Colourful Simplices and Octahedral Systems

Tamon Stephen

Simon Fraser University

Department of Mathematics

joint work with

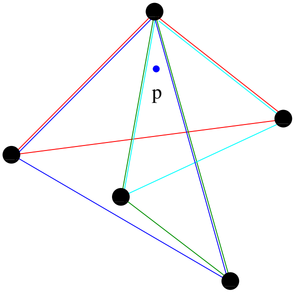
Antoine Deza and **Feng Xie**

The Fields Institute Retrospective Workshop on Discrete Geometry, Optimization and Symmetry, November 28th, 2013

Outline

- Simplicial Depth.
- Colourful Simplices.
- Lower Bounds for Colourful Simplicial Depth.
- Transversals, Octahedra and Octahedral Systems.
- Parity Tables and Enumeration Strategies.
- Subspace Coverings.
- Questions and Discussion.

- Given a set S of n points in $\mathbb{R}^d$, the simplicial depth of any point p with respect to S is the number of open simplices generated by points in S containing p. Denote this $\text{depth}_S(p)$ or just $\text{depth}(p)$.

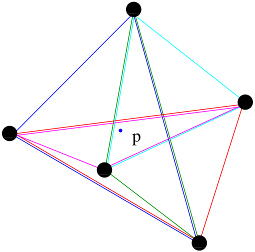


- We consider open rather than closed simplicial depth.

Deepest points

- Question: For fixed n and d, what are the possible values of the (monochrome) $\text{depth}_S(p)$?
- In particular, consider for a given S the quantity:

$$g(S) = \max_p \text{depth}_S(p)$$



- Then $g(S)$ is the maximum number of open simplices generated by S containing a given point.

Bounds for Deepest Points

- For a set S of n points in $\mathbb{R}^2$ the bounds are[1]:

$$n^3/27 + O(n^2) \leq g(S) \leq n^3/24 + O(n^2).$$

- Bárány showed that in dimension d:

$$\frac{1}{(d+1)^{d+1}}\binom{n}{d+1} + O(n^d) \leq g(S) \leq \frac{1}{2^d(d+1)!}n^{d+1} + O(n^d).$$

- The upper bound is tight.
- For fixed d, this gives the correct asymptotics in n. However the gap in constants is large.
- The lower bound has recently been improved by Gromov (2010), Karasev (2012) and Král', Mach and Serini (2012), ...

[1] Boros and Füredi (1984), but see Bukh, Matoušek and Nivasch (2010)

Simplicial Depth Context

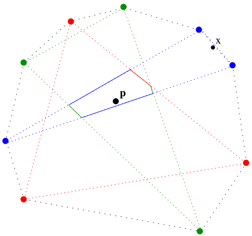
- The simplicial depth of p is an gives an idea of how representative p is of S. It is one of several measures studied by statisticians of the "depth" of a data point relative to a sample.

- A point of maximum simplicial depth can be considered to be a simplicial median. The simplicial median is a multidimensional generalization of the median of a set of numbers.

- The probability that p lies inside a random simplex chosen from S is: $\frac{\text{depth}_S(p)}{n^{d+1}}$.

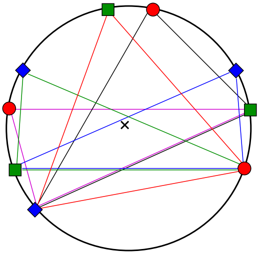
- The algorithmic problem of *finding* a simplex containing p is equivalent to the problem of finding a feasible basis in linear programming.

- Theorem (Bárány): if a point in $\mathbb{R}^d$ is in the convex hull of $(d+1)$ colourful sets, then it can be expressed as a convex combination of points of $(d+1)$ *different* colours.



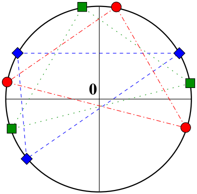
- This is a "Colourful" Carathéodory Theorem.
- We call the intersection of the $(d+1)$ colourful sets the core of the configuration.
- Note that it is not sufficient to have the point in the convex hull of some colour(s).

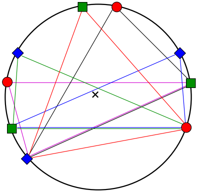
- Define a colourful configuration $\mathbf{S}$ to be a collection of $d+1$ sets of points $S_1, \ldots, S_{d+1}$ in $\mathbb{R}^d$.
- Define the colourful simplicial depth, denoted $\mathbf{depth_S}(p)$, of a point p with respect to a colourful configuration $\mathbf{S}$ to be the number of open colourful simplices from $\mathbf{S}$ containing p.



- Let $\mu(d)$ be the minimum colourful simplicial depth of a core point in dimension d.

