Chiral extensions of chiral polytopes

Gabe Cunningham

Daniel Pellicer

Abstract polytope

Abstract polytope

Abstract polytope

Abstract polytope $\longrightarrow$ combinatorial generalization of convex polytope

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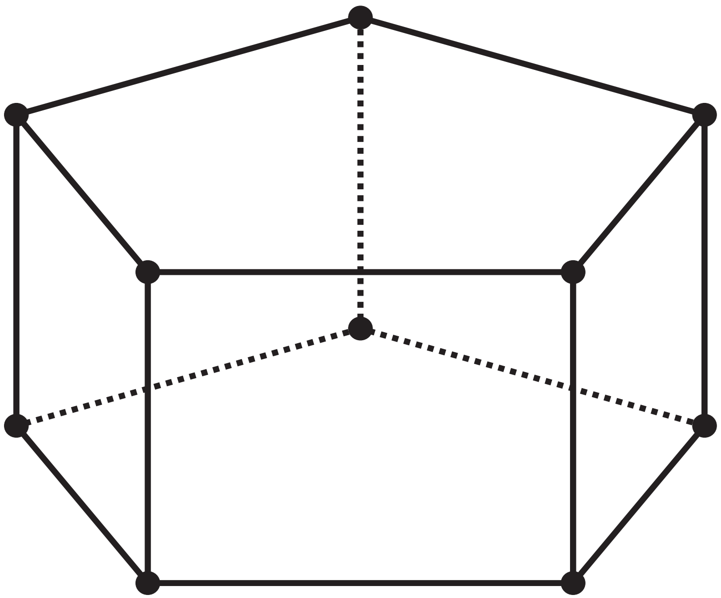
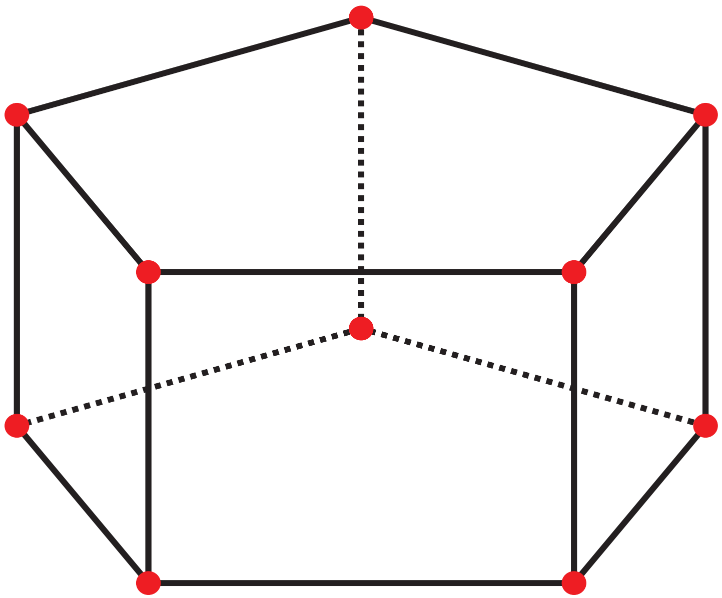
Abstract polytope $\longrightarrow$ combinatorial generalization of convex polytope



Abstract polytope

Abstract polytope $\longrightarrow$ combinatorial generalization of convex polytope



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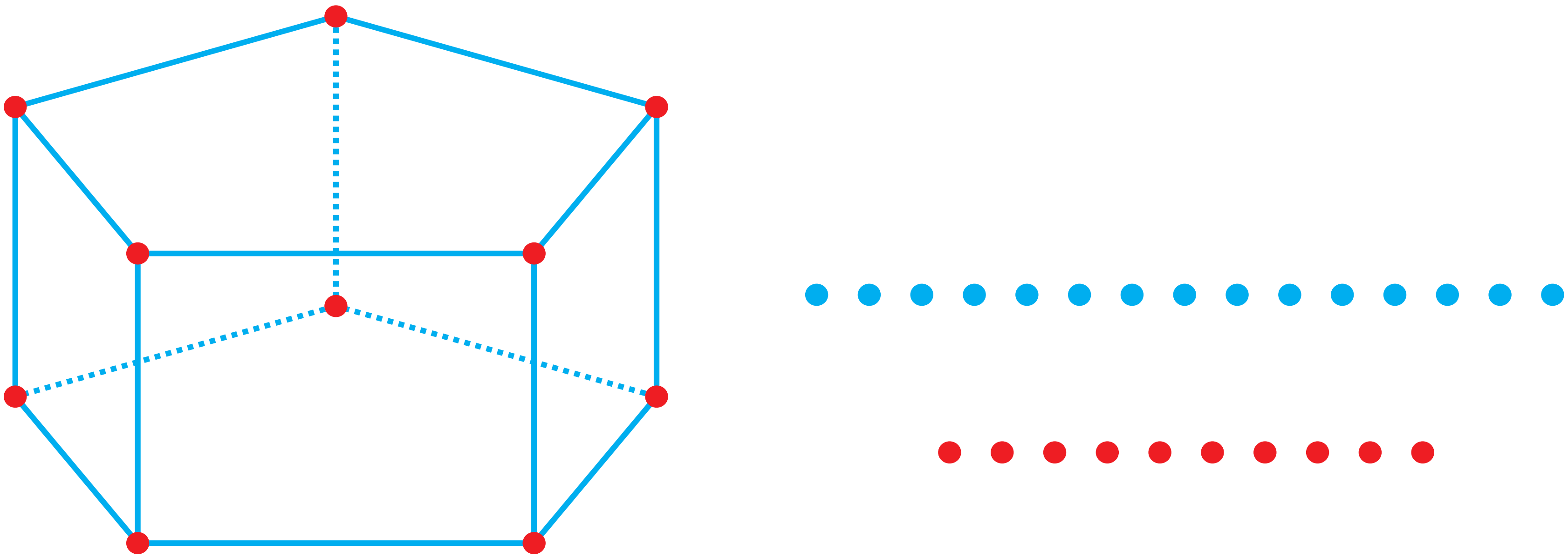

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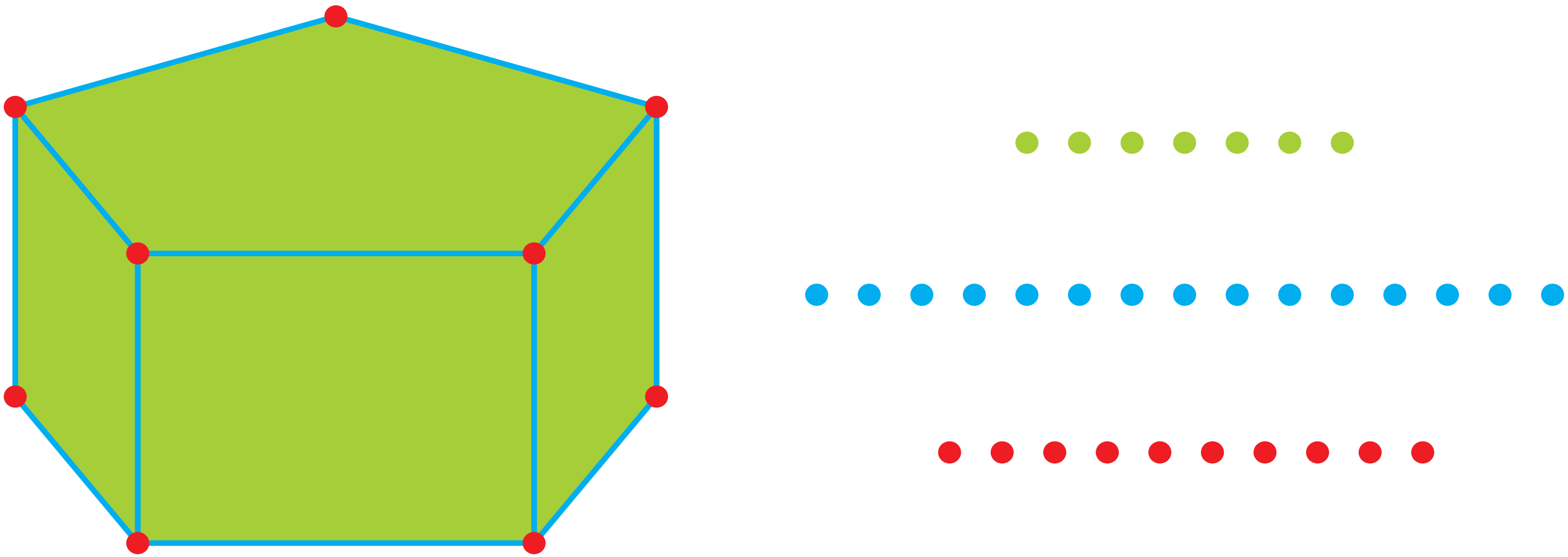

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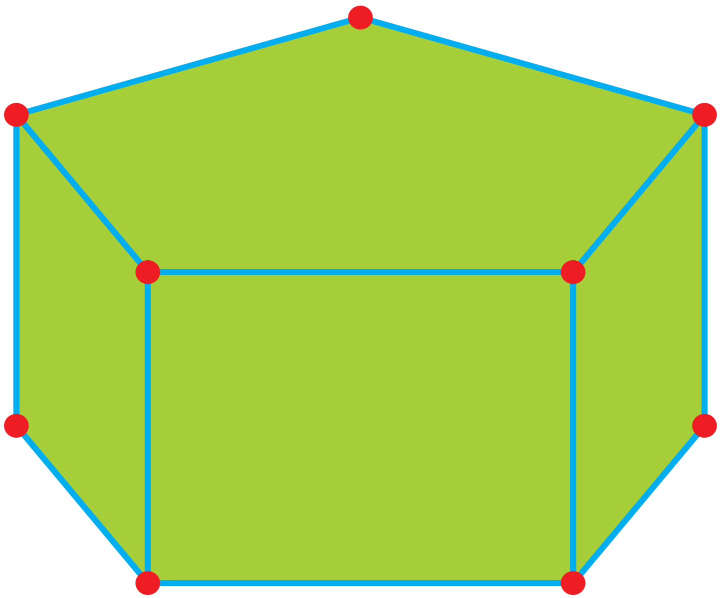

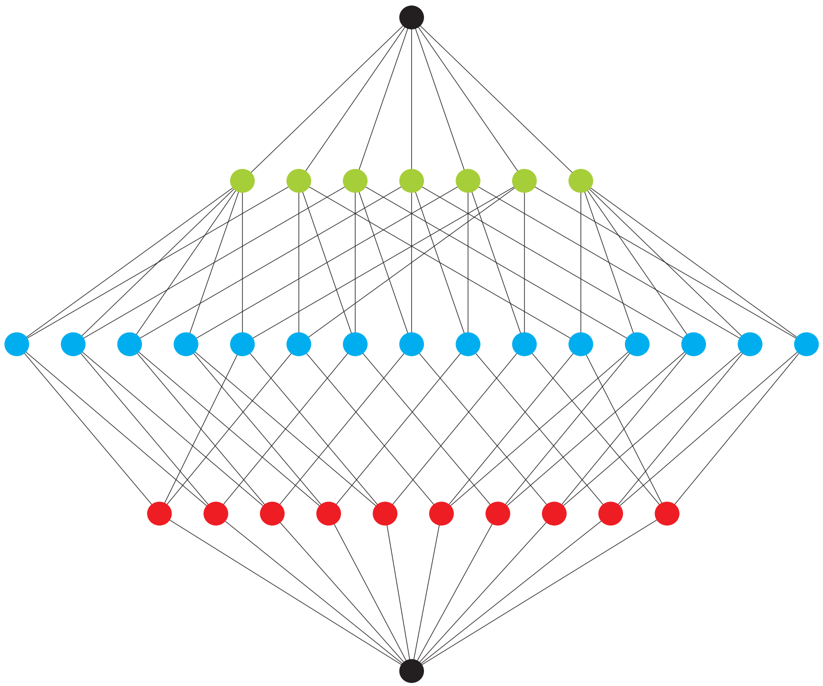

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- POSET

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- Unique maximal and minimal elements

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etc.

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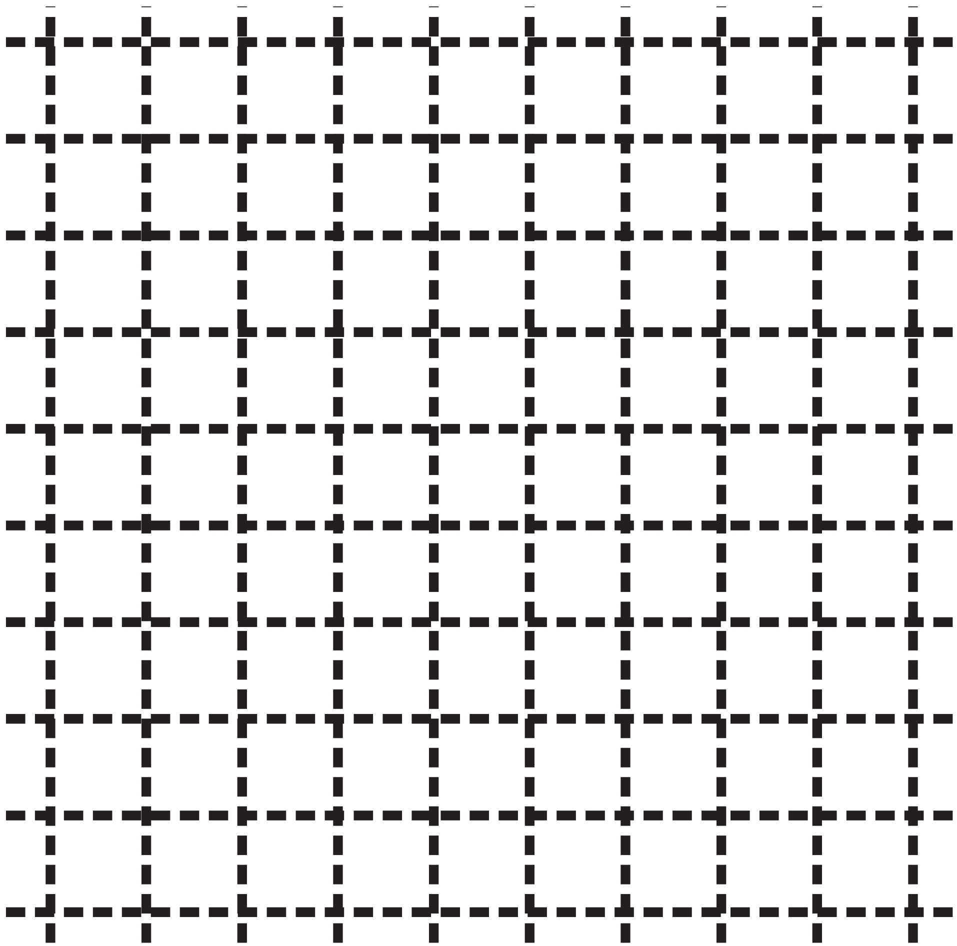

