Classification of Regular and Chiral Polytopes by Topology

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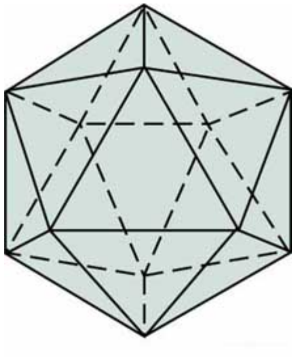
Classical Regular Polytopes — Review

Convex polytope: convex hull of finitely many points in $\mathbb{E}^n$

Key observation: topologically spherical, both globally and locally!

Regularity: flag transitivity of the symmetry group
(other equivalent definitions).

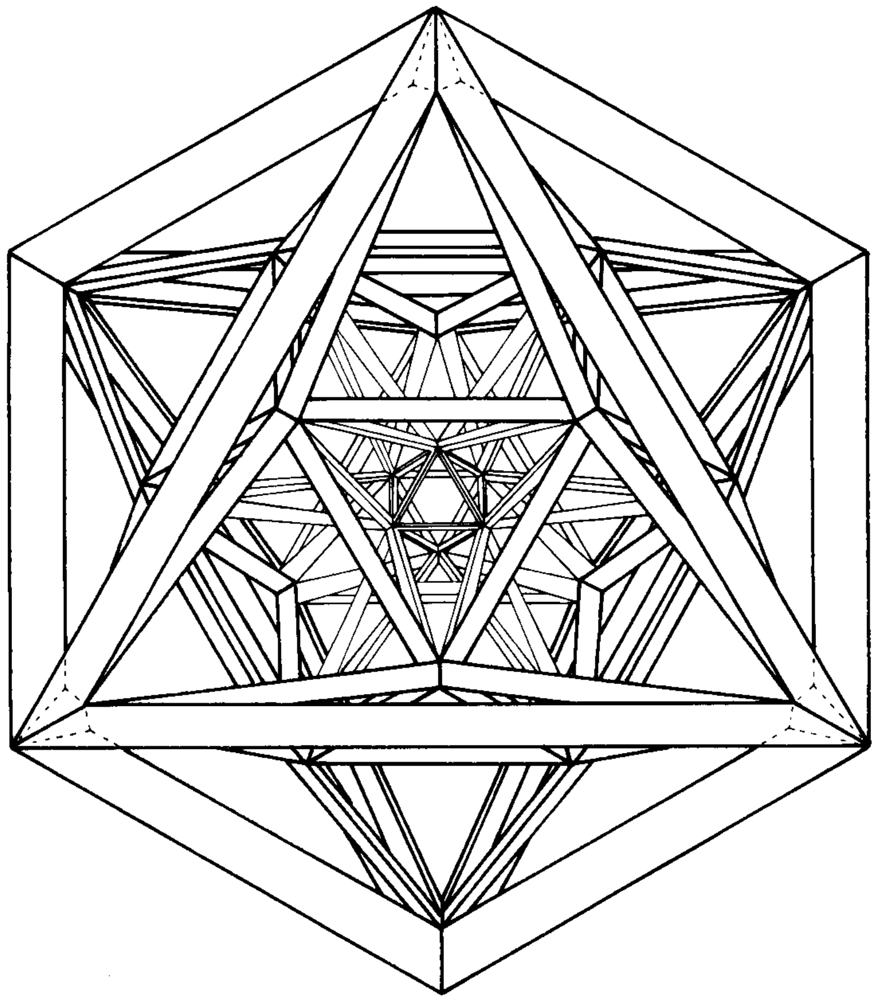
- n=2: polygons $\{p\}$ (Schläfli-symbol)
- n=3: Platonic solids $\{p, q\}$



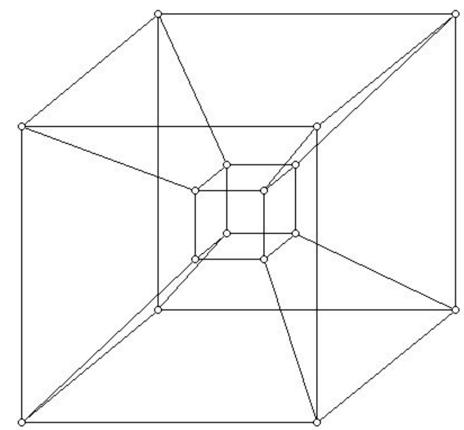
$\{3, 5\}$

DIMENSION n≥4

name	symbol	#facets	group	order
simplex	{3,3,3}	5	S_5	120
cross-polytope	{3,3,4}	16	B_4	384
cube	{4,3,3}	8	B_4	384
24-cell	{3,4,3}	24	F_4	1152
600-cell	{3,3,5}	600	H_4	14400
120-cell	{5,3,3}	120	H_4	14400
simplex	{3,. . . ,3}	n+1	S_{n+1}	$(n+1)!$
cross-polytope	{3,. . . ,3,4}	2^n	B_{n+1}	2^nn!
cube	{4,3,. . . ,3}	2n	B_{n+1}	2^nn!

