r^1, r^2)$ in its relative interior.



For every triple of possible points in the integer hull, try to find a compatible triangle.

# Compatible triangles

Triangle compatible with $r^1, r^2, r^3$:

- Compute integer hull of $f + cone(r^1, r^2)$, $f + cone(r^2, r^3)$, $f + cone(r^1, r^3)$, call them $T_1$, $T_2$, $T_3$ respectively.
- For each triple of extreme points $p^1, p^2, p^3$ of $T_1$, $T_2$, $T_3$, impose that we must have a triangle compatible with $r^1, r^2, r^3$ and with $p^1, p^2, p^3$ in each respective edge.
  - This can be done by solving a system of 6 nonlinear equations with 6 variables. Solved a priori using Gröbner basis and obtained a closed form solution.

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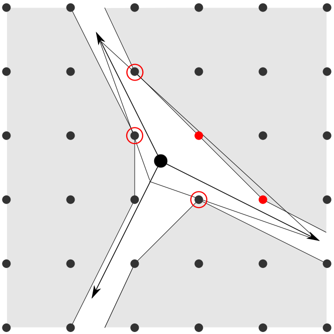

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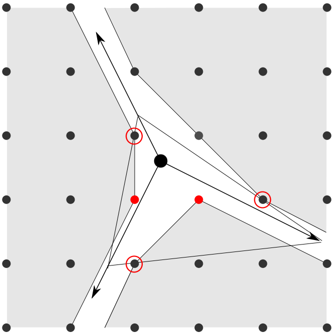

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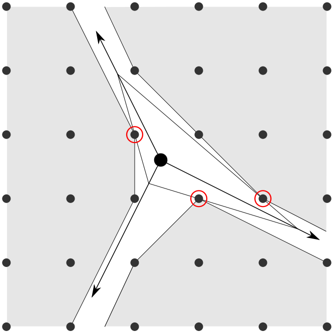
  - This can be done by solving a system of 6 nonlinear equations with 6 variables. Solved a priori using Gröbner basis and obtained a closed form solution.
- Check that it is lattice-free



# Compatible triangles

**Lemma (Chen, Cook, F., Steffy)**

*For a given set of rays $r^1, r^2, r^3$, if there exists a compatible maximal lattice-free triangle, then it is unique.*

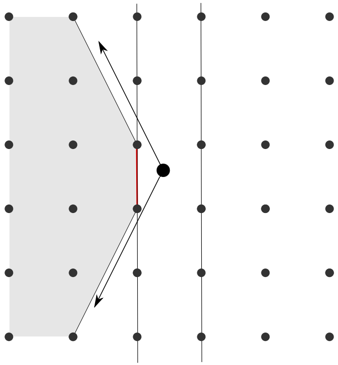
So it suffices to do as proposed until we get a compatible maximal lattice-free triangle.

# Compatible quadrilaterals

A similar approach to triangles.

## The ray condition: Non-compatible splits

For every pair $r^i, r^j$, compute the integer hull of $f + cone(r^i, r^j)$ and use the edges to generate splits:



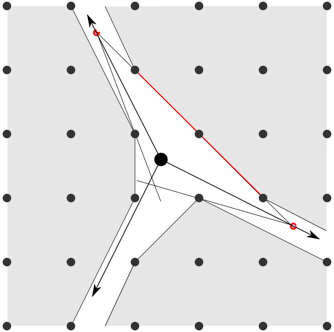
## The ray condition: Non-compatible triangles

**Lemma (Chen, Cook, F., Steffy)**

*Let $T$ be a noncompatible maximal lattice-free triangle satisfying the ray condition for $R_f(r^1, \ldots, r^k)$. Then, under some mild conditions, $T$ is not of type 3. Moreover, there is a maximal lattice-free triangle $T'$ that generates the same inequality and has two rays $r^i$ and $r^j$ pointing to distinct corners of it.*

# The ray condition: Non-compatible triangles

For every edge in each of the integer hulls, use it to determine three possible triangles:



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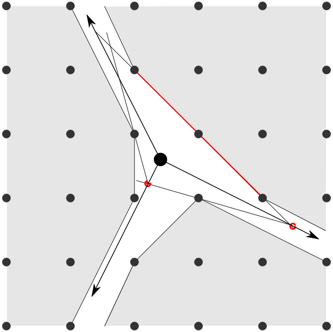