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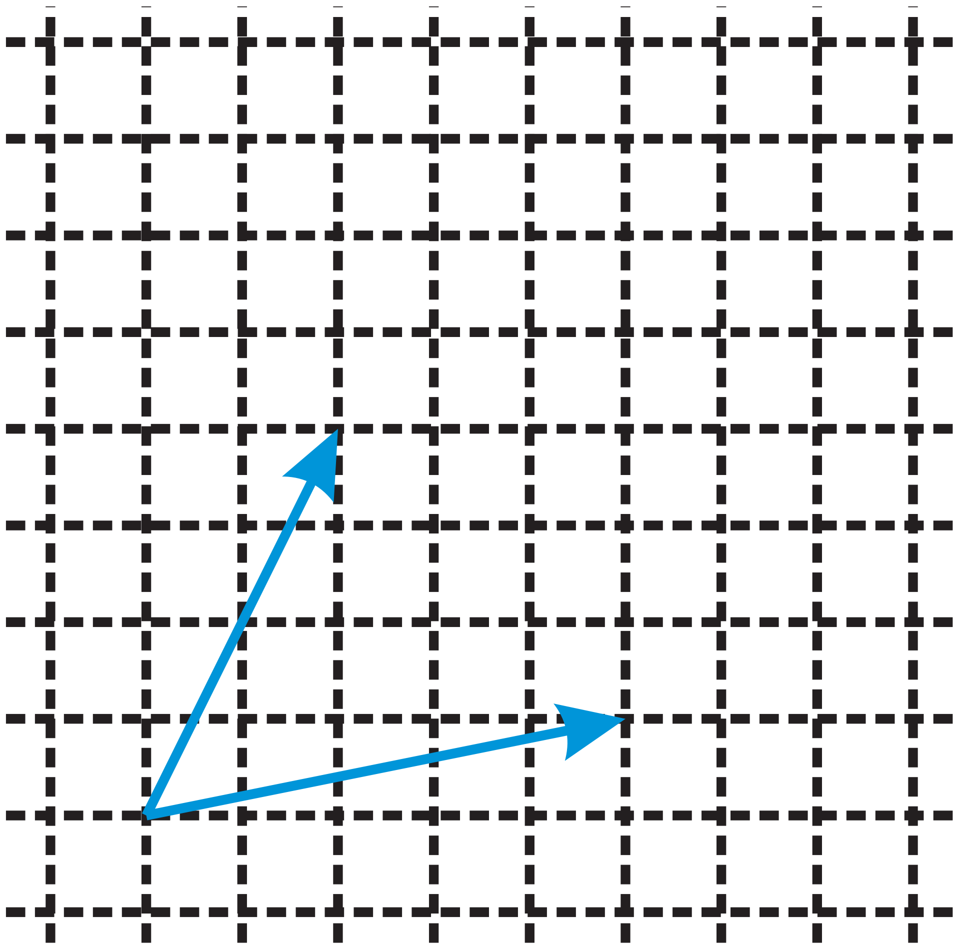

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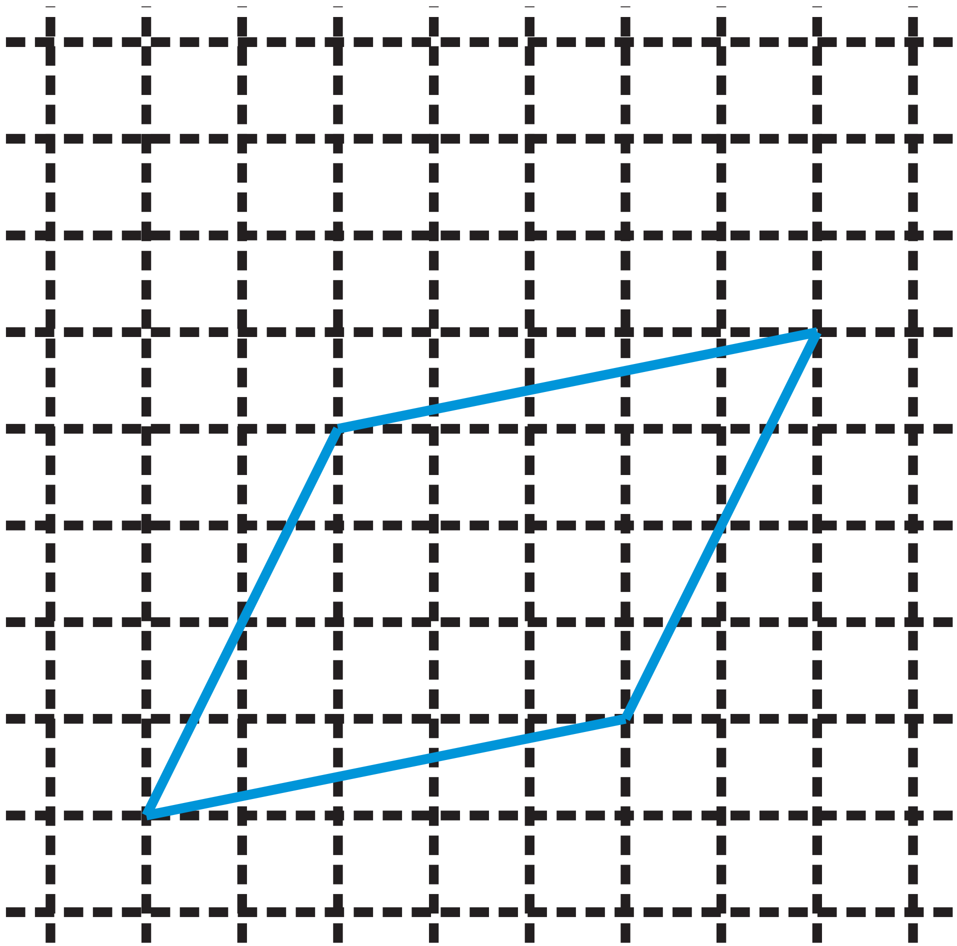

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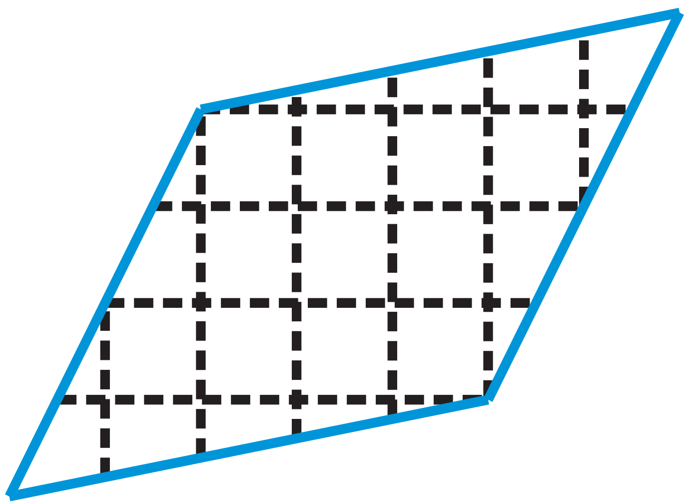
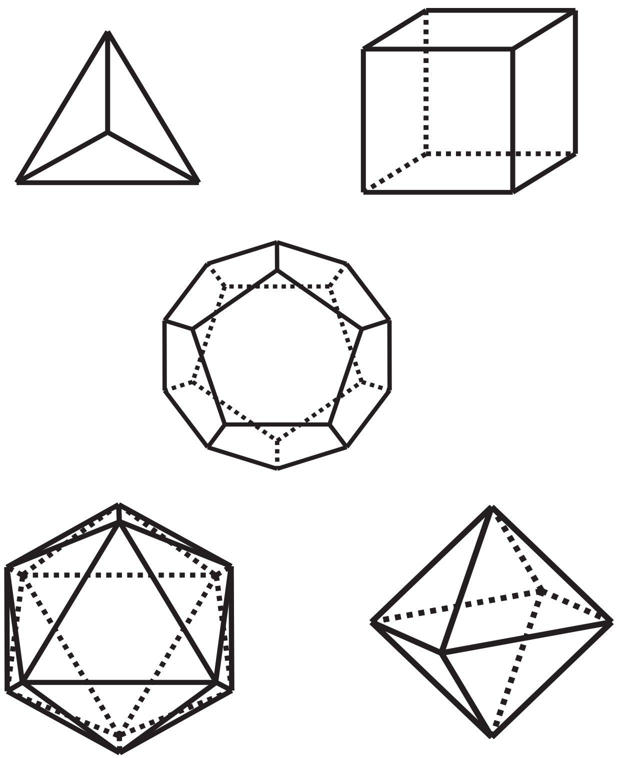


Regular polytopes



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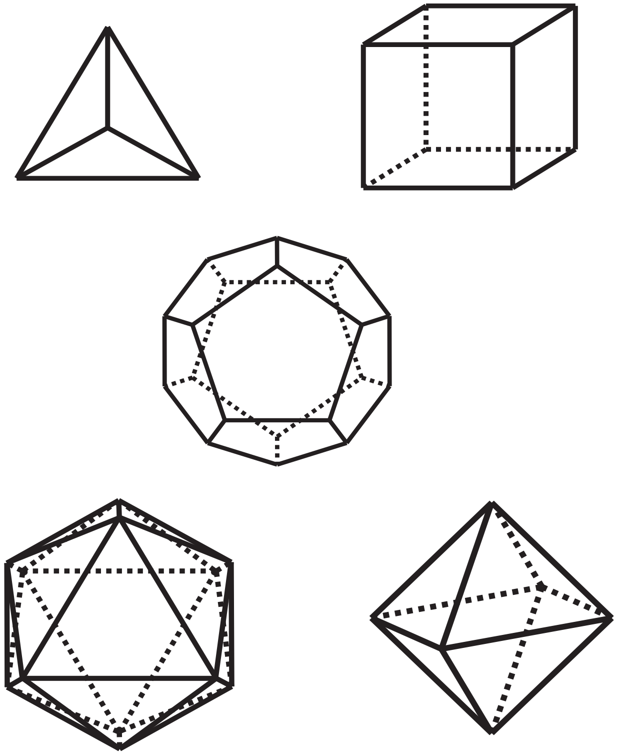

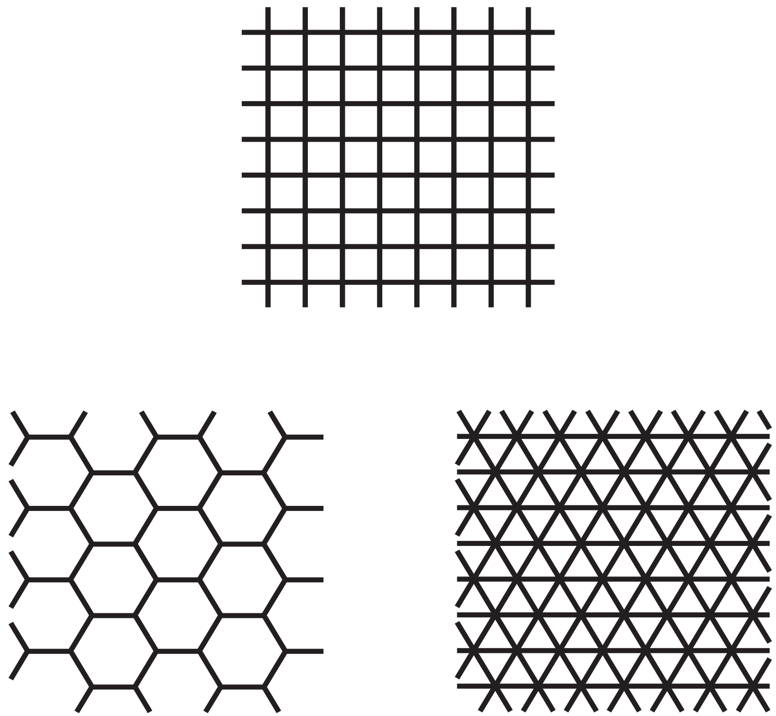

Regular polytopes

Flag

Regular polytopes

Flag $\longrightarrow$ maximal totally ordered subset

Regular polytopes

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Automorphism

Regular polytopes

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Automorphism $\longrightarrow$ order preserving bijection

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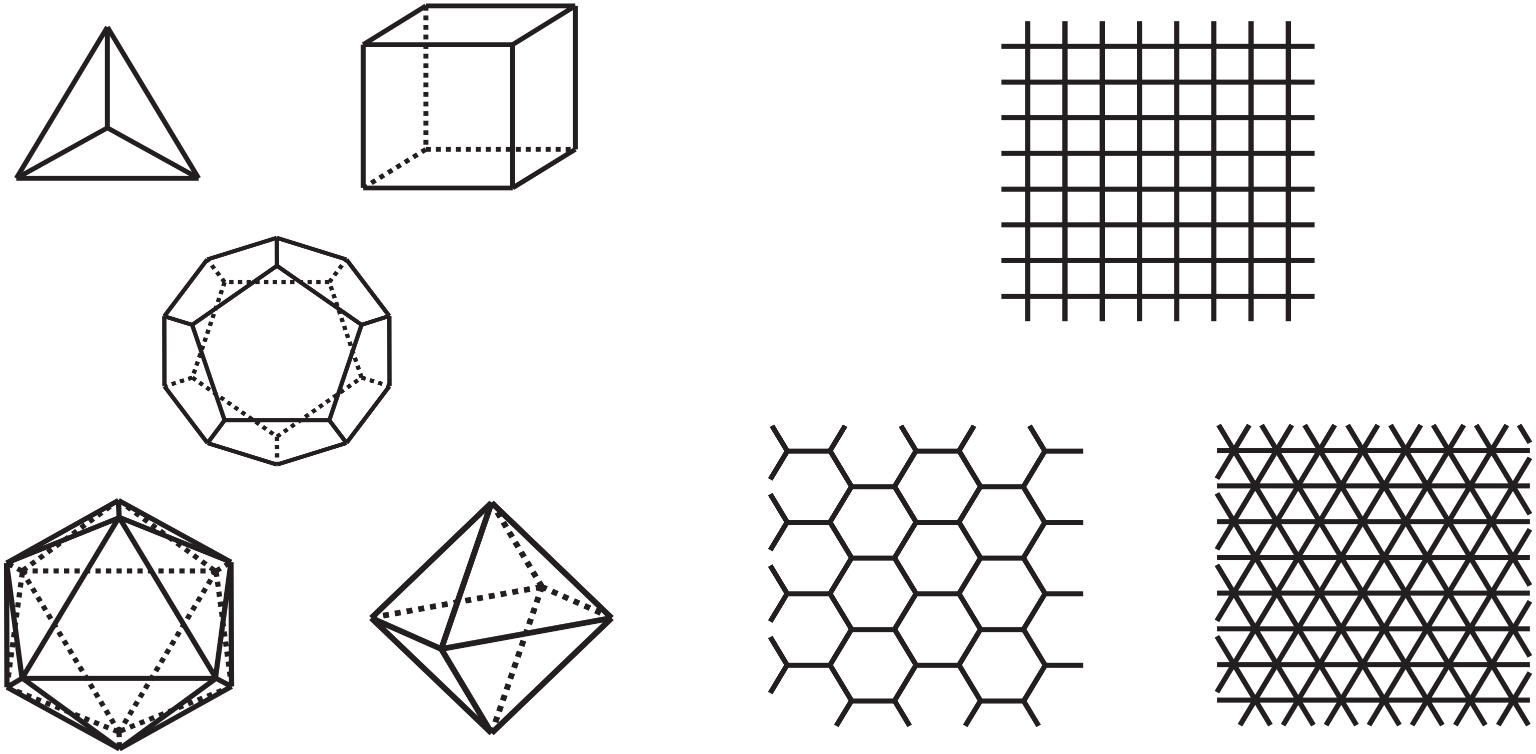
Automorphism $\longrightarrow$ order preserving bijection

Regular polytope $\longrightarrow$ automorphism group transitive on flags

Regular polytopes



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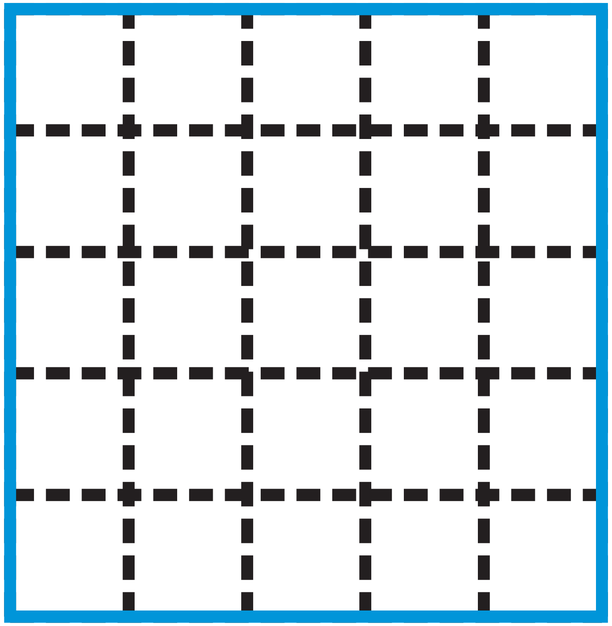