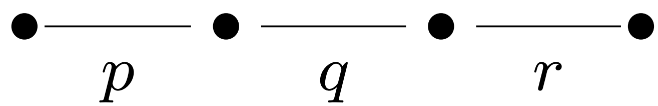


24-cell $\{3,4,3\}$
(with thickened edges)



4D cube $\{4,3,3\}$

Symmetry group of $\{p, q, r\}$ is the Coxeter group with string diagram



Presentation

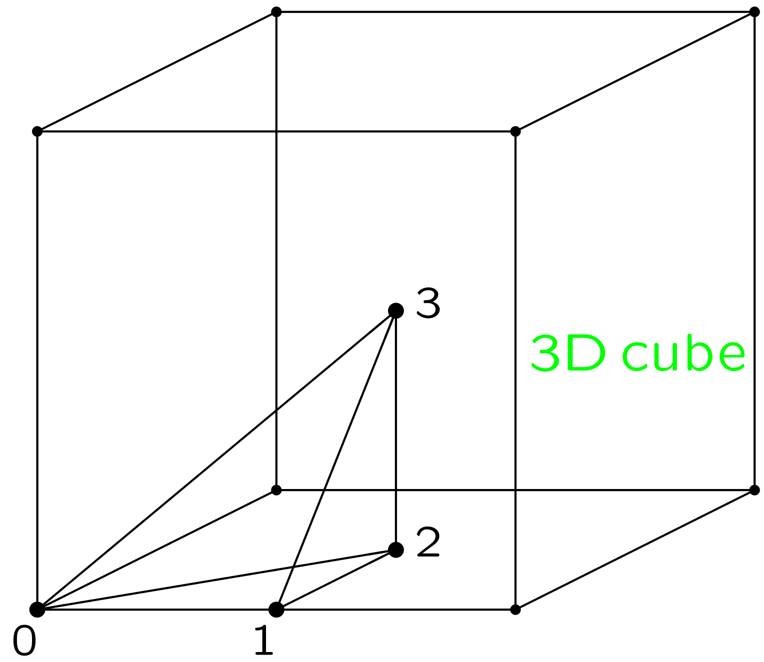
$$\rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = 1$$
$$(\rho_0\rho_1)^p = (\rho_1\rho_2)^q = (\rho_2\rho_3)^r = 1$$
$$(\rho_0\rho_2)^2 = (\rho_1\rho_3)^2 = (\rho_0\rho_3)^2 = 1$$

Generators are reflections in the walls of a fundamental chamber.

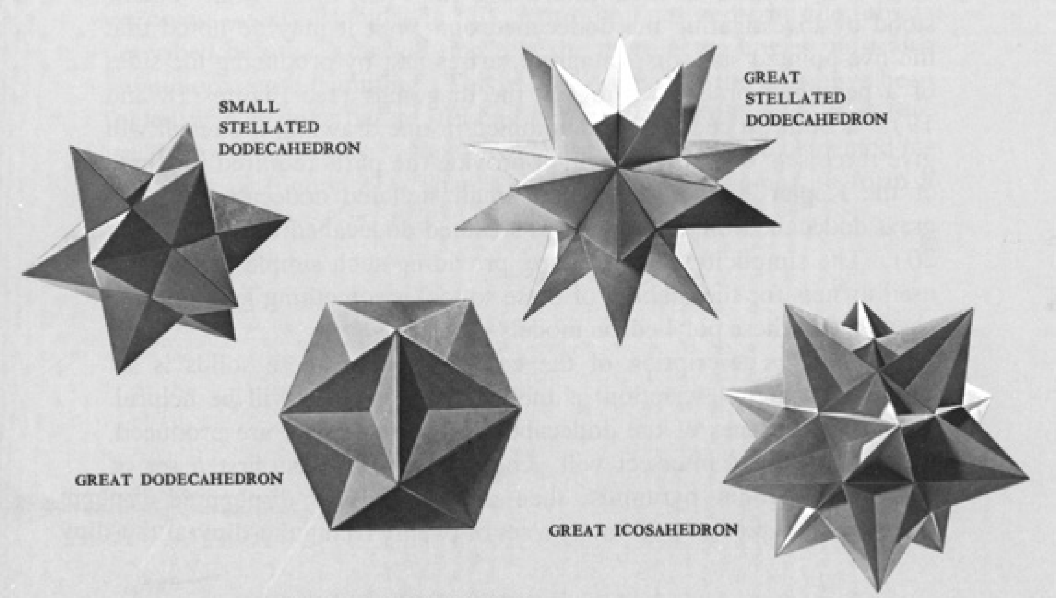
Presentation for 3-cube

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = 1$$

$$(\rho_0\rho_1)^4 = (\rho_1\rho_2)^3 = (\rho_0\rho_2)^2 = 1$$



• **Regular star-polyhedra — Kepler-Poinsot polyhedra** (Kepler 1619, Poinsot 1809). Cauchy (1813).