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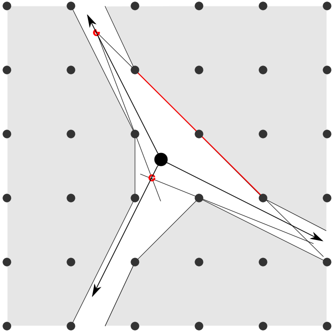

# All facet-defining inequalities

**Lemma (Chen, Cook, F., Steffy)**

*Assuming* $\text{cone}(r^1, \ldots, r^k) = \mathbb{R}^2$*, the procedures described generate all facet-defining inequalities for* $R_f(r^1, \ldots, r^k)$

# Lifting the nonbasic integer variables

Recall that we did the following relaxations:

1. Pick a subset of rows associated with basic integer variables
2. Relax the nonnegativity of the basic variables
3. Relax the integrality of the non-basic variables.

$$R_f^q(r^1, \ldots, r^k) = conv\left\{(x,s) \in \mathbb{Z}^q \times \mathbb{R}_+^k : x = f + \sum_{j=1}^{k} r^j s_j\right\}$$

What if we do not relax the integrality of non-basic variables?

$$conv\left\{(x,s) \in \mathbb{Z}^q \times \mathbb{R}_+^k : x = f + r^1 s_1 + \sum_{j=2}^{k} r^j s_j, s_1 \in \mathbb{Z}\right\}$$

What is the possible coefficient $\phi_B(r^1)$ for $s_1$?

Note that we may rewrite our set as:

$$conv\left\{(x,s) \in \mathbb{Z}^q \times \mathbb{R}_+^k : x + ts_1 = f + (r^1 + t)s_1 + \sum_{j=2}^{k} r^j s_j, s_1 \in \mathbb{Z}\right\}$$

for any $t \in \mathbb{Z}^q$.

Hence, one possible coefficient for $s_1$ will be $\phi_B(r^1) = \inf_{t\in\mathbb{Z}^q} \psi_B(r^1 + t) \le \psi_B(r^1)$.

(Trivial lifting. This is the coefficient obtained for integer variables in the GMI cut)

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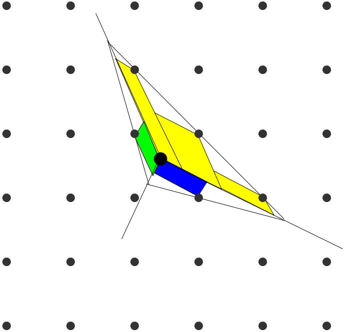
Question: When is this coefficient the smallest possible?

# Lifting in $\mathbb{R}^2$

Are there regions for which the trivial lifting is the smallest one?

For $t = 0$.



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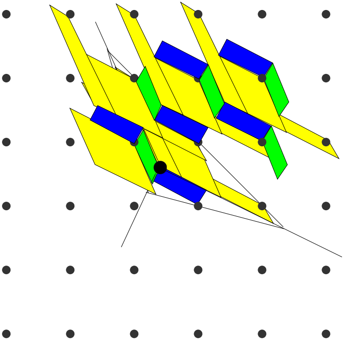

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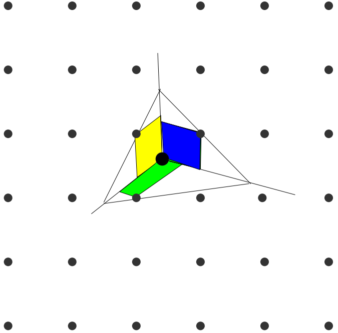

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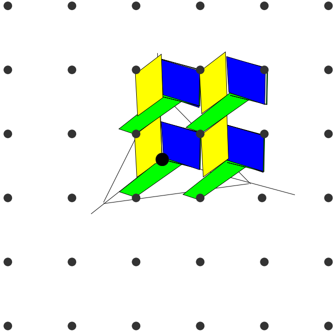

# Lifting

See:

- Dey and Wolsey '10
- Basu, Campelo, Conforti, Cornuéjols, Zambelli '10
- Basu, Cornuéjols, Köppe '11
- Conforti, Cornuéjols, Zambelli '11

# Conclusion

- Multi-row cutting planes are an important area of MIP that has nice connections to geometry
- Nice theoretical results and properties
- Other interesting results exist (e.g. what is the split/MIR rank of cuts, what different disjunctions can lead to these cuts)
- Some open problems:
  - Is there a characterization of lattice-free polyhedra in $\mathbb{R}^q$ for $q > 2$?
  - How do some of the results in 2D generalize?
  - Computationally, how do we choose the $q$ rows that will give us relaxations?

THANK YOU