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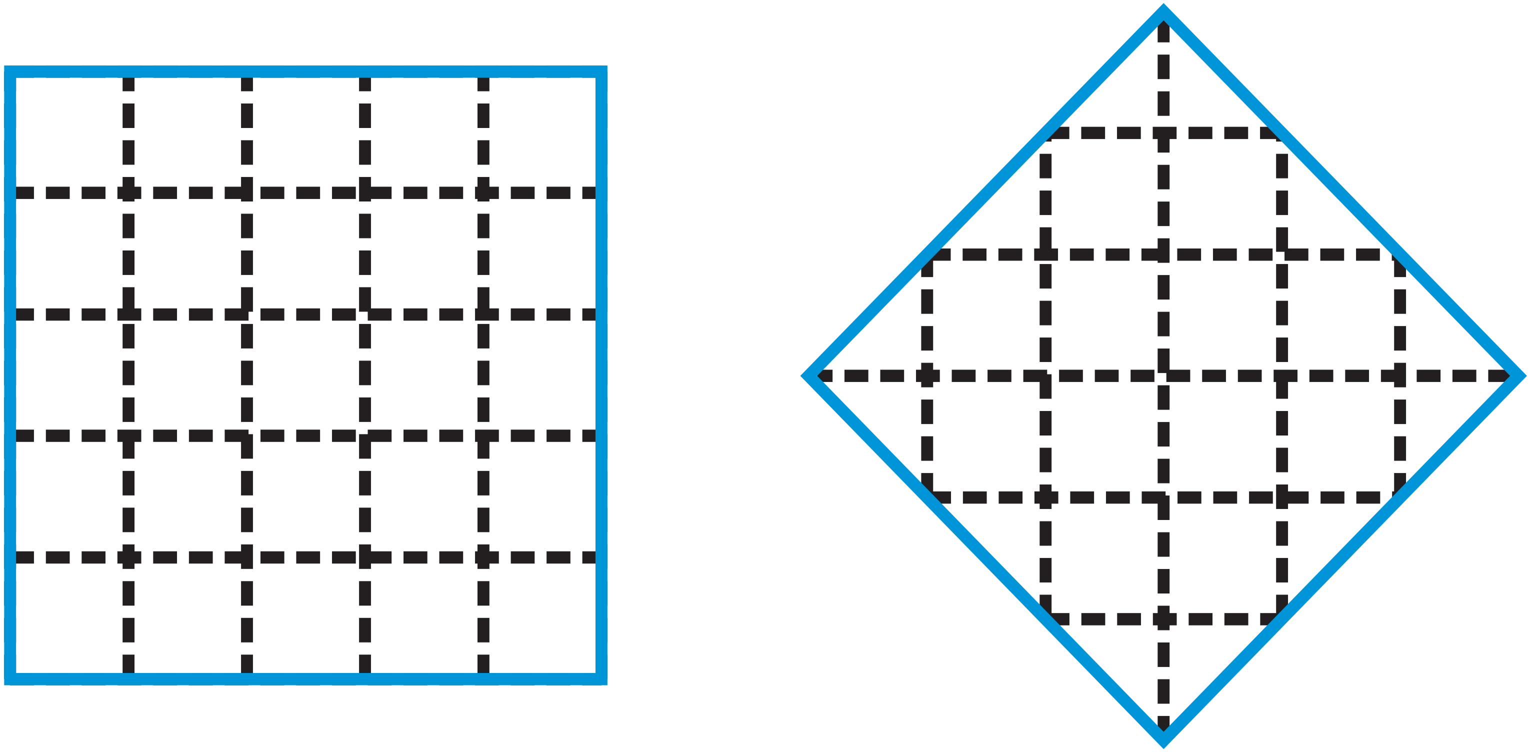

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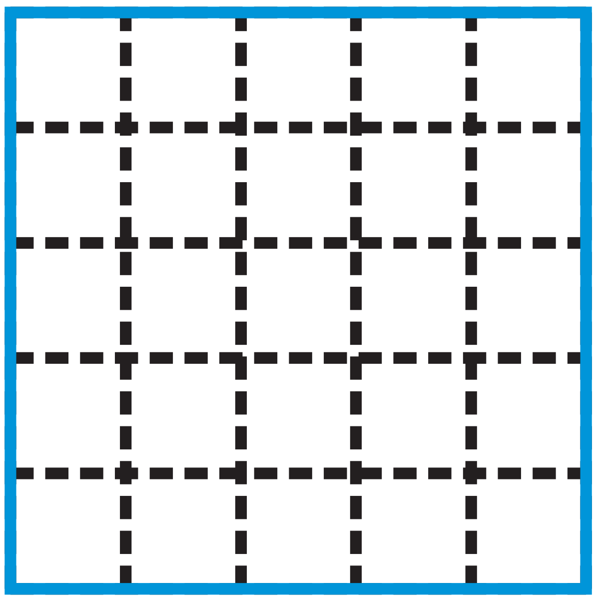

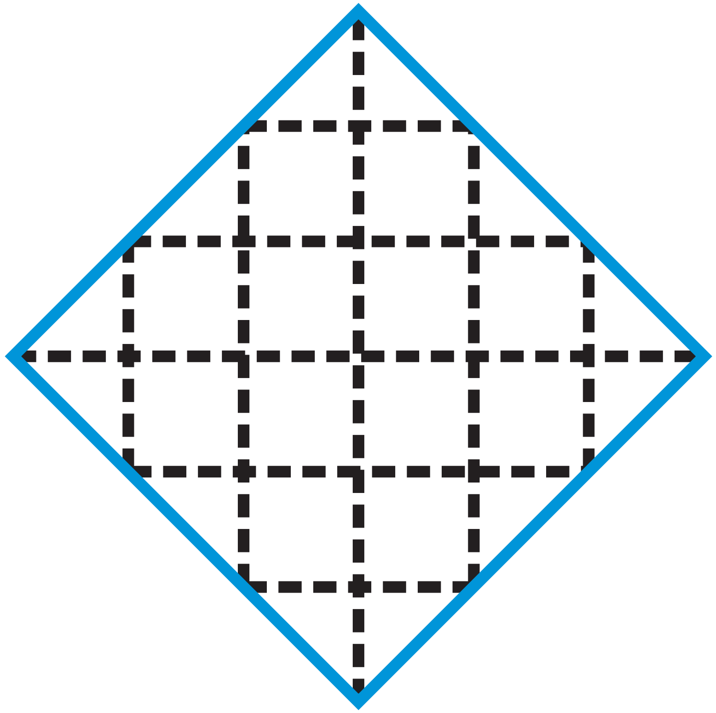

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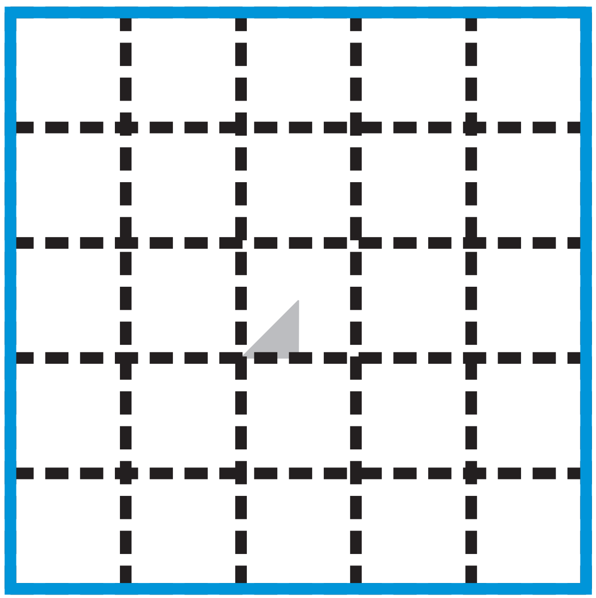

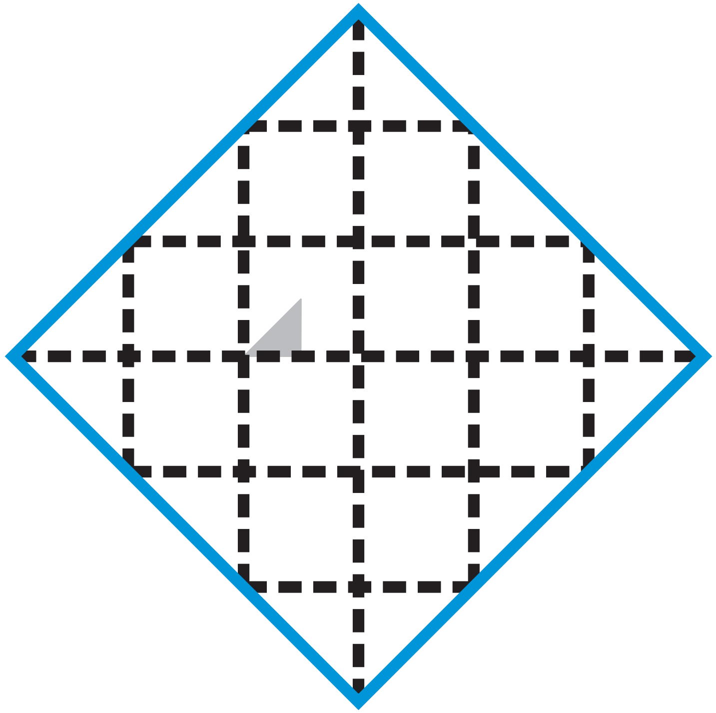

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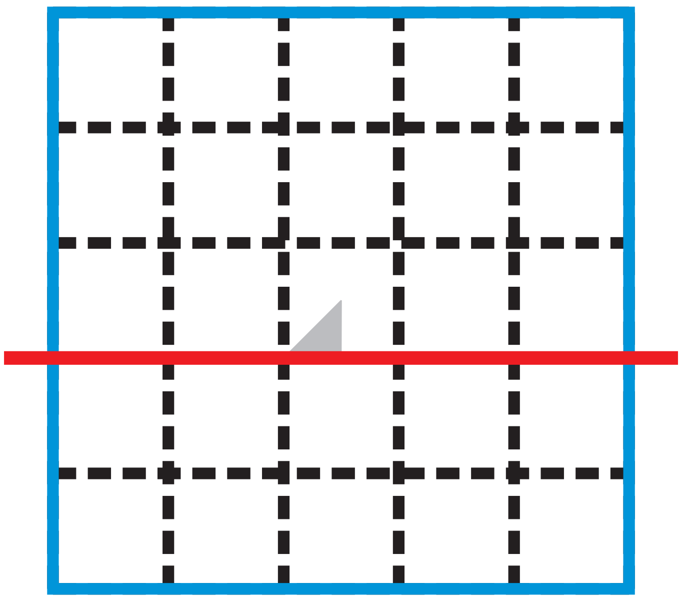

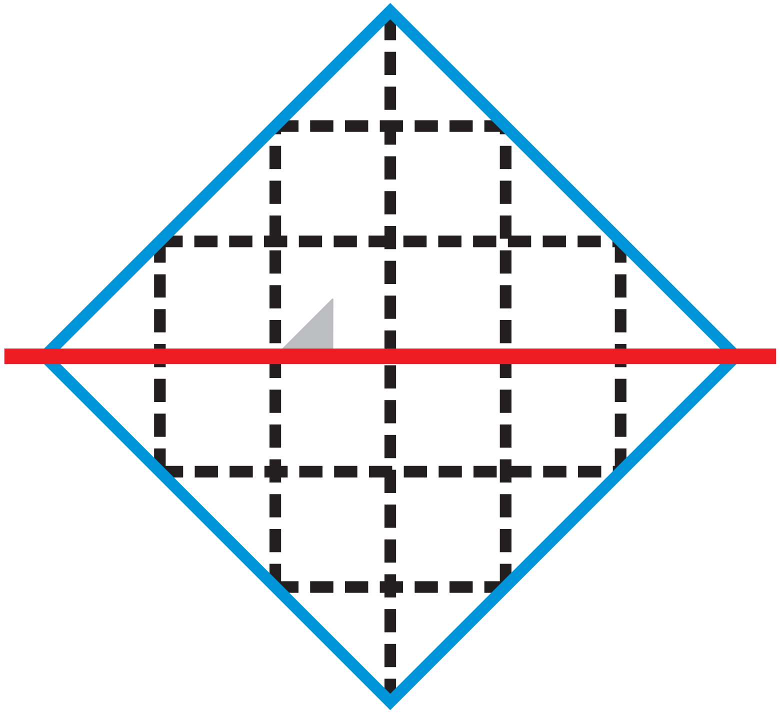

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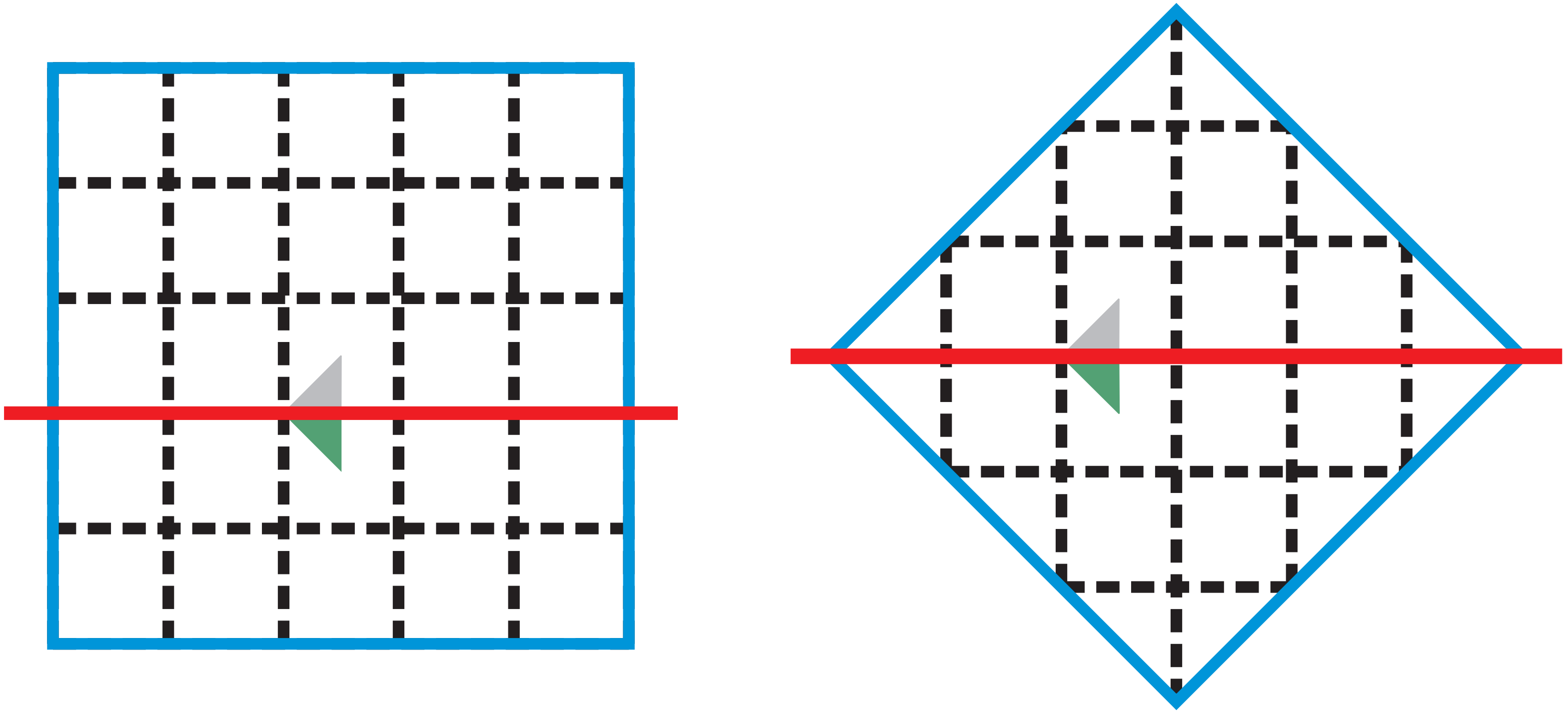