THE FOUR KEPLER-POINSOT SOLIDS

• **Ten regular star-polytopes in dimension 4. None in dimension > 4.**

Dim.	Symbol	f_0	f_{n-1}	Group
$n = 3$	$\{3, \frac{5}{2}\}$	12	20	H_3
	$\{\frac{5}{2}, 3\}$	20	12	
	$\{5, \frac{5}{2}\}$	12	12	
	$\{\frac{5}{2}, 5\}$	12	12	
$n = 4$	$\{3, 3, \frac{5}{2}\}$	120	600	H_4
	$\{\frac{5}{2}, 3, 3\}$	600	120	
	$\{3, 5, \frac{5}{2}\}$	120	120	
	$\{\frac{5}{2}, 5, 3\}$	120	120	
	$\{3, \frac{5}{2}, 5\}$	120	120	
	$\{5, \frac{5}{2}, 3\}$	120	120	
	$\{5, 3, \frac{5}{2}\}$	120	120	
	$\{\frac{5}{2}, 3, 5\}$	120	120	
	$\{5, \frac{5}{2}, 5\}$	120	120	
	$\{\frac{5}{2}, 5, \frac{5}{2}\}$	120	120	

Regular Star-Polytopes in $\mathbb{E}^n$ ($n \geq 3$)

Regular Honeycombs

Euclidean space

n=2: with triangles, hexagons, squares
{3,6}, {6,3}, {4,4}

n≥2: with cubes, {4,3,...,3,4}

n=4: with 24-cells, {3,4,3,3}
with cross-polytopes, {3,3,4,3}

Hyperbolic space

n=2: each symbol {p,q} with $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$

n=3: # =15 {3,5,3}, {4,3,5}, {5,3,5},{6,3,3}, ...

n=4: # =7 {5,3,3,4}, {5,3,3,5},{3,4,3,4}, ...

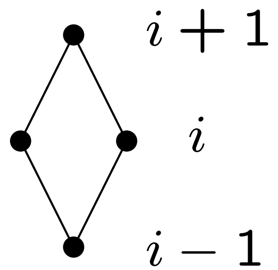
n=5: # =5 {3,3,4,3,3}, {3,3,3,4,3}, ...

n≥6: none

Abstract Polytopes P of rank n

P	ranked partially ordered set
i-faces	elements of rank i (= -1,0,1,...,n)
i=0	vertices
i=1	edges
i=n-1	facets

- Faces F_{-1}, F_n (of ranks -1, n)
- Each flag of P contains exactly n+2 faces
- P is connected
- Intervals of rank 1 are diamonds:

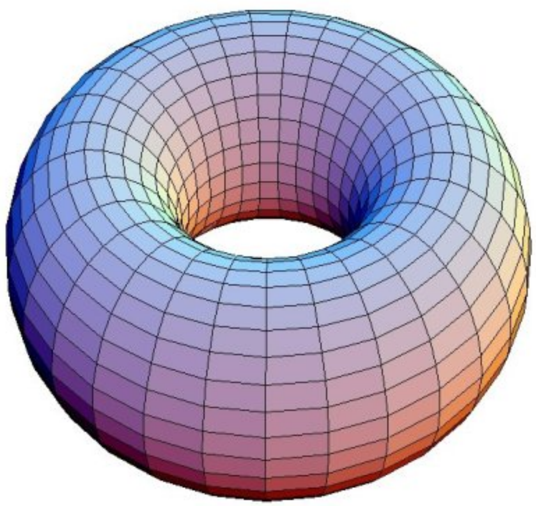


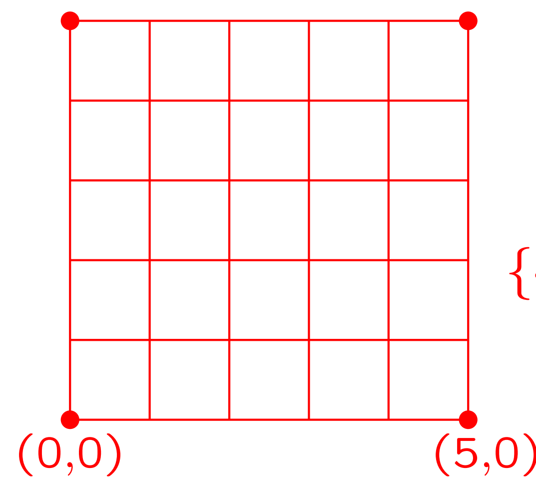
P is *regular* iff Γ(P) flag transitive.

P is *chiral* iff $\Gamma(P)$ has two orbits on the flags such that adjacent flags always are in different orbits.

Nothing new in ranks 0, 1, 2 (points, segments, polygons)!

Rank 3: maps (2-cell tessellations) on closed surfaces.





$\{4,4\}_{(5,0)}$

Rich history: Klein, Dyck, Brahana, Coxeter, Jones & Singerman, Wilson, Conder ....

Well-known: torus maps $\{4,4\}_{(b,c)}$,$\{3,6\}_{(b,c)}$,$\{6,3\}_{(b,c)}$.

Classification of regular and chirals maps by genus (Conder)

— orientable surfaces of genus 2 to 300

— non-orientable surfaces of genus 2 to 600

Rank $n \geq 4$: How about polytopes of rank 4 (or higher)?