- In a typical (random) situation, we expect to find $\mathbf{0}$ in around $\frac{(d+1)^{d+1}}{2^d}$ simplices.
- **Theorem**: There is a configuration of $d+1$ points in each of $d+1$ colours with $\mathbf{0}$ in the convex hull of each colour, but with $\mathbf{0}$ contained in only d^2+1 colourful simplices.
- **Conjecture**: This is minimal, i.e. $\mu(d) = d^2+1$ for all d.
- True for $d = 0, 1, 2, 3, 4$.
- **Example**: A 2-dimensional colourful configuration which contains $\mathbf{0}$ in only 5 simplices:





- The core of a colourful configuration is:

$$\bigcap_{i=1}^{d+1} \operatorname{conv}(S_i).$$

- We make the following assumptions:
  - We have $d+1$ points of each colour.
  - The points are in general position.
  - We have $\mathbf{0} \in \operatorname{int} \operatorname{core} \mathbf{S}$.

- By scaling the points, we assume without loss of generality that they lie on the unit sphere $\mathbb{S}^d \subset \mathbb{R}^d$.

Colourful Simplicial Depth Context

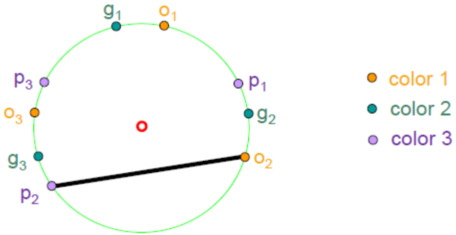
- The Colourful Carathéodory Theorem was originally proved by Bárány in the service of proving his lower bound for monochrome simplicial depth. This proof can be trivially modified to include a factor of $\mu(d)$ in the lower bound.
- There remains a probabilistic interpretation: the probability that p lies in a simplex whose vertices are sampled independently from the S_i's is: $\dfrac{\mathbf{depth_S}(p)}{|S_1| \cdot \ldots \cdot |S_{d+1}|}$.
- Given a colourful configuration with $\mathbf{0}$ in the core, the Colourful Linear Programming question of *efficiently* finding a colourful set of $(d+1)$ points containing $\mathbf{0}$ in their convex hull is an interesting problem whose complexity remains poorly understood.
- Recent research interest includes considering relaxed core conditions.

Lower bounds

- From Bárány (1982), we can deduce $\mu(d) \geq d+1$.
- Deza et al. (2006) show $\mu(d) \geq 2d$ and $\mu(2) = 5$.
- Quadratic lower bounds were independently obtained in Bárány and Matoušek (2007) and S. and Thomas (2008) using somewhat different methods. Additionally, Bárány and Matoušek showed that $\mu(3) = 10$.
- Deza, S. and Xie (2011): $\mu(d) \geq \lceil (d+1)^2/2 \rceil$.
- A computational approach described in this talk (2013) improves this by one in dimension 4.
- Deza, Meunier, and Sarrabezolles have recently announced proofs that $\mu(d) \geq \frac{d^2}{2} + \frac{7d}{2} - 8$ and $\mu(4) = 17$.

Transversals

- All these lower bounds depend on a key fact that we call the *Octahedron Lemma*. Octahedra are built from transversals.
- Fix a colour i. We call a set t of d points that contains exactly one point from each $\mathbf{S}_j$ other than $\mathbf{S}_i$ an $\widehat{i}$-transversal.

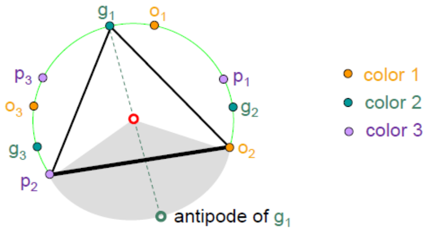


- In the picture, p_2 and o_2 form a $\widehat{2}$-transversal.

Image: A. Deza

Transversals and Antipodes

- Transversals are generators of colourful cones.

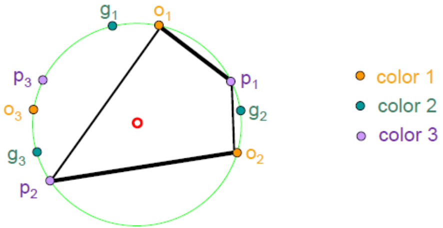


- An $\widehat{i}$-transversal and a point of colour i form a colourful simplex containing $\mathbf{0}$ if and only if the ray from $\mathbf{0}$ through the antipode of the point passes through the affine hyperplane generated by the transversal.

Image: A. Deza

Octahedra

- We call any pair of disjoint $\widehat{i}$−transversals an $\widehat{i}$*-octahedron*.



- These may or may not generate a geometric cross-polytope (d-dimensional octahedron).