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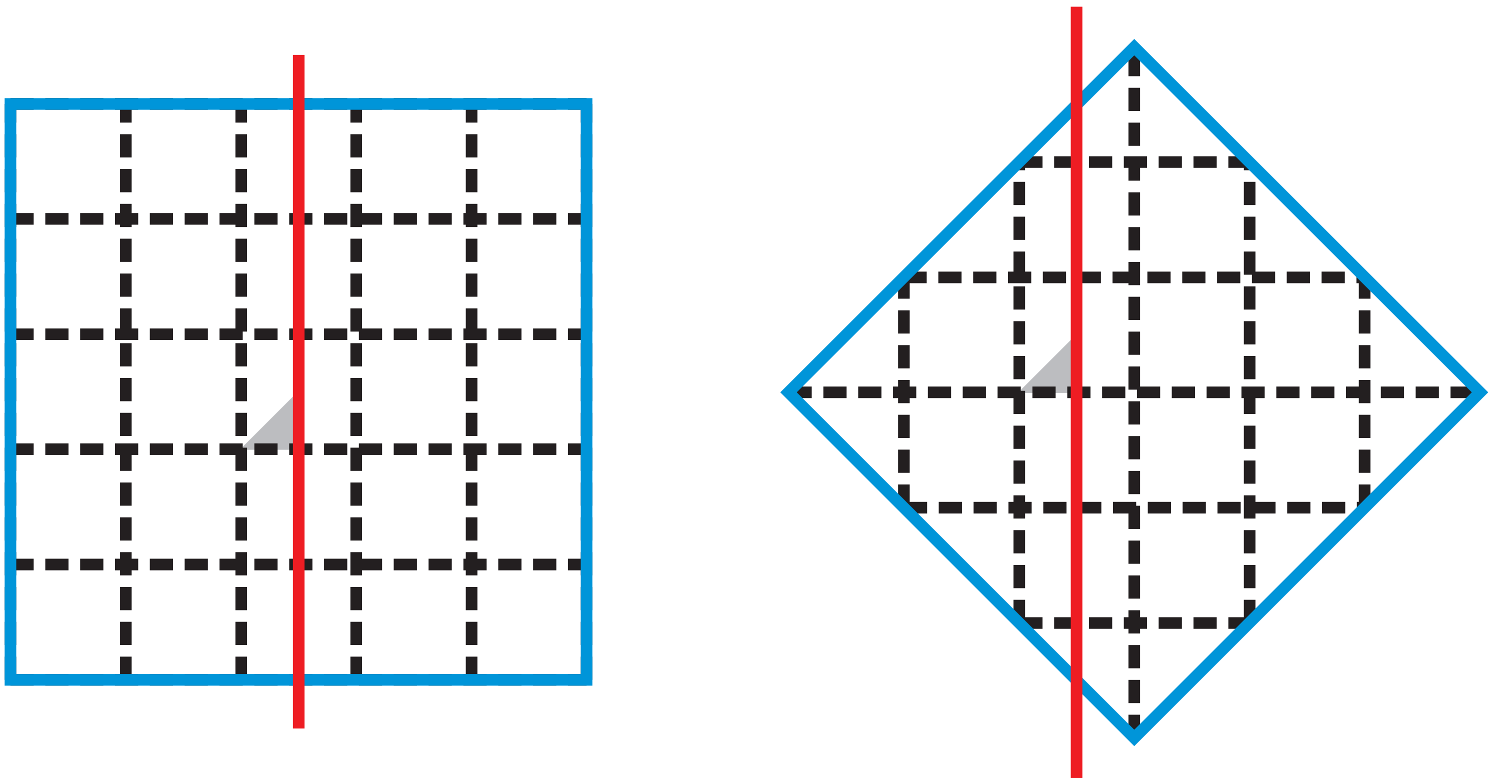

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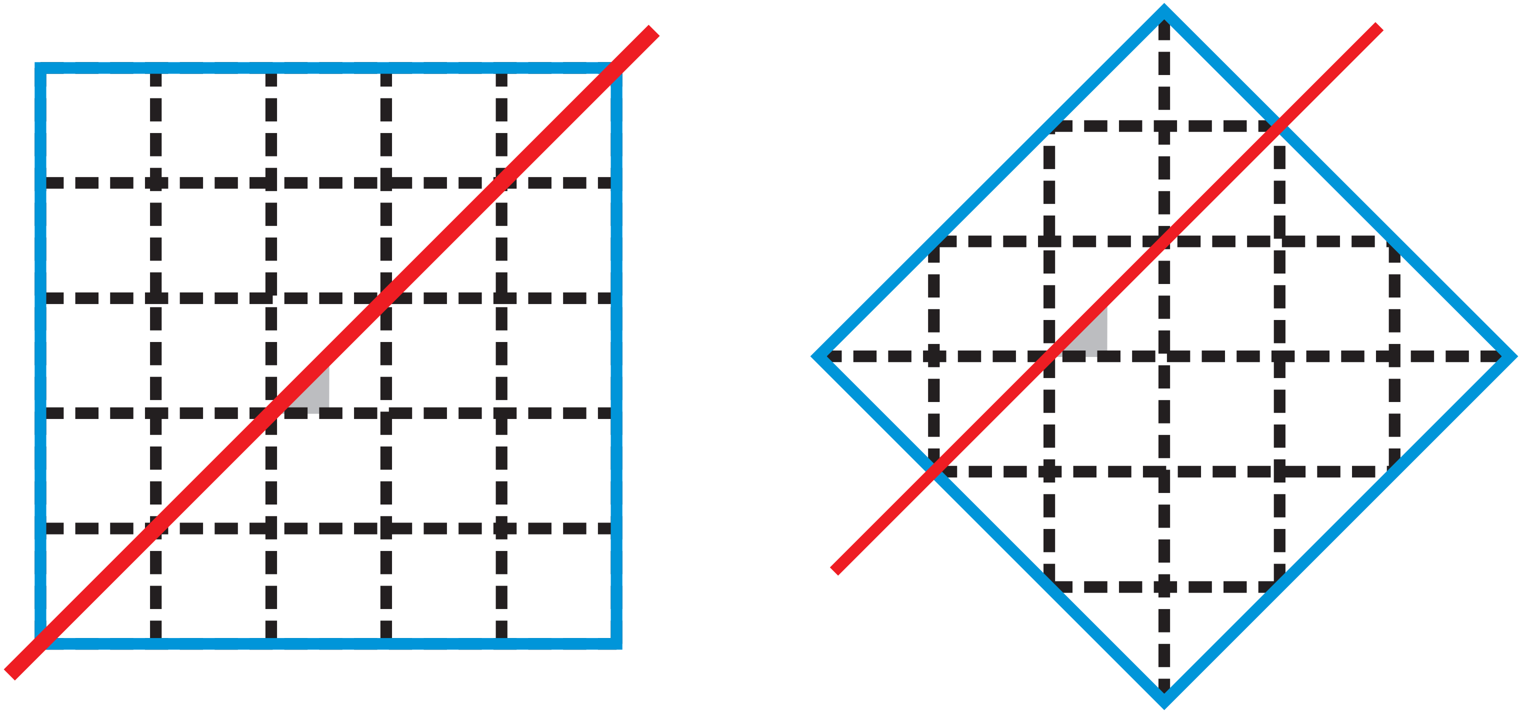


Regular polytopes

Regular polytope $\longrightarrow$ maximal symmetry by reflections

Regular polytopes

Regular polytope $\longrightarrow$ maximal symmetry by reflections

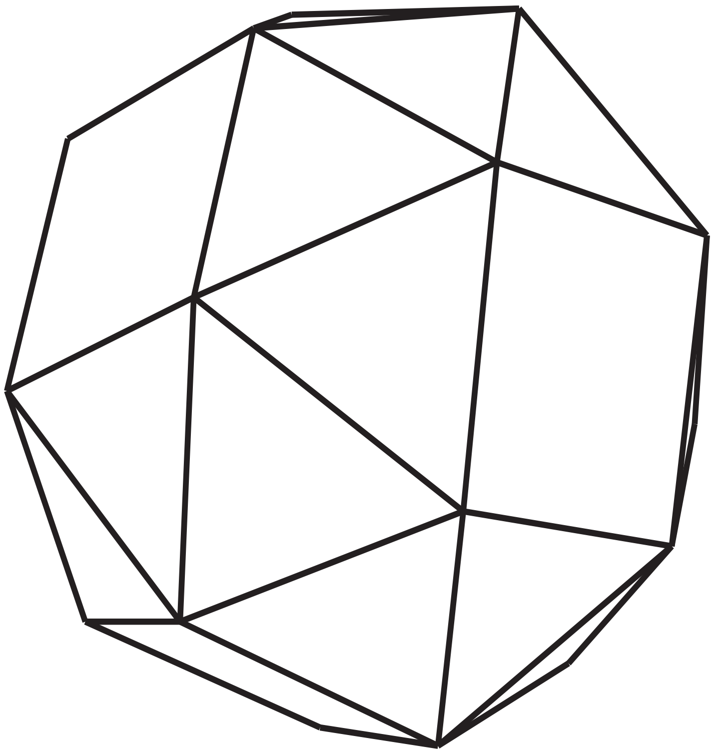
Chiral polytope $\longrightarrow$ maximal symmetry by rotation, but not by reflections

Chiral polytopes

Snub cube

Chiral polytopes

Snub cube



Chiral polytopes

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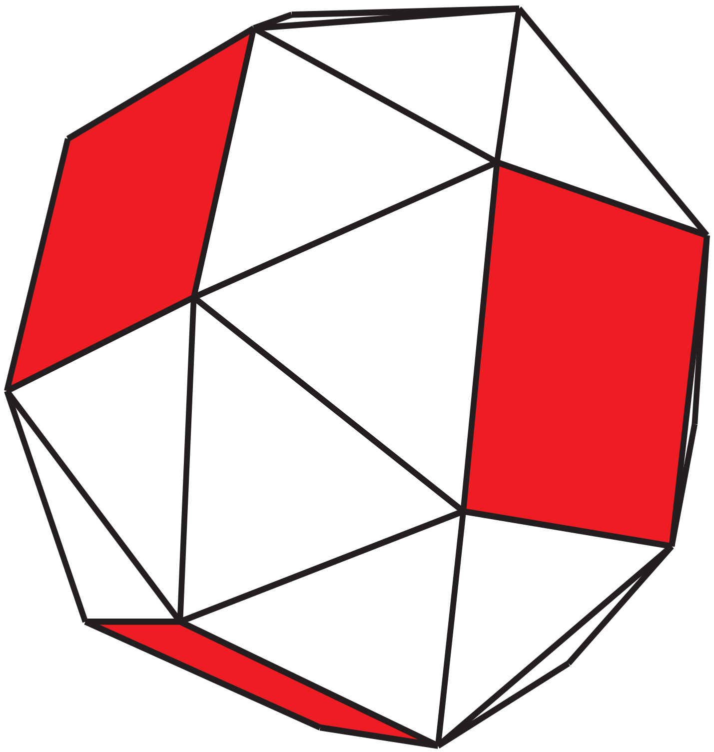

Chiral polytopes

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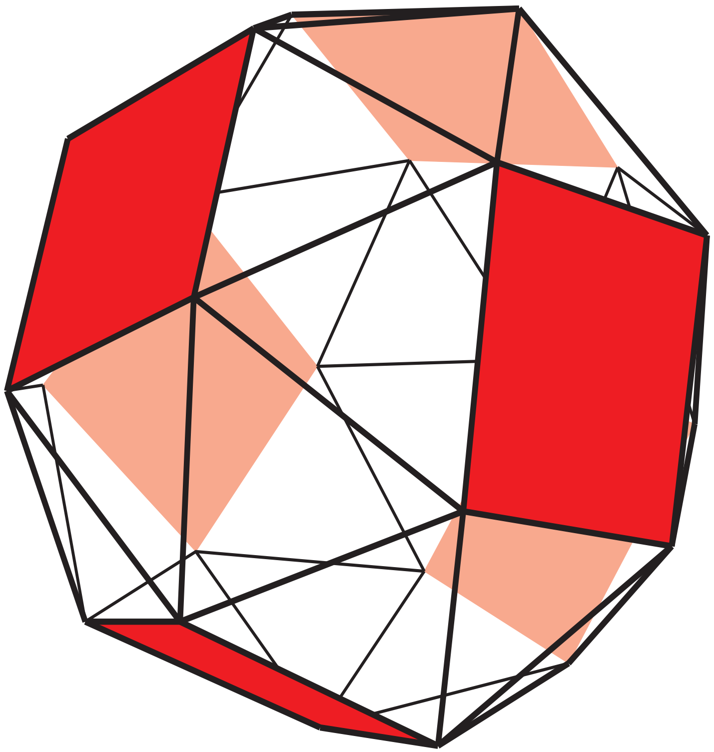

Chiral polytopes

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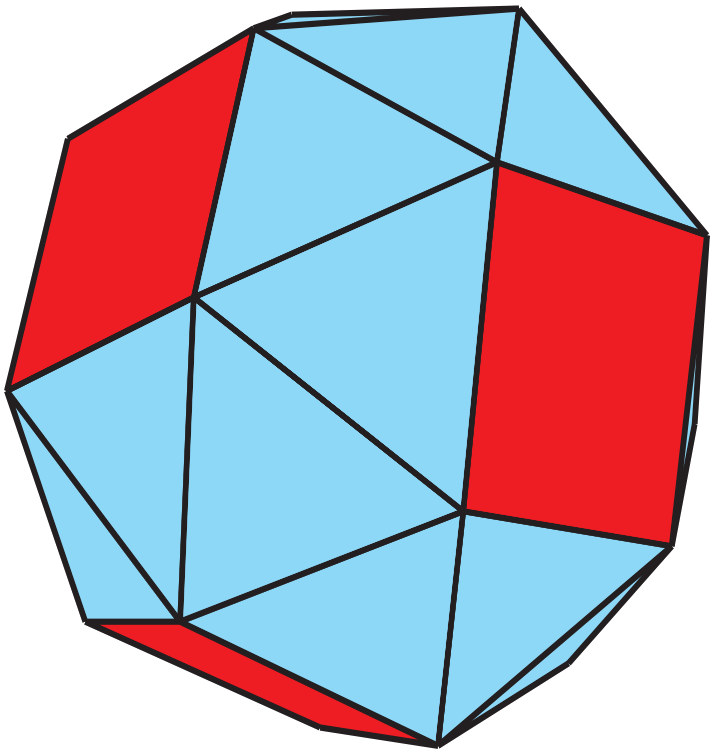

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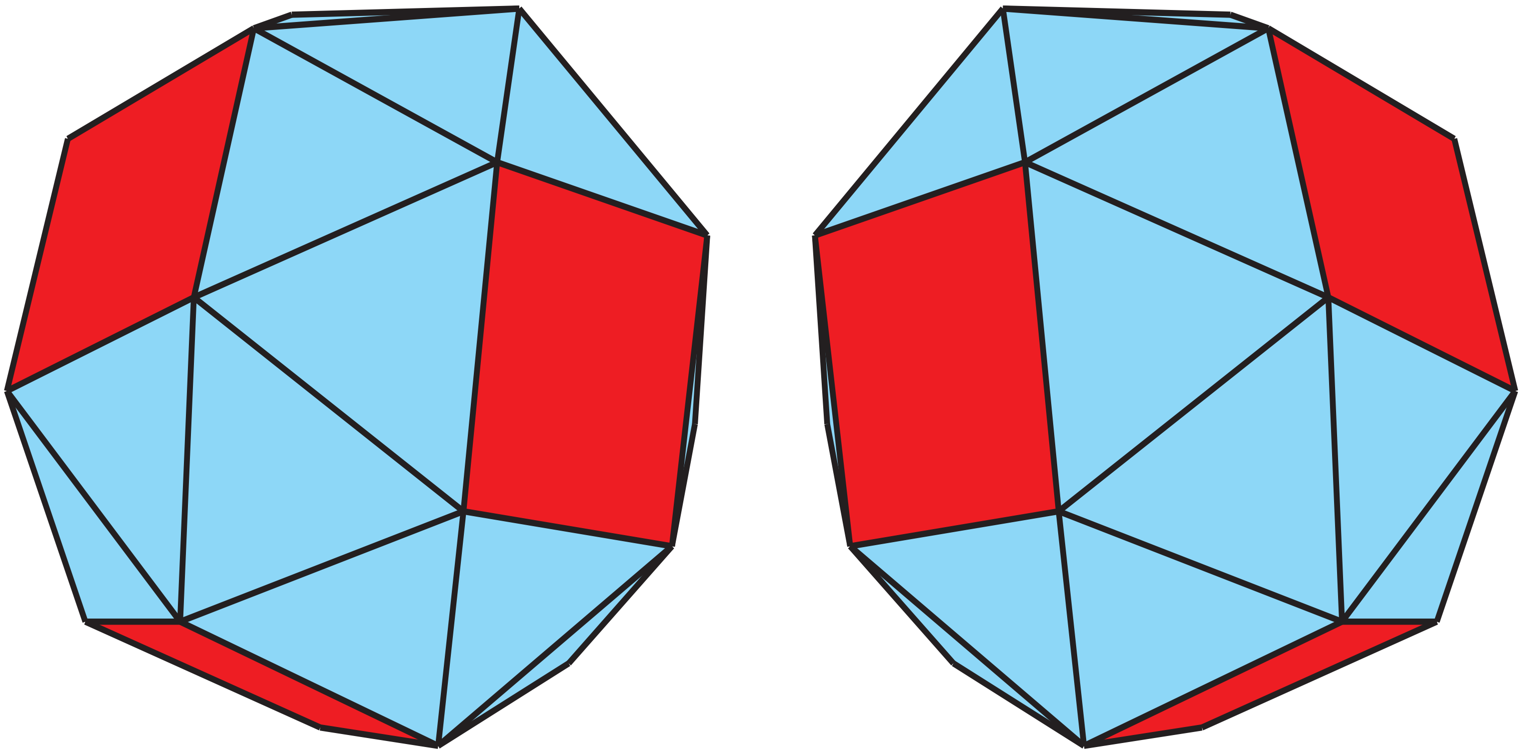