Local picture for a 4-polytope of type $\{4,4,3\}$

Facets: torus maps $\{4,4\}_{(s,0)}$ ($s \times s$ chessboard)

Vertex-figures: cubes $\{4,3\}$

2 tori meeting at each 2-face

3 tori surround each edge

6 tori surround each vertex

Problems: local — global; universal polytopes; finiteness.

regular polytopes $\Longleftrightarrow$ C-groups

C-group $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$

- $$\begin{cases} \rho_i^2 = (\rho_i \rho_j)^2 = 1 \ \ (|i-j| \geq 2) \\ (\rho_0\rho_1)^{p_1} = (\rho_1\rho_2)^{p_2} = \ldots = (\rho_{n-2}\rho_{n-1})^{p_{n-1}} = 1 \\ \text{\& in general additional relations!} \end{cases}$$

- *Intersection property* $\langle \rho_i \,|\, i \in I \rangle \cap \langle \rho_i \,|\, i \in J \rangle = \langle \rho_i \,|\, i \in I \cap J \rangle$

Polytope associated with Γ

j-faces — right cosets of $\Gamma_j := \langle \rho_i \mid i \neq j \rangle$

partial order: $\Gamma_j \varphi \leq \Gamma_k \psi$ iff $j \leq k$ and $\Gamma_j \varphi \cap \Gamma_k \psi \neq \emptyset$.

Quotient of the Coxeter group $\bullet \overset{}{\underset{p_1}{\text{———}}} \bullet \underset{p_2}{\text{———}} \bullet \cdots\cdots \bullet \underset{p_{n-1}}{\text{———}} \bullet$

Topological classification (of universal polytopes)

Classical case spherical or locally spherical

$\Updownarrow$

quotient of a regular tessellation in S^{n-1}, E^{n-1} or H^{n-1}

Grünbaum's Problem (mid 70's): Classify toroidal and locally toroidal regular polytopes.

Step 1: Tessellations on the $(n-1)$-torus (globally toroidal)

Step 2: Locally toroidally polytopes only in ranks $n = 4, 5, 6$.

A lot of progress! Enumeration complete for $n = 5$; almost complete for $n = 4$; conjectures for $n = 6$.

McMullen & S.; also Weiss, Monson

Toroids

Torus maps $\{4,4\}_{(b,c)}$,$\{3,6\}_{(b,c)}$,$\{6,3\}_{(b,c)}$. How about higher-dimensional tori?

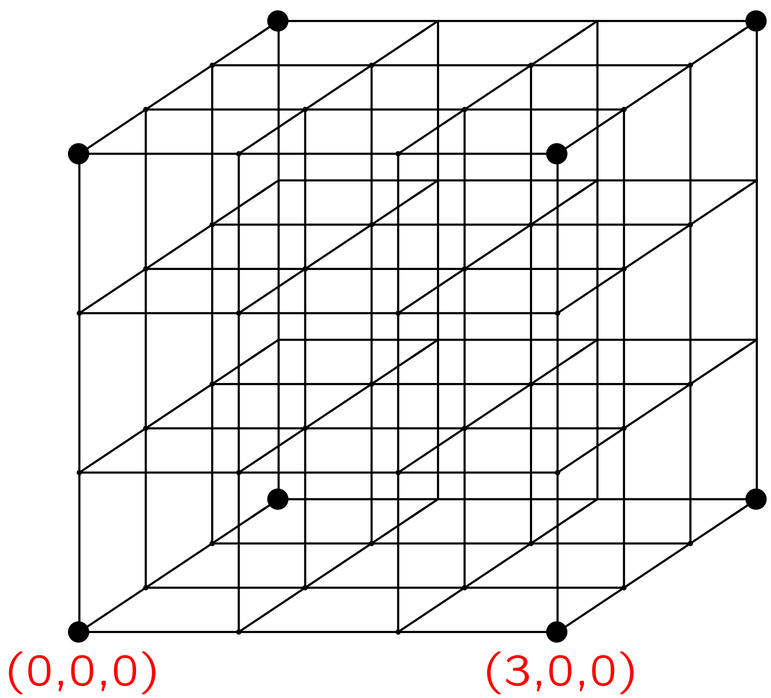
Tessellations $\mathcal{T}$ in euclidean space

- $n = 2$: with triangles, hexagons, squares, $\{3,6\}$, $\{6,3\}$, $\{4,4\}$
- $n \geq 2$: with cubes, $\{4,3,...,3,4\}$
- $n = 4$: with 24-cells, $\{3,4,3,3\}$; with cross-polytopes, $\{3,3,4,3\}$

Regular toroids of rank $n+1$ (McMullen & S.)

Quotients $\mathcal{T}/\Lambda$ of regular tessellations $\mathcal{T}$ in $\mathbb{E}^n$ by suitable lattices Λ.