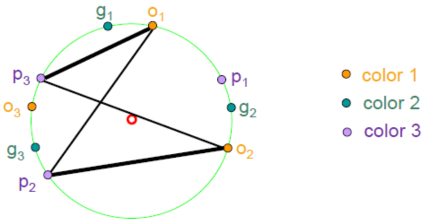
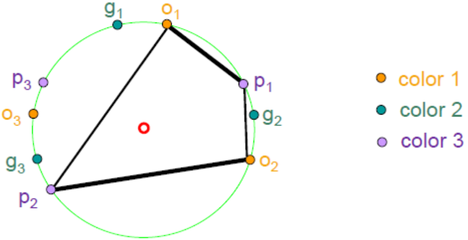
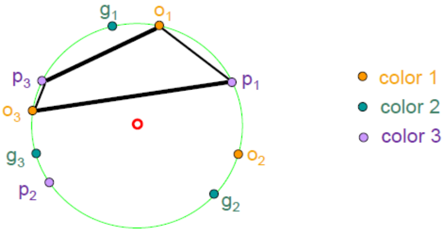


Image: A. Deza

Octahedral Lemma

- The **Octahedron Lemma**: Rays from **0** in *general position* always intersect the same parity of facets made from $\hat{i}$−transversals of any fixed $\hat{i}$-octahedron.





Images: A. Deza

From Geometry to Combinatorics

- A colourful configuration defines a $(d+1)$-uniform hypergraph on $\mathbf{S} = \cup_{i=0}^{d} \mathbf{S}_i$ by taking edges corresponding to the vertices of $\mathbf{0}$ containing colourful simplices. Call these configuration hypergraphs.
- A strong version of the Colourful Carathéodory Theorem implies that any configuration hypergraph $\mathcal{H}$ must satisfy **Property 1**: Every vertex of a configuration hypergraph $\mathcal{H}$ belongs to some edge of $\mathcal{H}$.
- The Octahedron Lemma gives that any configuration hypergraph $\mathcal{H}$ must satisfy **Property 2**: For any octahedron $\mathcal{O}$, the parity of the set of edges using points from $\mathcal{O}$ and a fixed point s_i for the $i\text{th}$ coordinate is the same for all choices of s_i.
- Call a hypergraph whose edges consist of one vertex from each of $(d+1)$ sets and satisfying Properties 1 and 2 a covering octahedral system.

Small Octahedral Systems

- One strategy for proving lower bounds is to show that there are no small covering octahedral systems.
- Let $\nu(d)$ be the smallest size of a non-trivial covering octahedral system. Then $\nu(d) \leq \mu(d) \leq d^2 + 1$. Conjecture: $\nu(d) = \mu(d) = d^2 + 1$.
- We begin by fixing a colour 0 and $d + 1$ disjoint $\widehat{0}$-transversals t_i for $i = 0, \ldots, d$.
- We include initial edge 00...0 and focus on three key quantities of a candidate covering octahedral system:
  - ℓ, the number of edges containing t_0. ?00...0
  - b the number of the octahedra formed from t_0 and t_i for some $i = 1, 2, \ldots, d$ that have odd parity. $t_0 * t_i$
  - j the minimum number of $\widehat{0}$-transversals that form an edge with any point of colour 0. 0??...?

- It is clear that for any covering octahedral system with d^2 or fewer edges we must have $1 \leq \ell, b, j \leq d$.

- The number of edges in the system is at least $j(d+1)$.

- We can get further inequalities by studying the tradeoffs between edges required to satisfy the odd parity octahedra and the even parity octahedra: $(b+\ell)(d+1) - 2b\ell$ and $j + b \geq d + 1$.

- Finally, if we choose colour 0 so as to minimize ℓ but still have $\ell \geq \frac{d+2}{2}$, then we also have that the number of edges is at least $d\ell + 1$.

- These inequalities combine to give $\nu(d) \geq \lceil (d+1)^2/2 \rceil$.

A Small Parity Table

- For a given $(d+1)$-uniform hypergraph, we can form a parity table that lists the parity of each point of colour 0 with respect to the octahedra generated by t_0 and each of the transversals $t_1, t_2, \ldots t_d$.
- Example: With $d = 4$, this is the parity table for the hypergraph with 3 edges: 00...0, 10...0, . . . , 20...0, i.e. $(0, t_0), (1, t_0)$ and $(2, t_0)$.

octahedron ↓ 0^{th} point →	0	1	2	3	4
$t_0 * t_1$	1	1	1	0	0
$t_0 * t_2$	1	1	1	0	0
$t_0 * t_3$	1	1	1	0	0
$t_0 * t_4$	1	1	1	0	0

- Only edges containing t_0 can change more than one entry in this table.