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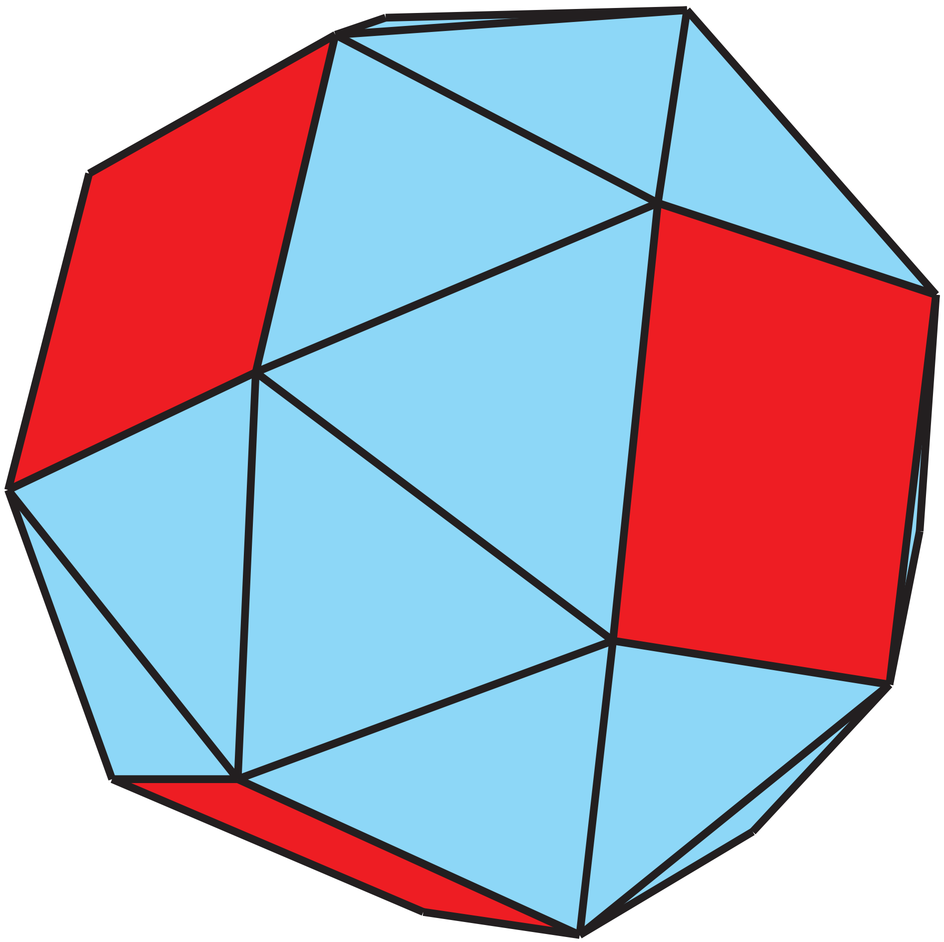

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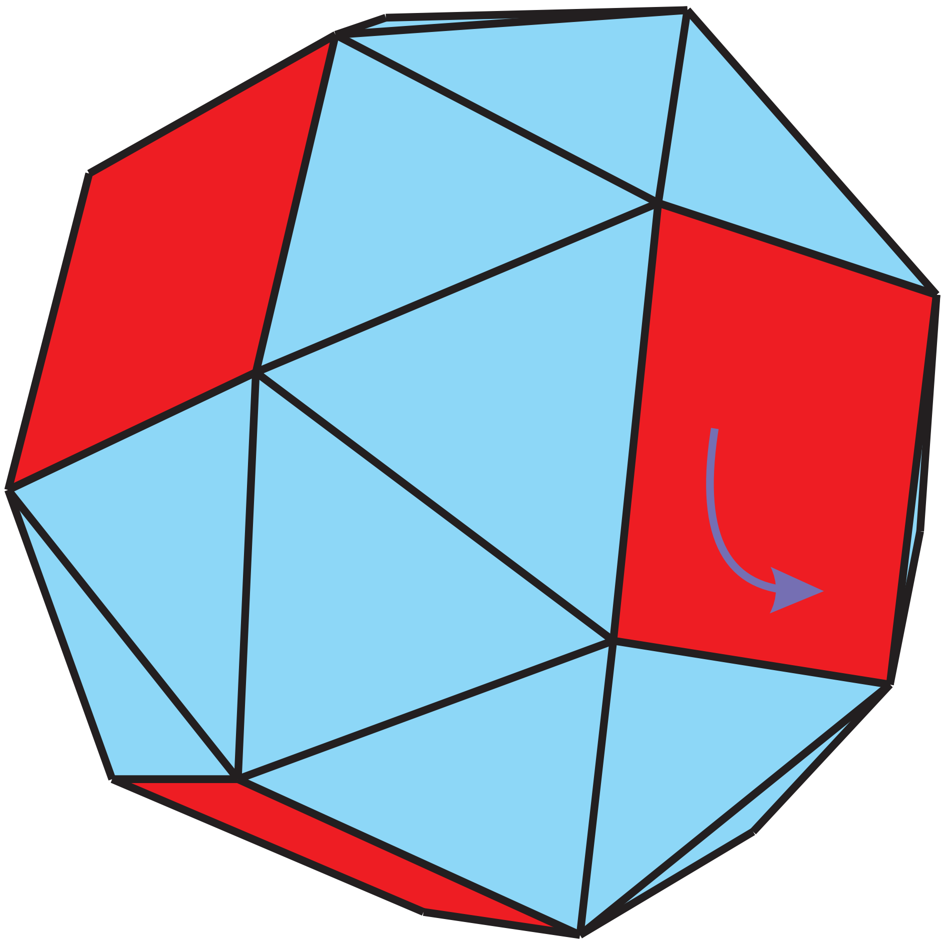

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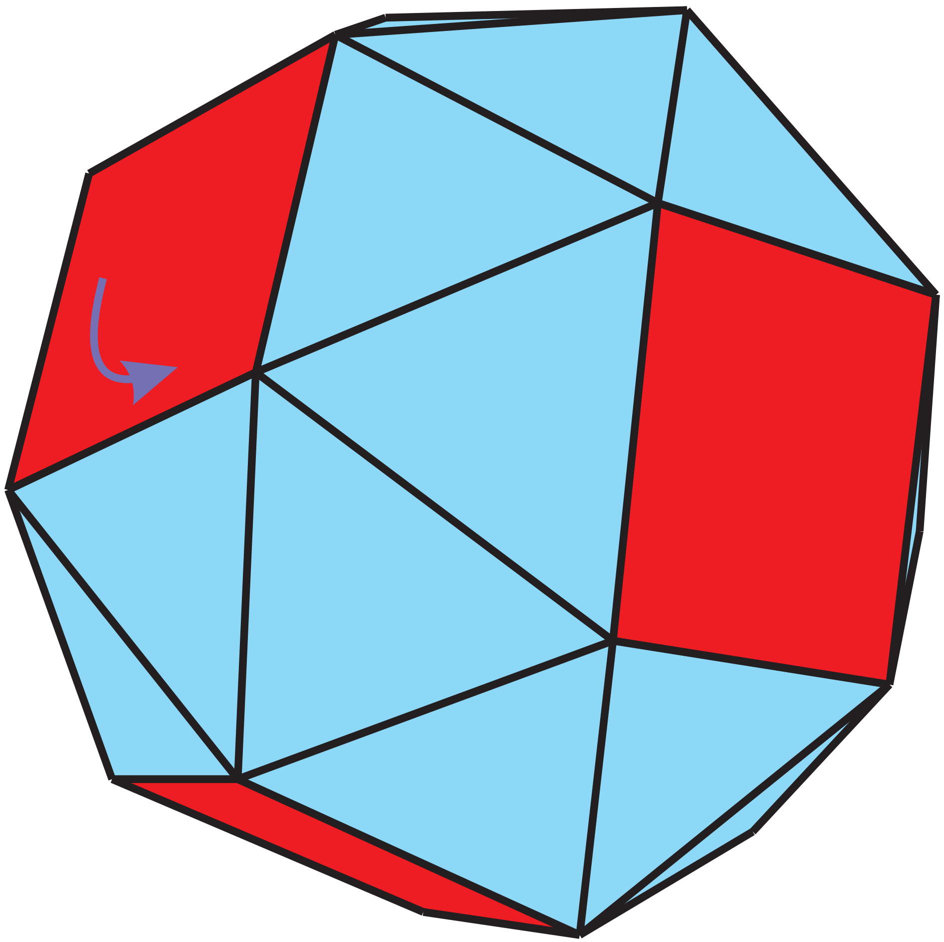

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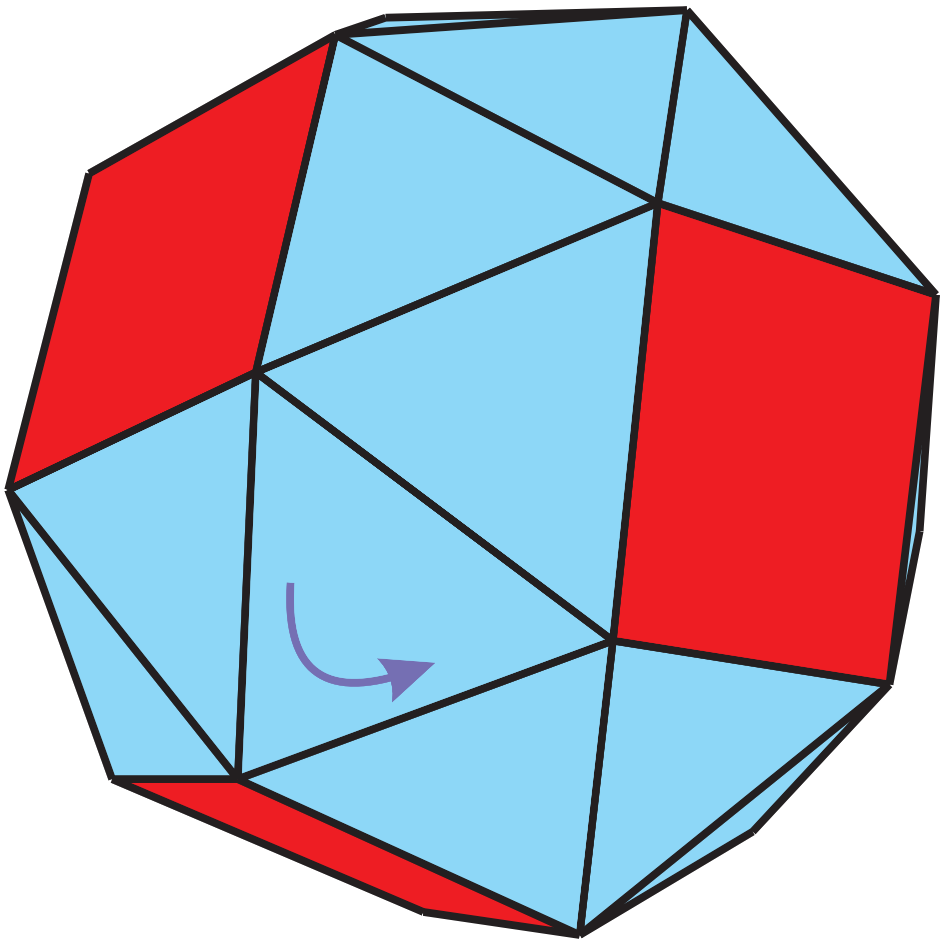

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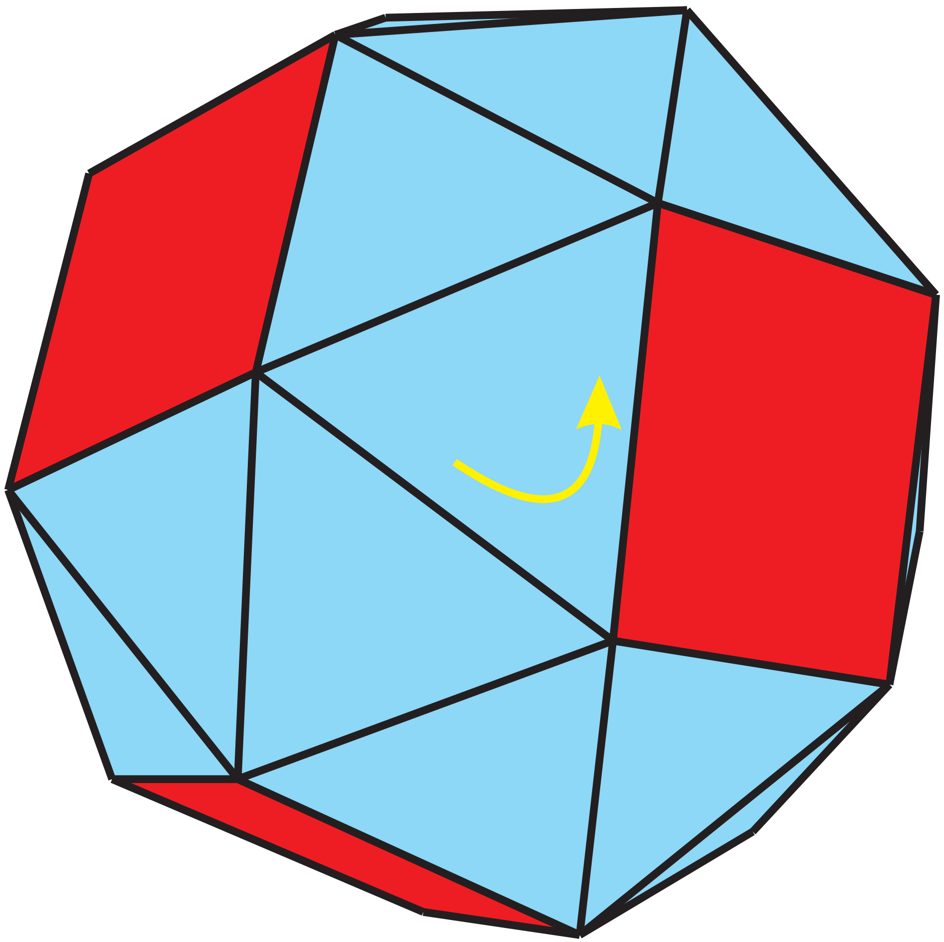