A toroid with 27 cubical facets on the 3-torus (rank 4)



Type $\{4,3,4\}_{(3,0,0)}$

$(\rho_0\rho_1\rho_2\rho_3\rho_2\rho_1)^3 = 1$

Cubical Toroids $\{4, 3^{n-2}, 4\}_{\mathbf{s}}$ on n-Torus

s	vertices	facets	order	lattice
$(s, 0, \ldots, 0)$	s^n	s^n	$(2s)^n \cdot n!$	$s\mathbb{Z}^n$
$(s, s, 0, \ldots, 0)$	$2s^n$	$2s^n$	$2^{n+1}s^n \cdot n!$	sD_n
$(s, \ldots, s)$	$2^{n-1}s^n$	$2^{n-1}s^n$	$2^{2n-1}s^n \cdot n!$	$2sD_n^*$

Standard relations for $\bullet \overset{}{\underset{4}{\text{——}}} \bullet \underset{3}{\text{——}} \bullet \ldots \bullet \underset{3}{\text{——}} \bullet \underset{4}{\text{——}} \bullet$

and the single extra relation

$$(\rho_0\rho_1 \ldots \rho_n\rho_{n-1} \ldots \rho_k)^{ks} = 1 \quad (k = 1, 2 \text{ or } n, \text{resp.})$$

Exceptional Toroids $\{3,3,4,3\}_{\mathbf{s}}$ on 4-Torus (up to duality)

s	vertices	facets	order	lattice
$(s,0,0,0)$	s^4	$3s^4$	$1152s^4$	sD_4 (self-reciprocal D_4)
$(s,s,0,0)$	$4s^4$	$12s^4$	$4608s^4$	sD_4

Standard relations for ● —3— ● —3— ● —4— ● —3— ●

and the single extra relation

$$\begin{cases} (\rho_0\,\sigma\,\tau\,\sigma)^s = 1 & \text{if } \mathbf{s} = (s,0,0,0), \\ (\rho_0\,\sigma\,\tau)^{2s} = 1 & \text{if } \mathbf{s} = (s,s,0,0), \end{cases}$$

where $\sigma = \rho_1\,\rho_2\,\rho_3\,\rho_2\,\rho_1$ and $\tau = \rho_4\,\rho_3\,\rho_2\,\rho_3\,\rho_4$.

Locally Toroidal Regular Polytopes

- universal polytopes = {facets,vertex-figures}

Rank n=4

$\{\{4,4\}_{\mathbf{s}},\{4,3\}\}$,

$\{\{4,4\}_{\mathbf{s}},\{4,4\}_{\mathbf{t}}\}$,

$\{\{6,3\}_{\mathbf{s}},\{3,r\}\} \quad (r=3,4,5)$,

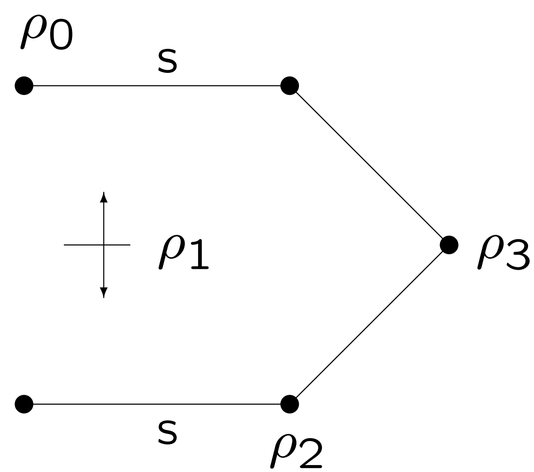
$\{\{6,3\}_{\mathbf{s}},\{3,6\}_{\mathbf{t}}\}$,

$\{\{3,6\}_{\mathbf{s}},\{6,3\}_{\mathbf{t}}\}$,

where $\mathbf{s}=(s,0)$ or (s,s) and $\mathbf{t}=(t,0)$ or (t,t).

Locally toroidal 4-polytopes $\{\{4,4\}_{(s,0)},\{4,3\}\}$

Coxeter group W_s



$$(\rho_0\rho_1\rho_2\rho_1)^s = 1$$

$\Gamma_s := \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle \cong W_s \rtimes C_2$ is the correct group!

The universal polytope is finite iff $s = 2$ or $s = 3$.

The polytope for $s = 3$ (with group $S_6 \rtimes C_2$) can be realized by a tessellation on $\mathbb{S}^3$ consisting of 20 tori (Grünbaum and Coxeter & Shephard).

More on Rank 4

s	v	f	g	Group
$(2,0)$	4	6	192	$D_4 \ltimes S_4$
$(3,0)$	30	20	1440	$S_6 \times C_2$
$(2,2)$	16	12	768	$C_2 \wr D_6$

The finite polytopes $\{\{4,4\}_{\mathbf{s}},\{4,3\}\}$**,** $\mathbf{s} = (s,0),(s,s)$**.**

$\mathbf{s}$	$\mathbf{t}$	v	f	g	Group
$(2,0)$	(t,t), $t \geq 2$	4	$2t^2$	$64t^2$	$(D_t \times D_t \times C_2 \times C_2) \rtimes (C_2 \rtimes C_2)$
$(2,0)$	$(2m,0)$, $m \geq 1$	4	$4m^2$	$128m^2$	$(C_2 \times C_2) \rtimes [4,4]_{(2,0)}$ if $m = 1$; $(D_m \times D_m) \rtimes [4,4]_{(2,0)}$ if $m \geq 2$
$(3,0)$	$(3,0)$	20	20	1440	$S_6 \times C_2$
$(3,0)$	$(4,0)$	288	512	36864	$C_2 \wr [4,4]_{(3,0)}$
$(3,0)$	$(2,2)$	36	32	2304	$(S_4 \times S_4) \rtimes (C_2 \times C_2)$
$(2,2)$	$(2,2)$	16	16	1024	$C_2^4 \rtimes [4,4]_{(2,2)}$
$(2,2)$	$(3,3)$	64	144	9216	$C_2^6 \rtimes [4,4]_{(3,3)}$
$(3,0)$	$(5,0)$	19584	54400	3916800	$Sp_4(4) \times C_2 \times C_2$