Repairing the Small Parity Table

The choice of b dictates the required parities of the octahedra $t_0 * t_i$ for $i = 1, \ldots, d$. Without loss of generality, these can be 1 for $i = 0, 1, \ldots, b-1$ and 0 for $i = b, b+1, \ldots, d$. Then given b, the parity table corresponding to the hypergraph must have b constant rows of ones, followed by $d - b$ constant rows of zeros. In the case where $d = 4$ and $b = 2$, this would be

octahedron ↓ 0^{th} point →	0	1	2	3	4
$t_0 * t_1$	1	1	1	1	1
$t_0 * t_2$	1	1	1	1	1
$t_0 * t_3$	0	0	0	0	0
$t_0 * t_4$	0	0	0	0	0

Thus, starting from the hypergraph consisting of the edges 00...0, 10...0 and 20...0 (previous overhead), we need to add at least 10 additional edges to get the proper parity table for $b = 2$.

Exclusion via Enumeration

- We implemented an enumeration scheme to improve the bound (slightly) for $d = 4$.
- We start by fixing a choice of (ℓ, b, j).
- Beginning with an empty hypergraph, add edges initially as required by ℓ, these are unique up to symmetry.
- Then repair the parity table. At each stage we add one of the 15 edges that flip a single entry in the table.
- Next we try to add edges using the fact that a covering octahedral system with d^2 or fewer edges cannot have any isolated edges that differ from all other edges of the hypergraph in more than one vertex.
- As a last resort we may have to add arbitrary edges.

A Large Parity Table

- An octahedral system needs to satisfy an enormous number of parity conditions simultaneously.
- Call a set of $(d+1)$ points, one of each colour, a full transversal.
- Meunier and Deza (2013) reformulate Property 2 elegantly as **Property 2'**: For any pair of full transversals, the number of edges from the octahedral system that are contained in the pair must always be even.
- For the edge $T_0 := t_0 \cup \{0\}$ alone, there are d^{d+1} such parity conditions that must be satisfied.

Fixing a Large Parity Table

- Consider now building an octahedral system beginning with T_0 and adding additional edges.
- With T_0 alone, all d^{d+1} parity conditions fail.
- Adding an edge will flip exactly d^k parity conditions, where k is the number of 0's in the edge.
- This immediately gives the fact that any octahedral system with d^2 or fewer edges must not contain any isolated edges: if T_0 is isolated, the number of parity conditions fixed by adding an edge is at most d^{d-1}, thus we require at least d^2 additional edges.

The Parity Cube

- Rather than simply counting parity conditions, we should exploit their natural structure.
- Each full transversal is indexed by a point in $\{1, 2, \ldots, d\}^{d+1} \subseteq \mathbb{R}^{d+1}$. We call this the parity cube.
- The effect of adding edge e to the configuration is to flip all parity conditions in the subspace defined by the equations $x_i = e_i$ for each non-zero entry in e.

  So, for example with $d = 4$, including edge 12020 changes exactly the d^2 parity conditions of points in the subspace $\{x_0 = 1, x_1 = 2, x_3 = 2\}$.

  The initial edge T_0 changed the entire $(d + 1)$-dimensional parity cube, while an edge disjoint from T_0 will change a single parity condition, i.e. a 0-dimensional subspace.

Subspace Coverings

Thus the problem of fixing the parity conditions for T_0 can be viewed as a subspace covering problem (modulo 2). Required is to cover (modulo 2) the points of the parity cube, i.e. $\{1, 2, \ldots, d\}^{d+1}$, by non-trivial coordinate subspaces.

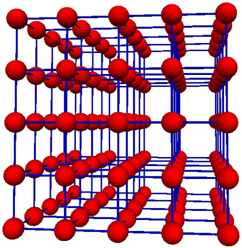


Image: HPC REU @ UMBC

Subspace Coverings

- For the cover to satisfy Property 1, we note that if edge e contains point $i \geq 1$ of colour j, then the related subspace satisfies $x_j = i$. The 0 points of each colour are in T_0, so we need merely to require that the subspace cover includes at least one subspace contained in each of the $d(d+1)$ hyperplanes $x_j = i$ for $i = 1, \ldots, d$ and $j = 0, \ldots, d$.
- Thus we would like to find such a (mod 2) subspace cover of minimal size.
- If we drop the (mod 2) condition, an inductive approach should show that such a cover requires at least d^2 subspaces.
- Unfortunately with the (mod 2) condition, there are subspace covers of size $d^2/2 + O(d)$, which do not appear to to arise from octahedral systems.

Questions and Discussion

- A gap remains even for $d = 5$.
- Deza, Meunier and Sarrabezolles show that some covering octahedral systems are not realizable via colourful configurations. However, it remains possible that $\mu(d) = \nu(d)$ for all d.
- Can we get lower bounds analogous to the lower bound for the monochrome $g(S)$ for the maximum colourful simplicial depth of a point in colourful configuration? (The point is not necessarily in the core.)
- There is interesting recent progress on the monochrome depth problem.
- How to compute colourful simplicial depth efficiently?
- The complexity of Colourful Linear Programming.

Thank you!