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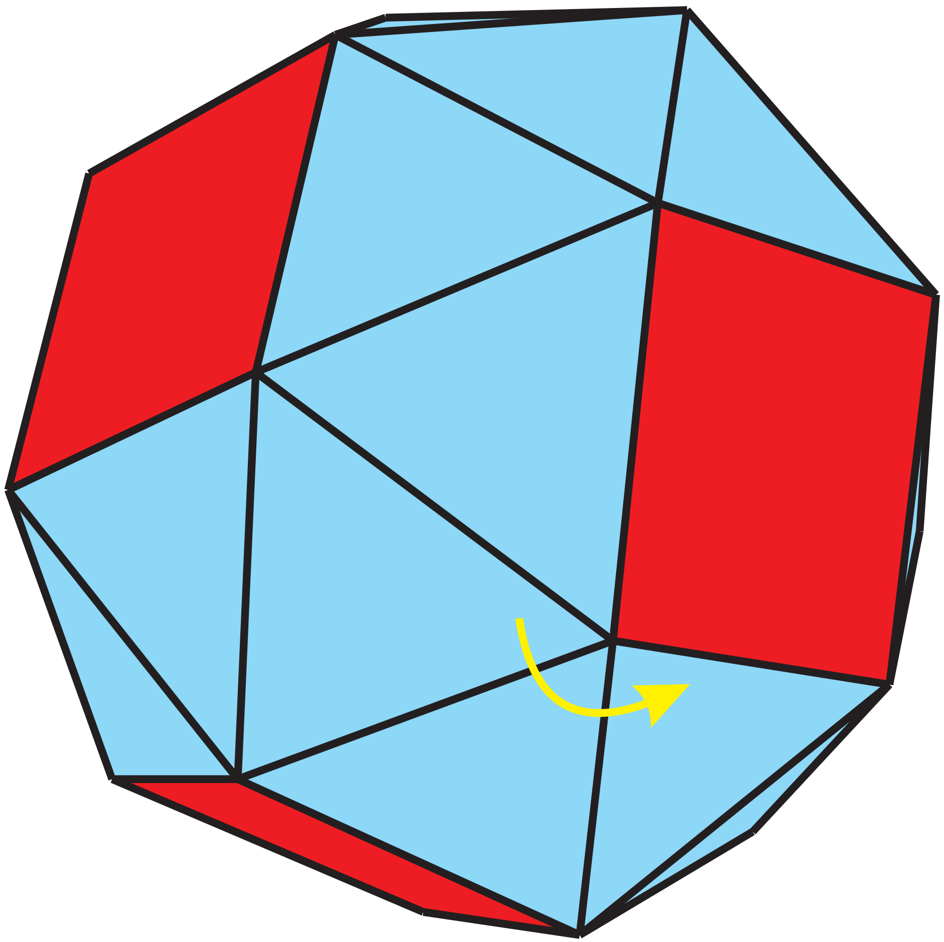

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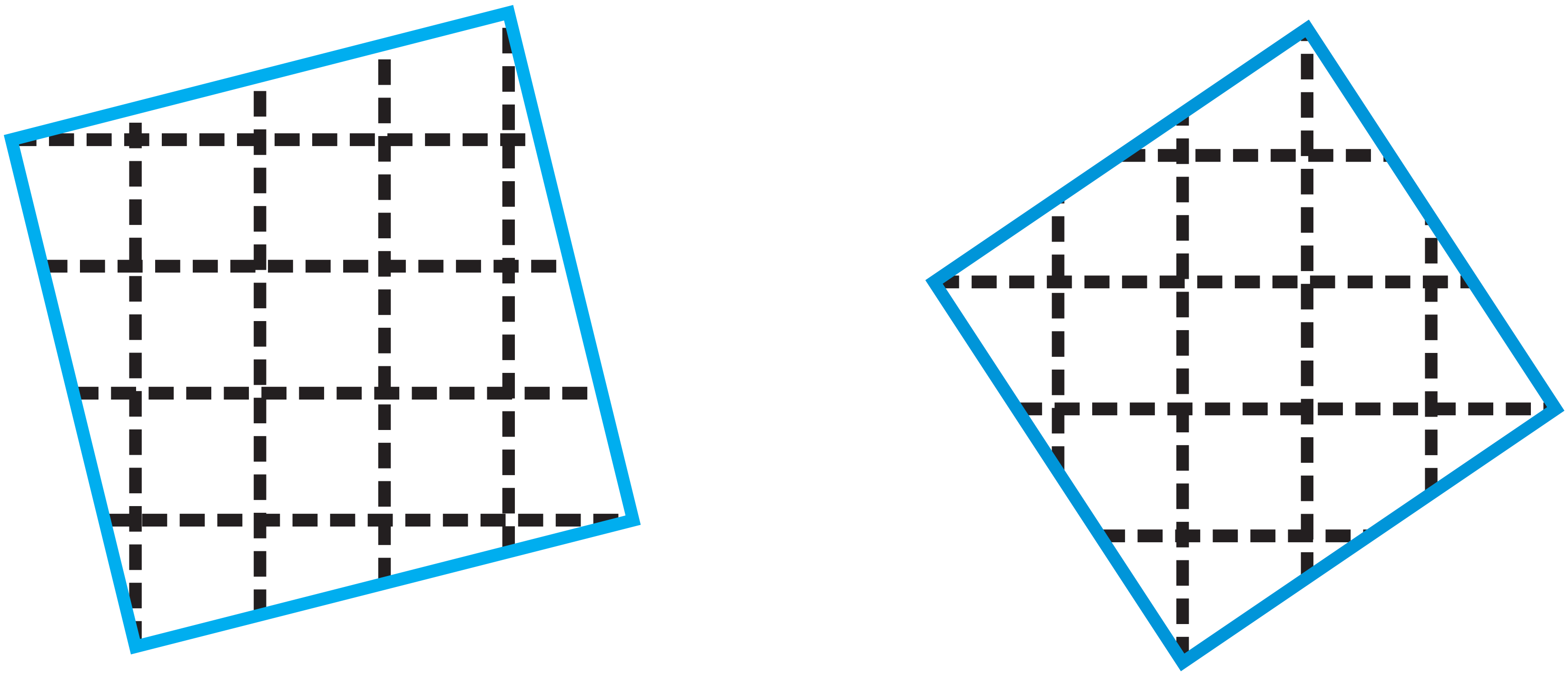

Chiral 3-polytopes

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M. Conder, P. Dobcsányi, I. Hubard, D. Lemmans, R. Nedela, J. E. Schulte, ŠiráňT. Tucker, ... $\longrightarrow$ various aspects

Higher rank chiral polytopes

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A. Breda, M. Conder, G. Cunningham, I. Hubard, E. O'Reilly, E. Schulte, A. Weiss $\longrightarrow$ other approaches

Higher rank chiral polytopes

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No toroidal n-polytopes for $n \neq 3$

For rank $n \geq 6$, chiral n-polytopes seem to be BIG

Higher rank chiral polytopes

From a group theoretical perspective,

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Conder, Hartley, Hubard, Leemans, Schulte $\longrightarrow$ finite almost simple groups

Higher rank chiral polytopes

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Conder, Hartley, Hubard, Leemans, Schulte $\longrightarrow$ finite almost simple groups

Conder, Hubard, O'Reilly, Pellicer $\longrightarrow$ construction of chiral n-polytopes with symmetric or alternating automorphism groups

Higher rank chiral polytopes

Chiral n-polytopes from $(n-1)$-polytopes

Higher rank chiral polytopes

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The facets of a chiral polytope may be either regular or chiral

The facets of the facets of a chiral polytope are regular

If a chiral n-polytope has chiral facets, it is not the facet of an $(n+1)$-chiral polytope

Higher rank chiral polytopes

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Is every finite d-polytope with regular facets the facet of a FINITE chiral $(d+1)$-polytope?

Automorphism groups

$$\Gamma(\mathcal{P}) = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$$

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Automorphism groups

Theorem Given a group $\Gamma = \langle \sigma_1, \ldots, \sigma_{d-1} \rangle$ satisfying $(\sigma_i \sigma_{i+1} \cdots \sigma_j)^2 = Id$ and the intersection condition, it is the automorphism group of a unique regular or chiral n-polytope

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- $\mathcal{P}$ is regular if and only if there is an automorphism

$$\begin{array}{rcl} \sigma_1 & \mapsto & \sigma_1^{-1} \\ \sigma_2 & \mapsto & \sigma_1^2 \sigma_2 \\ \sigma_k & \mapsto & \sigma_k \quad k \geq 3 \end{array}$$

PR graphs

- PR $\longrightarrow$ Permutation representation

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Embedding $\varepsilon : \Gamma(\mathcal{P}) \to S_n$

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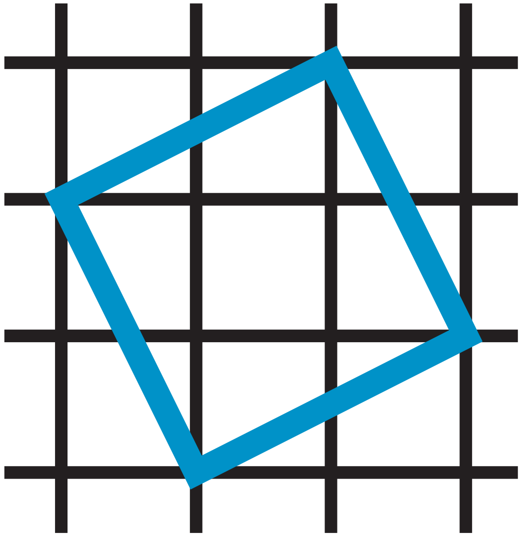
Involutions $\tau_{i,i+1} := \sigma_i \sigma_{i+1}$ may replace σ_i or σ_{i+1}

PR graphs



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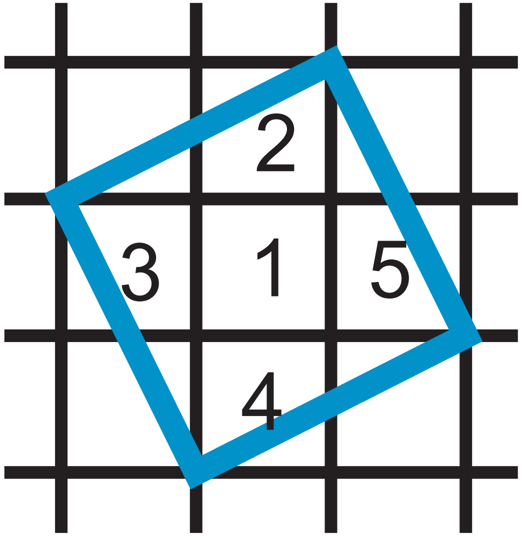

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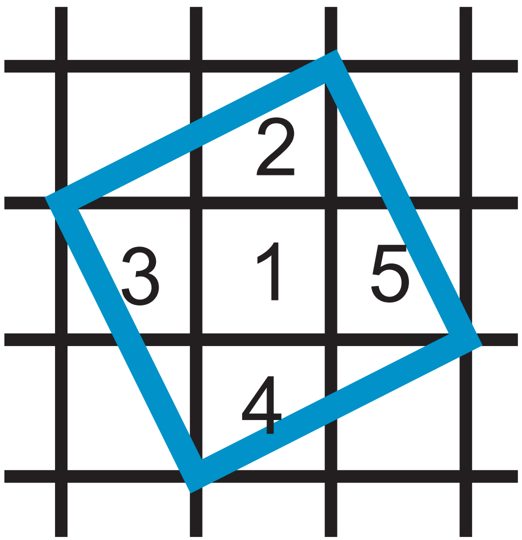

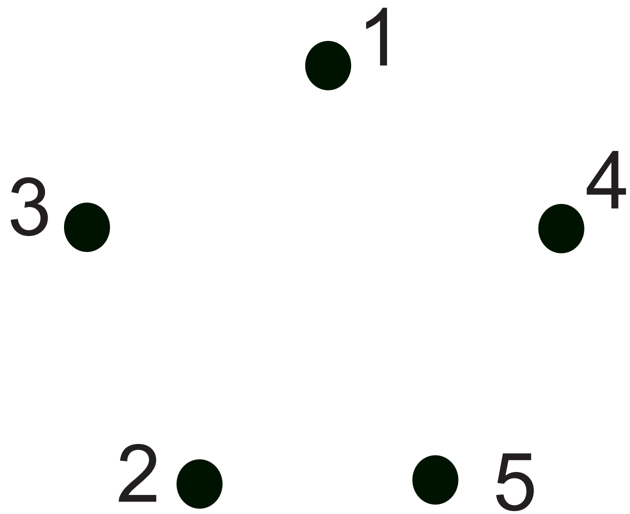

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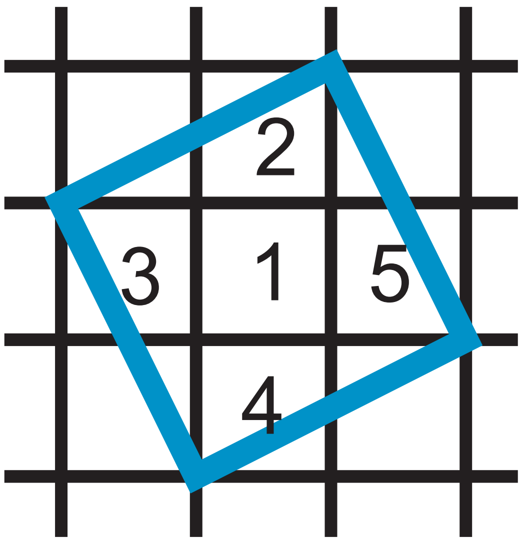

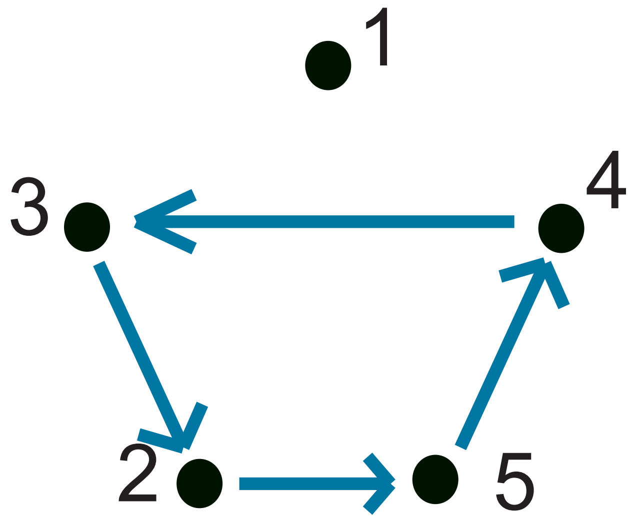

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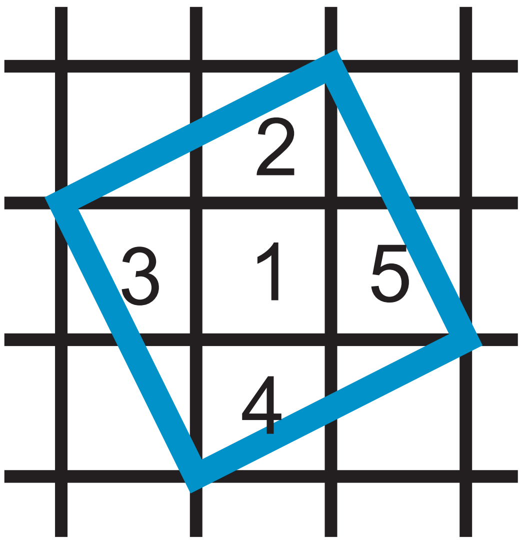

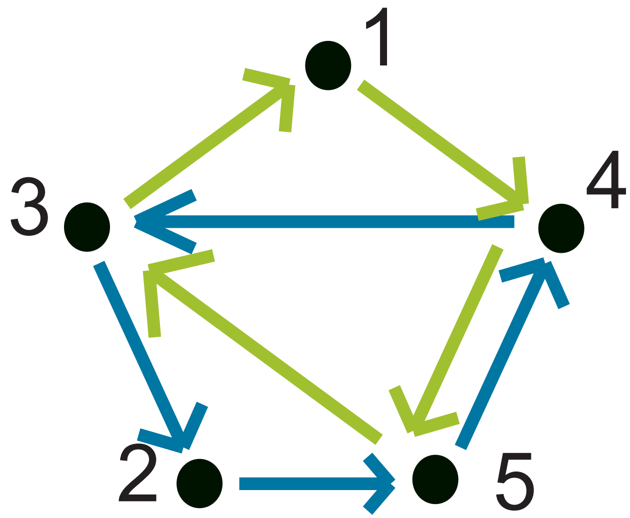

PR graphs

- Cayley PR graph $\longrightarrow \Gamma(\mathcal{P})$ is seen as a permutation group on one orbit of flags of $\mathcal{P}$