The finite polytopes $\{\{4,4\}_{\mathbf{s}}, \{4,4\}_{\mathbf{t}}\}$

(except $\{\{4,4\}_{(s,0)}, \{4,4\}_{(t,0)}\}$, with s, t odd and distinct)

Conjecture

The universal polytopes $\{\{4,4\}_{(s,0)},\{4,4\}_{(t,0)}\}$, with s,t odd and distinct, are finite iff the regular tessellation $\{s,t\}$ is spherical (that is, iff $(s,t)=(3,5),(5,3)$.)

Case $(s,t)=(3,5)$: $\quad Sp_4(4)\times C_2\times C_2$.

Still more on Rank 4

r	s	v	f	g	Group
3	$(2,0)$	10	5	240	$S_5 \times C_2$
	$(3,0)$	54	12	1296	$[1\ 1\ 2]^3 \rtimes C_2$
	$(4,0)$	640	80	15360	$[1\ 1\ 2]^4 \rtimes C_2$
	$(2,2)$	120	20	2880	$S_5 \times S_4$
4	$(1,1)$	12	8	288	$S_3 \rtimes [3,4]$
	$(2,0)$	16	16	768	$[3,3,4] \rtimes C_2$
5	$(2,0)$	240	600	28800	$[3,3,5] \rtimes C_2$

The finite polytopes $\{\{6,3\}_{\mathrm{s}},\{3,r\}\}$
($\mathrm{s}=(s,0),(s,s)$ and $r=3,4,5$).

Thm The universal regular 4-polytope $\{\{6,3\}_{(s,0)},\{3,6\}_{(t,0)}\}$ exists for all $s,t \geq 2$. In particular, it is finite if and only if $(s,t) = (2,k)$ or $(k,2)$, with $k = 2,3,4$. In this case, its group is $[1\,1\,2]^k \rtimes (C_2 \times C_2)$, of order $480, 108 \cdot 4!, 256 \cdot 5!$ if $k = 2,3,4$, respectively.

Thm The universal regular 4-polytope $\{\{6,3\}_{(s,s)},\{3,6\}_{\mathbf{t}}\}$, with $\mathbf{t} = (t,0)$ or (t,t), exists for all $s,t \geq 2$. In particular, it is finite if and only if $s = 2$ and $\mathbf{t} = (2,0)$; in this case, its group is $S_5 \times S_4 \times C_2$.

Somewhat open: $\{\{3,6\}_{\mathbf{s}},\{6,3\}_{\mathbf{t}}\}$

Locally toroidal regular polytopes (cont.)

Rank $n = 5$

s	vertices	facets	group	order
$(2,0,0)$	24	8	$C_2^3 \rtimes F_4$	9216
$(2,2,0)$	48	32	$C_2^5 \rtimes F_4$	36864
$(2,2,2)$	1536	2048	$(C_2^6 \rtimes C_2^5) \rtimes F_4$	2359296

Finite polytopes $\{\{3,4,3\},\{4,3,4\}_{\mathbf{s}}\}$

(with $\mathbf{s} = (s,0,0), (s,s,0), (s,s,s)$**)**

F_4

Locally toroidal regular polytopes (cont.)

Rank $n = 6$ (first type)

s	vertices	facets	order
$(2,0,0,0)$	20	960	368640
$(2,2,0,0)$	160	30720	11796480
$(3,0,0,0)$	780	189540	72783360

Conjectured finite polytopes of type

$\{\{3,3,3,4\},\{3,3,4,3\}_{\mathbf{s}}\}$

Rank $n = 6$ (second type)

s	**t**	vertices	facets	order
$(2,0,0,0)$	$(t,0,0,0)$ (t even)	32	$2t^4$	$36864t^4$
$(2,0,0,0)$	$(t,t,0,0)$ (t even)	32	$8t^4$	$147476t^4$
$(2,2,0,0)$	$(2,2,0,0)$	2048	2048	150994944
$(3,0,0,0)$	$(3,0,0,0)$	2340	2340	218350080

Conjectured finite polytopes of type

$$\{\{3,3,4,3\}_{\mathbf{s}},\{3,4,3,3\}_{\mathbf{t}}\}$$

Rank $n = 6$ (third type)

s	**t**	vertices	facets	order
$(s, 0, 0, 0)$ (s even)	$(2, 0, 0, 0)$	$3s^4$	16	$18432s^4$
$(s, s, 0, 0)$	$(2, 0, 0, 0)$	$12s^4$	16	$73728s^4$
$(s, 0, 0, 0)$ (s even)	$(2, 2, 0, 0)$	$6s^4$	64	$73728s^4$
$(s, s, 0, 0)$ (s even)	$(2, 2, 0, 0)$	$24s^4$	64	$294912s^4$
$(2, 0, 0, 0)$	$(2, 2, 2, 2)$	384	1024	18874368
$(2, 0, 0, 0)$	$(4, 0, 0, 0)$	12288	65536	1207959552
$(3, 0, 0, 0)$	$(3, 0, 0, 0)$	2340	780	72783360

Conjectured finite polytopes of type

$\{\{3, 4, 3, 3\}_{\mathbf{s}}, \{4, 3, 3, 4\}_{\mathbf{t}}\}$

Open Problem

Classify all locally toroidal chiral polytopes!