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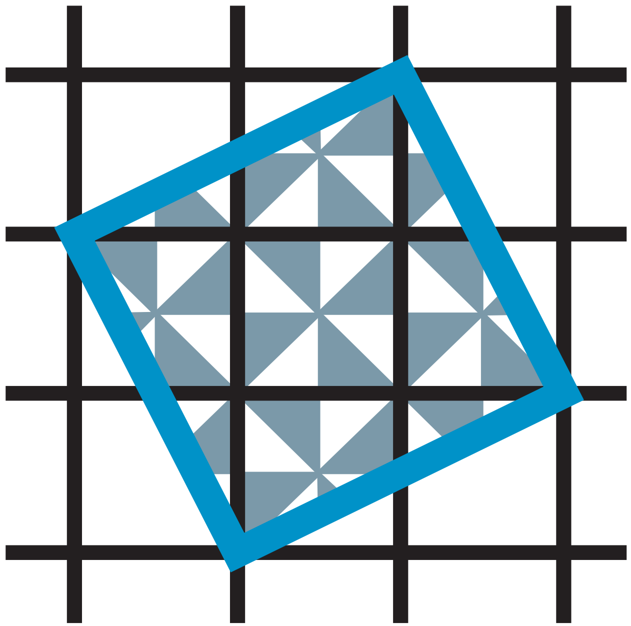

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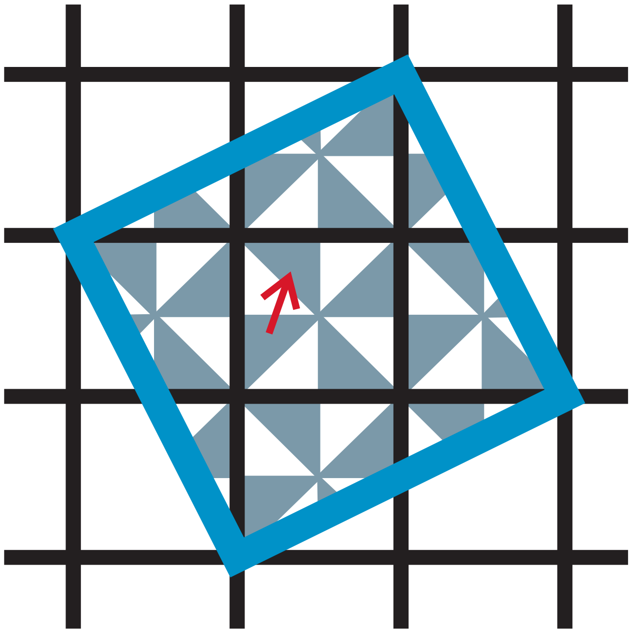

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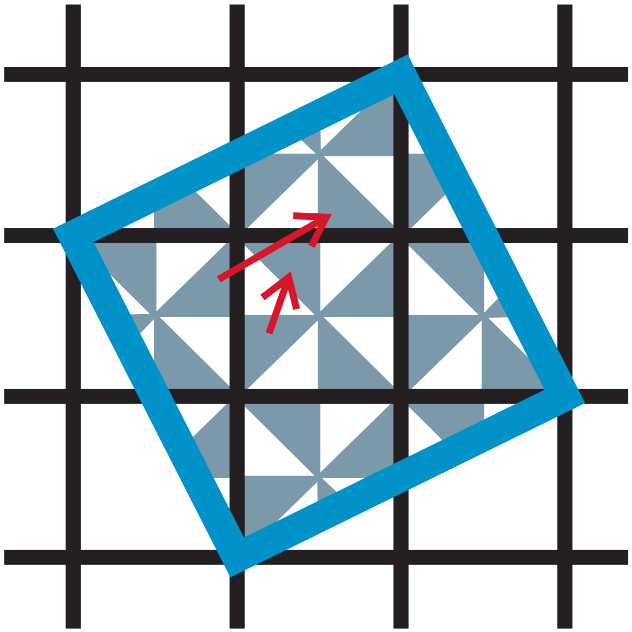

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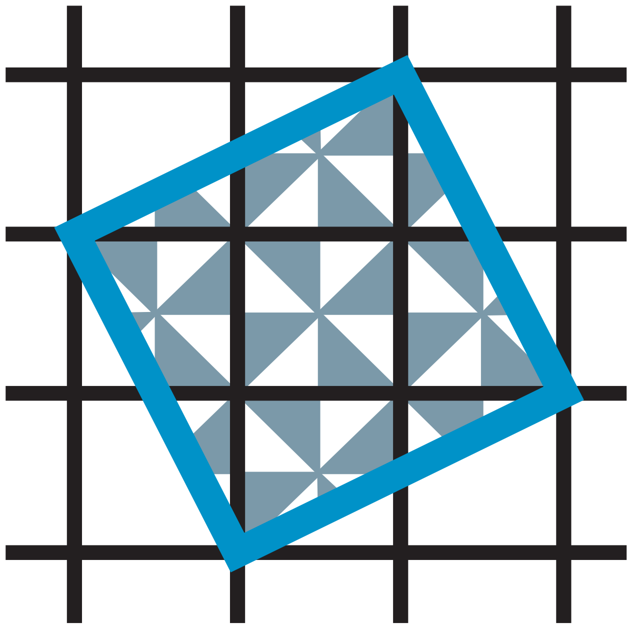

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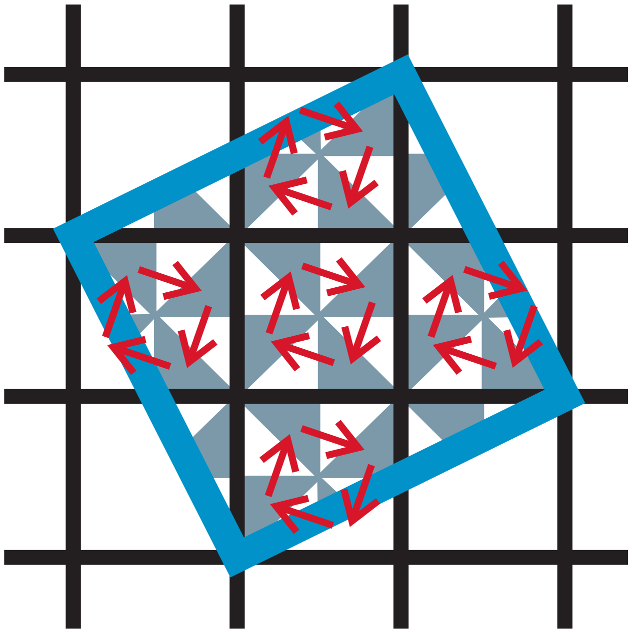

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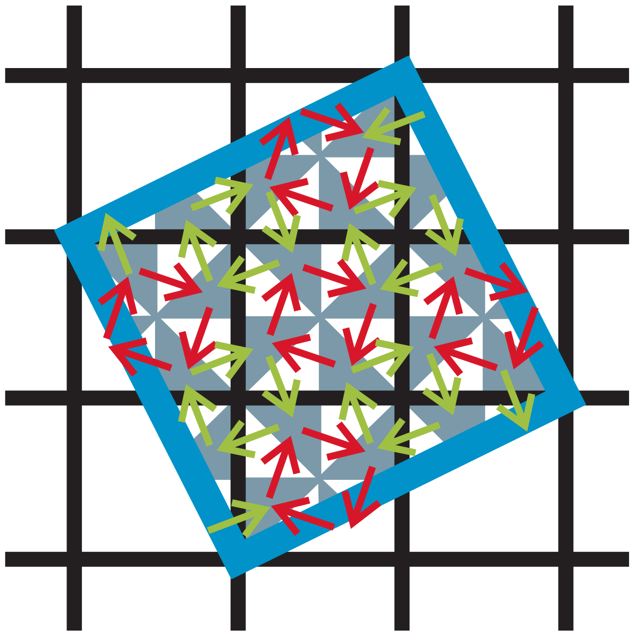

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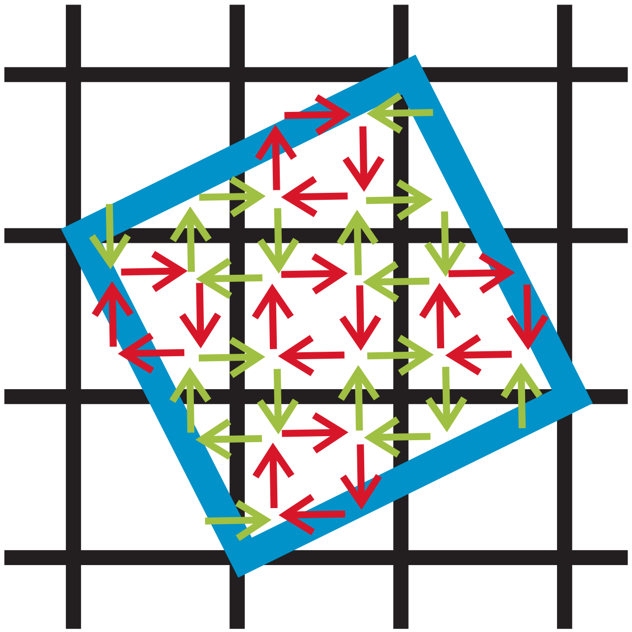

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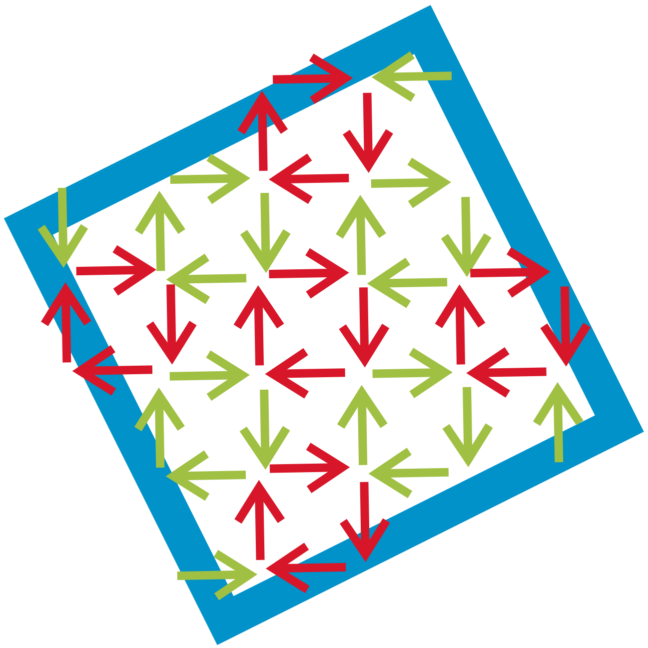

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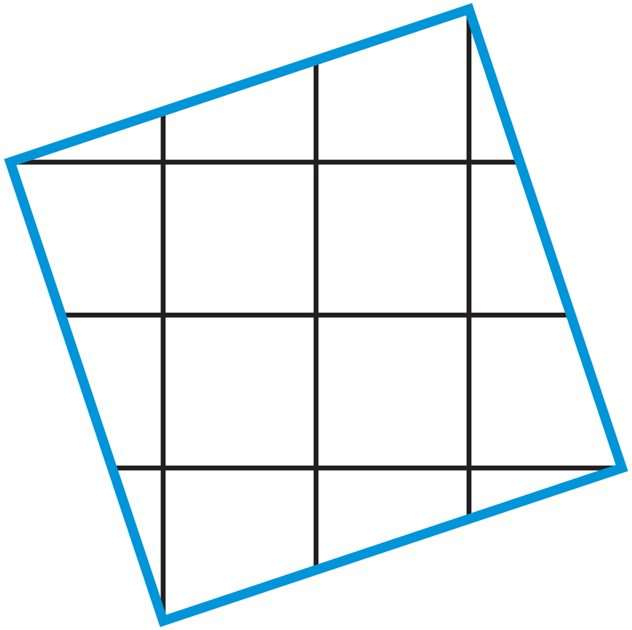

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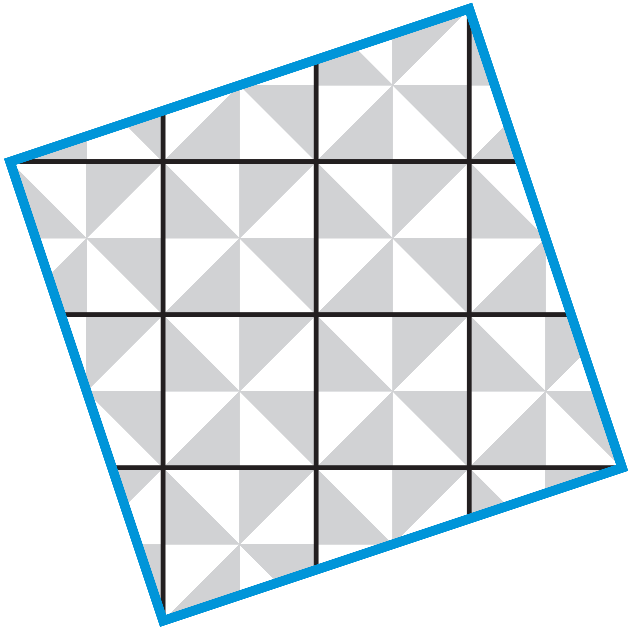

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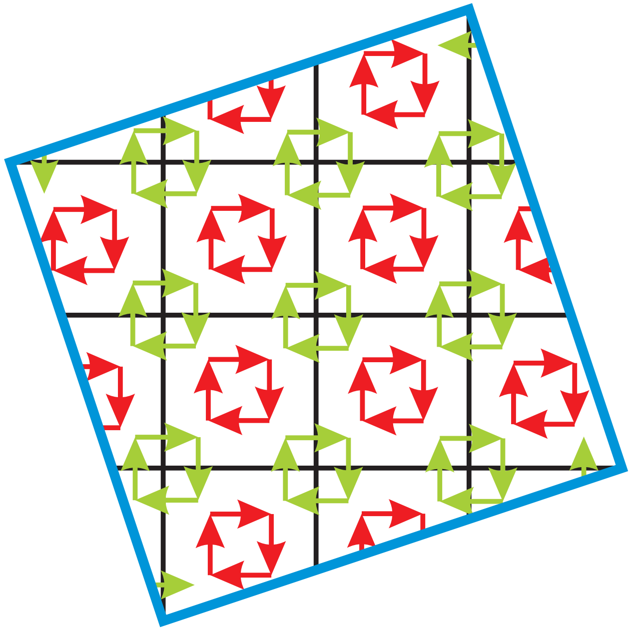

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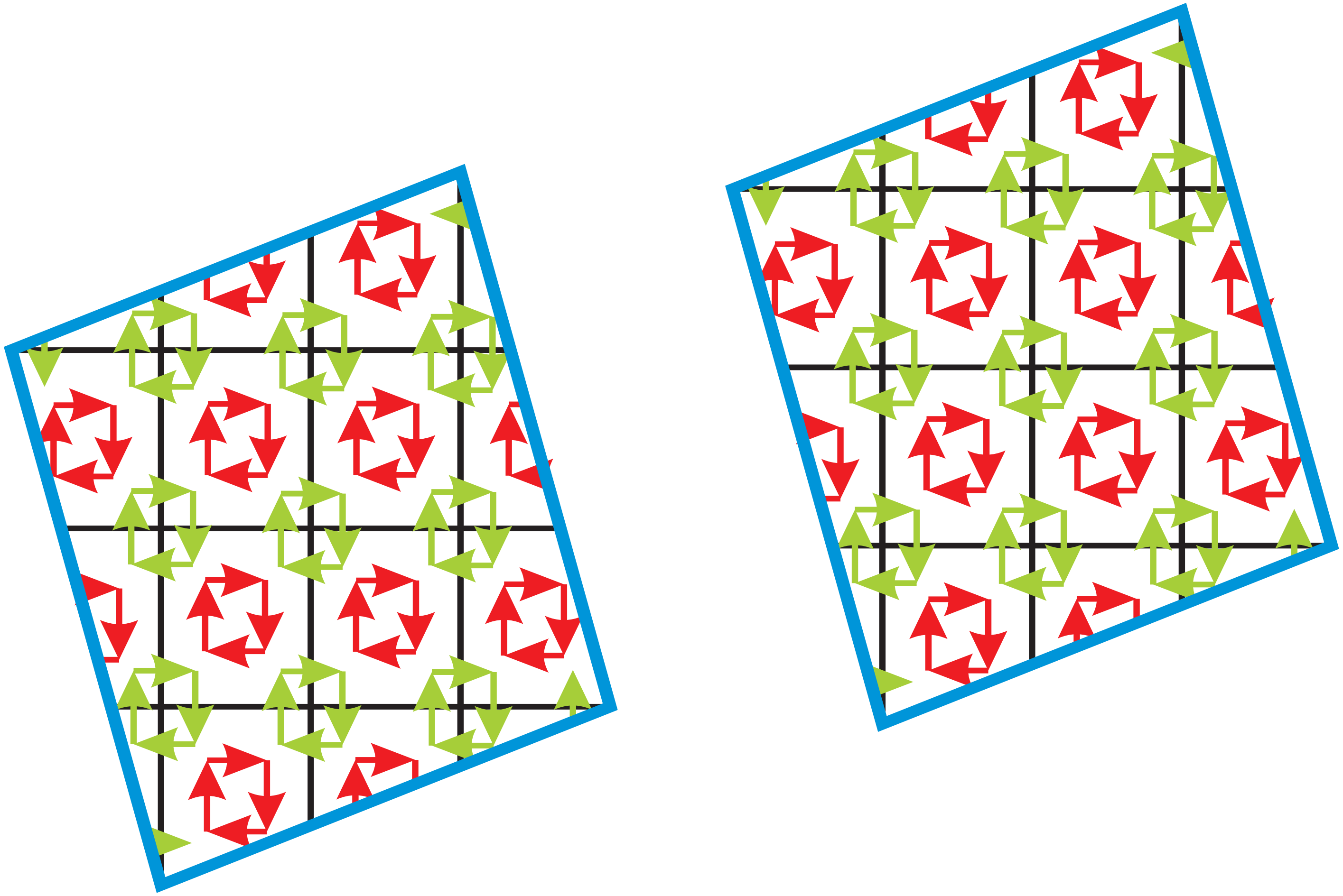