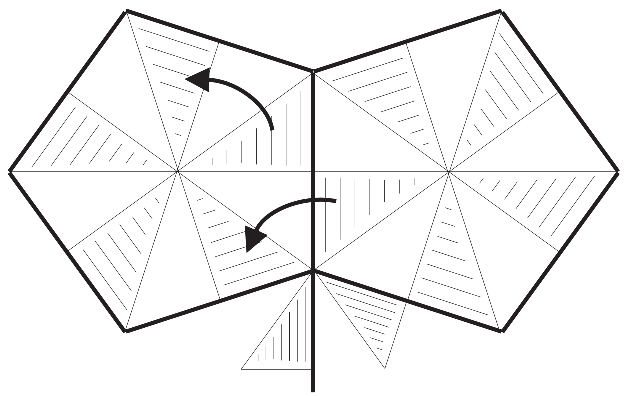
Rank 4: $\{\{4,4\}_{(b,c)},\{4,3\}\}$, $\{\{4,4\}_{(b,c)},\{4,4\}_{(e,f)}\}$, **.....**

Almost completely open!

Chirality

$\Gamma(P)$ has 2 flag-orbits, represented by adjacent flags!

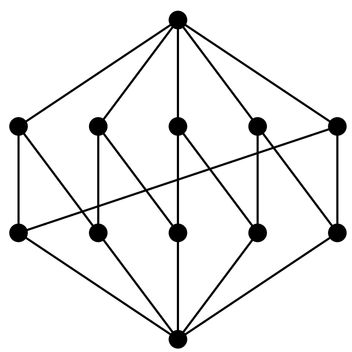
- Rank 3: Lots of chiral torus maps! Occurrence very sporadic, at least for small genus g (next for $g = 7$).



Generators σ_1, σ_2 for type $\{p, q\}$ in rank 3

$$\sigma_1^p = \sigma_2^q = (\sigma_1\sigma_2)^2 = 1 \quad \& \quad \text{generally more relations.}$$

Local definition: P not regular, but for some *base* flag $\Phi := \{F_1, F_0, \ldots, F_n\}$ there exist $\sigma_1, \ldots, \sigma_{n-1} \in \Gamma(\mathsf{P})$ such that σ_i fixes each face in $\Phi \setminus \{F_{i-1}, F_i\}$ and cyclically permutes consecutive i-faces in the section F_{i+1}/F_{i-2}.



$i+1$

i

$i-1$

$i-2$

σ_i cyclically permutes vertices (edges) of p_i-gon

Two enantiomorphic forms: Chiral polytopes occur in a "right-hand" and a "left-hand" version, distinguished by the choice of base flag.

Rank 4

Generators $\sigma_1, \sigma_2, \sigma_3$ for type $\{p, q, r\}$ in rank 4

Standard relations

$$\sigma_1^p = \sigma_2^q = \sigma_3^r = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = 1$$

Example: The universal $\{\{4,4\}_{(b,c)}, \{4,3\}\}$ has extra relation

$$(\sigma_1^{-1}\sigma_2)^b(\sigma_1\sigma_2^{-1})^c = 1$$

Intersection property

$$\langle\sigma_1\rangle \cap \langle\sigma_2\rangle = \langle\epsilon\rangle = \langle\sigma_2\rangle \cap \langle\sigma_3\rangle, \quad \langle\sigma_1, \sigma_2\rangle \cap \langle\sigma_2, \sigma_3\rangle = \langle\sigma_2\rangle$$

Polytopes associated with the groups

Regular polytopes: Γ generated by $\rho_0, \ldots, \rho_{n-1}$

j-faces: right cosets of $\Gamma_j := \langle \rho_i \mid i \neq j \rangle$

Chiral polytopes: Γ generated by $\sigma_1, \ldots, \sigma_{n-1}$

j-faces: right cosets of

$$\Gamma_j := \begin{cases} \langle \sigma_2, \ldots, \sigma_{n-1} \rangle \text{ if } j = 0, \\ \langle \{\sigma_i \mid i \neq j, j+1\} \cup \{\sigma_j \sigma_{j+1}\} \rangle \text{ if } j = 1, \ldots, n-2, \\ \langle \sigma_1, \ldots, \sigma_{n-2} \rangle \text{ if } j = n-1. \end{cases}$$

Partial order in both cases:

$\Gamma_j \varphi \leq \Gamma_k \psi$ iff $j \leq k$ and $\Gamma_j \varphi \cap \Gamma_k \psi \neq \emptyset$.

Rank 4

Plenty of locally toroidal chiral 4-polytopes. (Coxeter, Weiss & S., Monson, Nostrand; 1990's and earlier.)

Key idea: Relevant hyperbolic Coxeter groups have nice representations as groups of Möbius transformations over $\mathbb{Z}[i]$, $\mathbb{Z}[\omega]$, ... Rotation subgroups have generators like $\sigma_1, \sigma_2, \sigma_3$.

Then construct polytopes by modular reduction of the corresponding groups of 2×2 matrices.

Example: Take rotation subgroup of $\bullet \underset{4}{\rule{2em}{0.4pt}} \bullet \underset{4}{\rule{2em}{0.4pt}} \bullet \underset{3}{\rule{2em}{0.4pt}} \bullet$ and work over $\mathbb{Z}_m$, where -1 is a quadratic residue mod m. Gives chiral polytopes of type $\{\{4,4\}_{(b,c)},\{4,3\}\}$ with $m = b^2 + c^2$, $(b,c) = 1$ and group $PSL_2(\mathbb{Z}_m)$ or $PSL_2(\mathbb{Z}_m) \rtimes C_2$. (Work modulo the ideal in $\mathbb{Z}[i]$ generated by $b + ic$.)

Higher ranks

- Lots of finite examples in "low ranks" by Conder, Hubard & Pisanski; Breda, Jones & S.; Conder & Devillers, ...

- Finite examples for every rank $n \geq 3$ (Pellicer, 2009)!

- Extension problem: Chiral n-polytope P as the facet of a chiral $(n+1)$-polytope Q? Facets of P regular!

(a) Universal: $\Gamma(Q) = \Gamma(P) *_{\Gamma^+(F)} \Gamma(F)$ (Weiss & S., 1994)

(b) Finite Q, if P is finite. (Cunningham & Pellicer, 2013)

- n-torus is the only compact euclidean space form with regular or chiral tessellations. Chirality only when $n = 2$! (Hartley, McMullen & S., 1999)

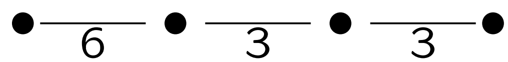
..... The End .....

Thank you

Abstract

The past three decades have seen a revival of interest in the study of polytopes and their symmetry. The most exciting new developments all center around the concept and theory of abstract polytopes. The lecture gives a survey of currently known topological classification results for regular and chiral polytopes, focusing in particular on the universal polytopes which are globally or locally toroidal. While there is a great deal known about toroidal regular polytopes, there is almost nothing known about the classification locally toroidal chiral polytopes.

Example: P $= \{\{6,3\}_{(s,s)}, \{3,3\}\}$



extra relation: $(\rho_2(\rho_1\rho_0)^2)^{2s} = 1$

Polytopes of type {6,3,r}

1. Normal subgroup W of Γ(P) of finite index!

2. "Locally unitary" representation

$$\varphi : W \mapsto GL_m(C)$$

which preserves a hermitian form h on C^m.

3. Finiteness of P is decided by h!

$$\boxed{P = \{\{6,3\}_{(s,s)}, \{3,3\}\}}$$

$$\begin{cases} W = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle \\ \sigma_i^2 = (\sigma_i \sigma_j)^3 = 1 \\ (\sigma_i \sigma_j \sigma_k \sigma_j)^s = 1 \end{cases}$$

GROUP: $\Gamma(P) = W \ltimes S_4$

$$\rho_0 = \sigma_1, \ \rho_1 = \tau_1, \ \rho_2 = \tau_2, \ \rho_3 = \tau_3$$

Structure of $W = W_s$?

REPRESENTATION

$$\varphi: \quad W \mapsto GL_4(C)$$

$$\sigma_i \mapsto S_i \ (i = 1, 2, 3, 4),$$

where

$$S_i(x) = x - 2\, h(x, e_i)\, e_i$$

HERMITIAN FORM:

$e_1, \ldots, e_4$ canonical basis of C^4

$$h(x,y) := \sum_{i=1}^{4} x_i \overline{y}_i - \sum_{i \neq j} c_{ij} x_i \overline{y}_j \ ,$$

$$\langle S_i, S_j, S_k \rangle \cong [1\ 1\ 1]^s \ (\text{“locally unitary”}).$$

Choice of c_{ij}:

$$c_{12} = c_{34} = c_{31} = \frac{e^{2\pi i/s}}{2}$$

$$c_{23} = c_{24} = c_{41} = \frac{e^{-2\pi i/s}}{2}$$

Situation: W acts on C^4 as a reflection group

Theorem: W finite iff h positive definite

Classification of unitary reflection groups: Shephard, Todd, Coxeter, Cohen

Consequence: $\{\{6,3\}_{(s,s)},\{3,3\}\}$ finite
iff h positive definite

$$\det(h) = \frac{1}{16}(-9 - 16\cos\frac{2\pi}{s} - 2\cos\frac{4\pi}{s})$$

$$\text{h is } \begin{cases} \text{positive definite for } s = 2 \\ \text{positive semi-definite for } s = 3 \\ \text{indefinite for } s \geq 4 \end{cases}$$

<u>Thm:</u> $P := \{\{6,3\}_{(s,s)}, \{3,3\}\}$ exists for each $s \geq 2$, and P is finite iff $s = 2$.

<u>s=2:</u> $\Gamma(P) = S_5 \times S_4$,
$W = S_5 = \langle (1\,5), (2\,5), (3\,5), (4\,5) \rangle$