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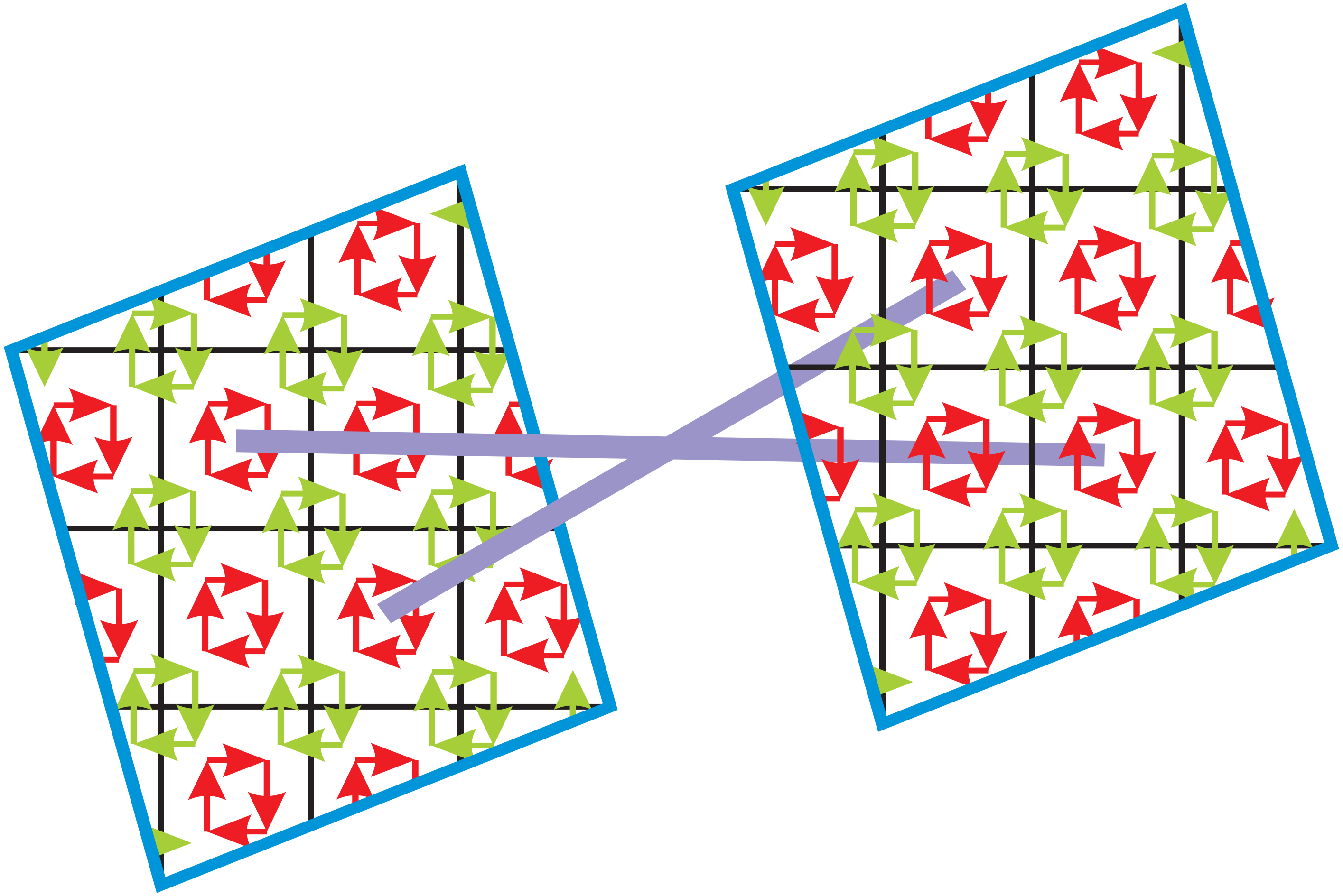

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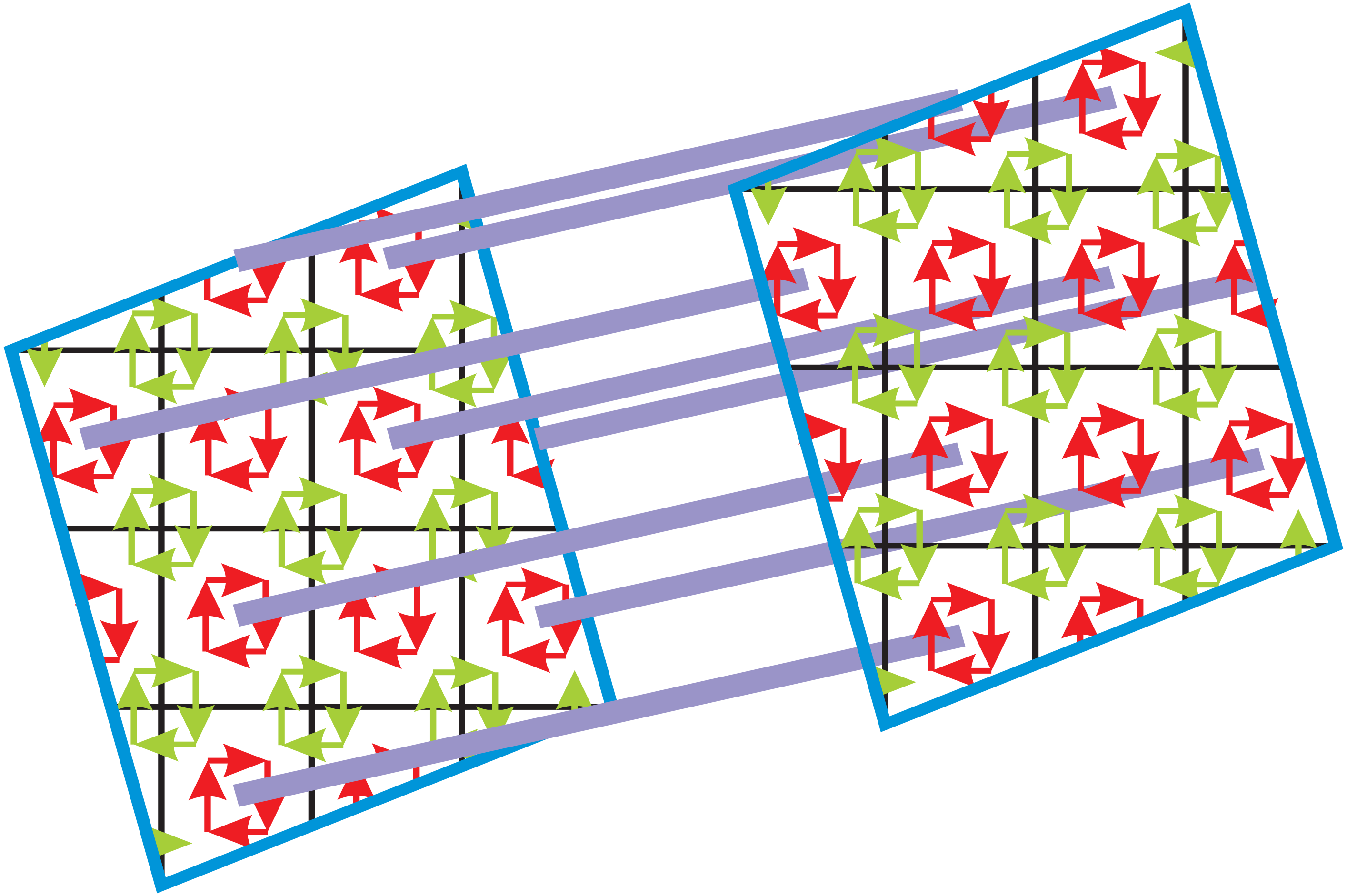

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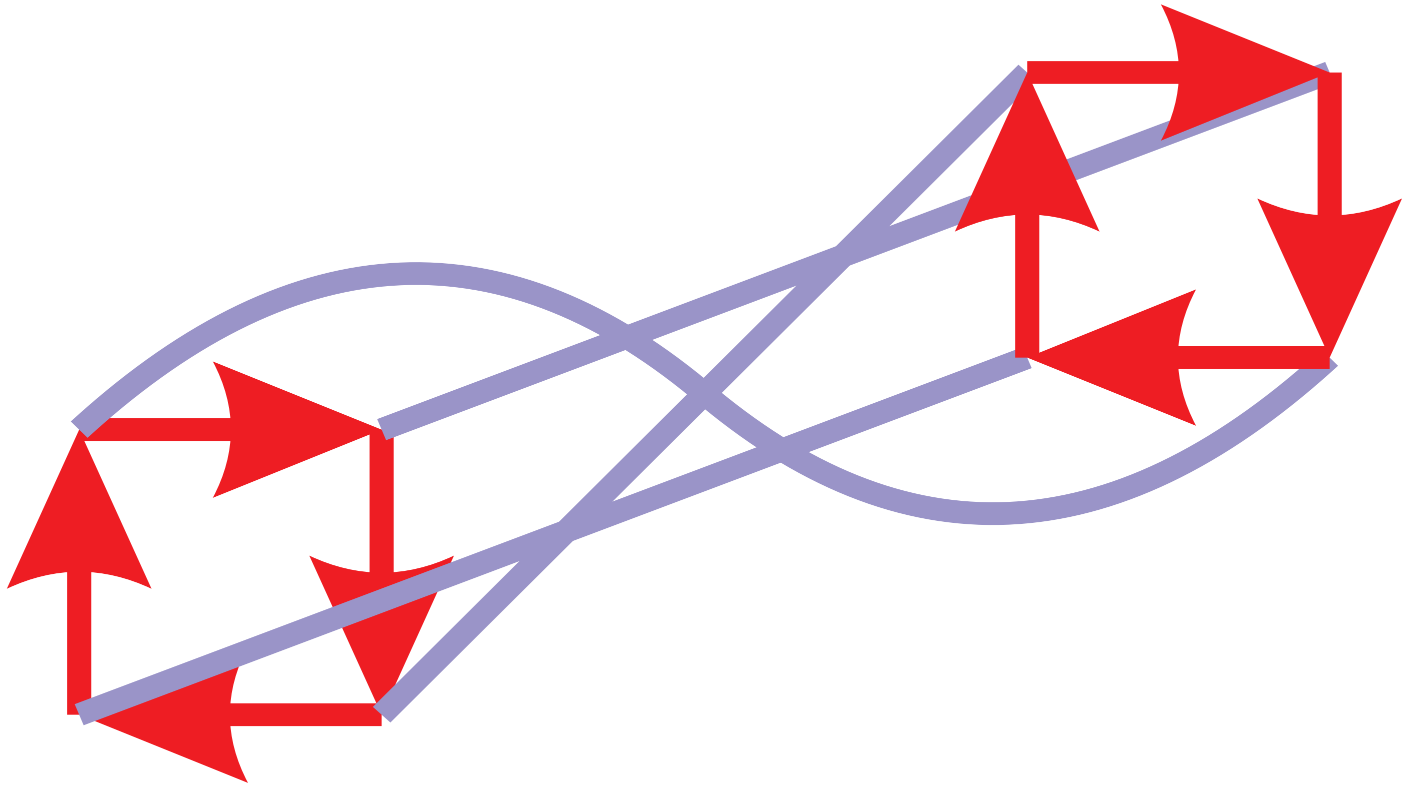

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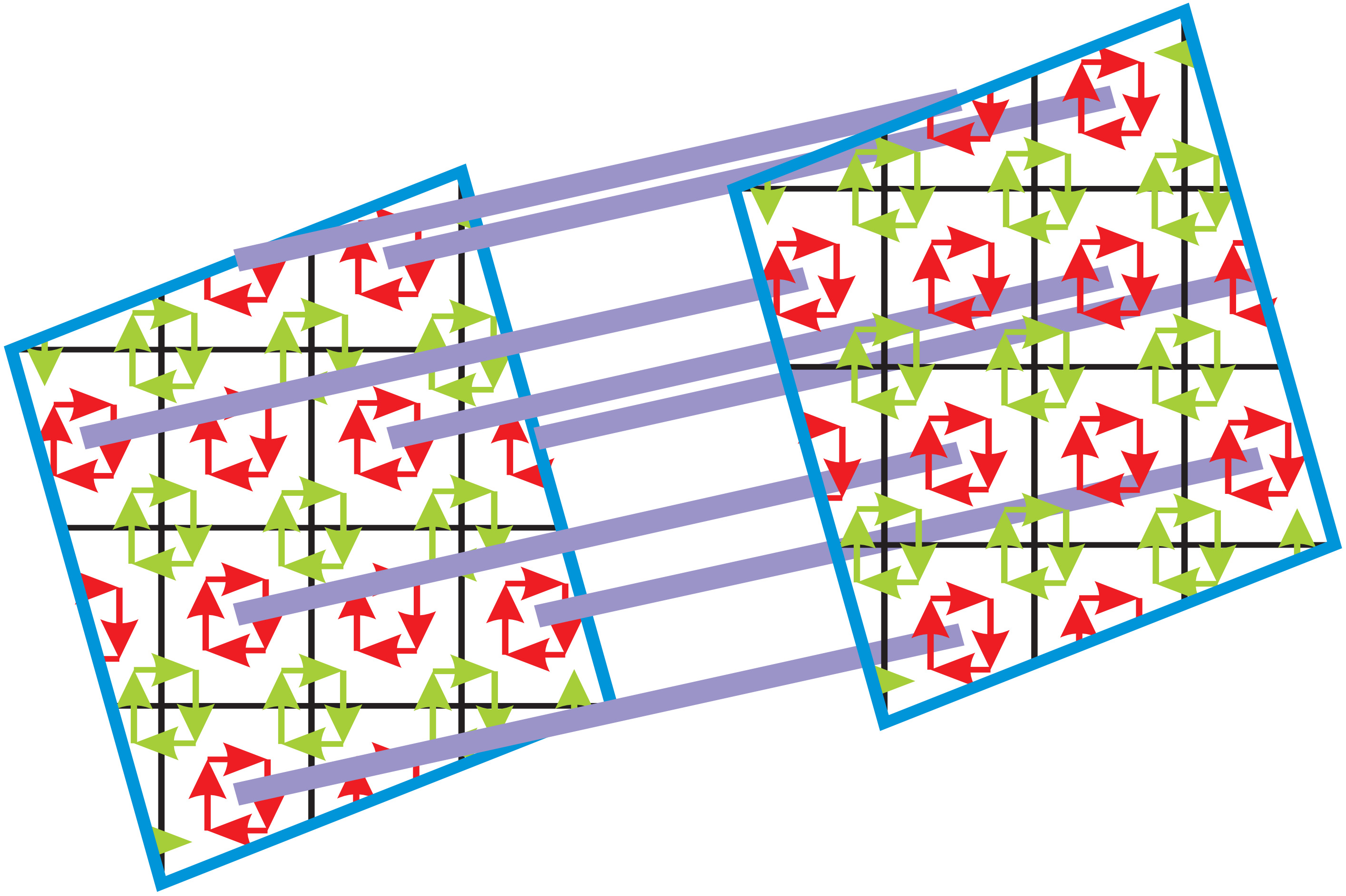

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